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WITTGENSTEIN ON EQUINUMEROSITY AND SURVEYABILITY

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Summary

This paper aims to connect two of Wittgenstein's arguments against Logicism. The 'modality argument' is directed at the Frege/Russell-definition of numbers in terms of one-one correlations. According to this argument, it is only when the Fs and Gs are few in number that one can know that they can be one-one correlated without knowing their numbers. Wittgenstein's 'surveyability argument' purports to show that only a limited portion of arithmetic can actually be proven within *Principia Mathematica*. For proof-constructions within this system quickly become unsurveyable and thereby loose their cogency. As we shall argue, the role of visualisation in proofs plays a fundamental role in both arguments.

1. *Introduction*

In this paper, we wish to draw attention to a close link between two important arguments in Wittgenstein's philosophy of mathematics, namely his argument against the use of the notion of one-one correlation in the Frege-Russell definition of numbers, which we call here the 'modality argument', and his notorious 'surveyability argument'. The fact that these two arguments are closely related points to a nexus in his philosophy of mathematics, a point where it would stand or fall. The former argument has received less attention in the secondary literature, but both have at any rate failed to convince. Alas, this is not the place to mount a defence, should we wish to, because it is important first to understand correctly the nature of those arguments. We limit ourselves here to this last task only—but this task should be seen, however, as part of a larger investigation of the core claims of Wittgenstein's philosophy of mathematics. We shall thus first propose in some detail an interpretation of the modality argument and then briefly show how it is related to the surveyability argument.

2. *The modality argument*

It is useful, in order to understand the point of Wittgenstein's argument against one-one correlation, to recall some details of the Frege-Russell definition of numbers. In §63 of *Grundlagen der Arithmetik*, Frege introduced a cardinality operator, 'the number of Fs', which is nowadays written:

$$\mathrm{N}x : \mathrm{F}x.$$

He introduced this operator with a contextual definition, which is known as 'Hume's principle', according to which 'the number of Fs is equal to the number of Gs if and only if they are in a one-one correlation' (here: F ≈ G):

$$\mathrm{F} \approx \mathrm{G} \leftrightarrow \mathrm{N}x : \mathrm{F}x = \mathrm{N}x : \mathrm{G}x$$

The idea behind this definition is that it provides an all important criterion of identity, i.e., a criterion for our being able to recognize again the same number. The key here is thus Frege's definition of 'equinumerosity' (§§71–72), which reads like this: F and G are 'equinumerous' just in case there is a relation R such that every object belonging to F—Frege would say 'falling under F'—has the relation R to a unique object belonging to G and every object falling under G is such that there is a unique object belonging to F which also has the relation R to it. To get the definition going, we need the notion of 'unique existence':

$$\exists! x\ \mathrm{H}x =_{\mathrm{def}} \exists x\ (\mathrm{H}x\ \&\ \forall y\ (\mathrm{H}y \rightarrow y = x))$$

So the definition reads formally as:

$$\mathrm{F} \approx \mathrm{G} =_{\mathrm{def}} \exists \mathrm{R}\ ((\forall x\ (\mathrm{F}x \rightarrow \exists! y\ (\mathrm{G}y\ \&\ \mathrm{R}xy))\ \&\ (\forall x\ (\mathrm{G}x \rightarrow \exists! y\ (\mathrm{F}y\ \&\ \mathrm{R}xy)))$$ [1]

1. We omit details that are of no importance here, e.g., the fact that one can show that '≈' is reflexive, symmetric and transitive, etc.

With this notion Frege can then define, in §73, the number of Fs in terms of 'classes of classes':

$$\mathrm{N}x : \mathrm{F}x =_{\mathrm{def}} \{\mathrm{G}: \mathrm{G} \approx \mathrm{F}\}$$

And from there he can go on defining natural numbers. For example, a number such as 2 is defined in terms of the class of all classes that are in one-one correlation with a given pair. In *Principia Mathematica*, Russell and Whitehead proceed in a similar fashion to obtain the same definition, albeit in the rather complicated syntax of their type theory. Hence the name 'Frege-Russell definition'—it is a key to their 'logicism'.

Hume's principle is a biconditional, but Frege provides an argument that might properly be called 'philosophical' to the effect that the direction that really counts is from left to right, i.e., from the fact that there is a one-one correlation '$\mathrm{F} \approx \mathrm{G}$' to the sameness of number, or '$\mathrm{N}x : \mathrm{F}x = \mathrm{N}x : \mathrm{G}x$'. Frege's argument at §§64–68 involves, however, showing the priority of 'The line *a* is parallel to *b*' over 'The direction of *a* is the same as the direction of *b*', and this gets him into some further difficulties into which we need not get into. The reason for his having to provide an argument here is in the end rather simple: in order for Hume's principle to serve in a convincing manner for the definition of natural numbers, one must, for fear of circularity, use some other notion that does not involve numbers; one-one correlation, he argues, is just this *prior* notion. Indeed, one can correlate a bunch of cups and saucers to see that they are equal in number, without knowing what that number is. To find out what that number is, one would correlate them with natural numbers, i.e., count them.

Wittgenstein discussed the Frege-Russell definition on numerous occasions in his writings and lectures, from the 'middle period' up to and including the 1939 lectures on the foundations of mathematics.[2] Among his numerous remarks, the modality argument plays a central role.[3] One early occurrence of it is in Wittgenstein's conversations with Schlick and Waismann (January 1931):

2. See LFM, 157f. For this reason it would be wrong to dismiss his remarks as merely pertaining to the apparently discredited 'middle period', a typical but exegetically unwarranted move.

3. For the modality argument itself, see Wittgenstein 2003, 373f.; WVC, 164f.; PR, §118; BT, 415; PG, 355f.; AWL, 148f., 158, 161ff. One should note that Wittgenstein hardly ever refers to Frege, but discusses at length the specifics of Russell's own version. There is no need to get into this, however, within the context of this paper.

> In Cambridge[4] I explained the matter to my audience in this way: Imagine I have a dozen cups. Now I wish to tell you that I have got just as many spoons. How can I do it?
>
> If I had wanted to say that I allotted one spoon to each cup, I would not have expressed what I meant by saying that I have just as many spoons as cups. Thus it will be better for me to say, I can allot the spoons to the cups. What does the word "Can" mean here? If I meant it in the physical sense, that is to say, if I mean that I have the physical strength to distribute the spoons among the cups—then you would tell me, We already knew that you were able to do that. What I mean is obviously this: I can allot the spoons to the cups because there is the right number of spoons. But to explain this I must presuppose the concept of number. It is not the case that a correlation defines number; rather, number makes a correlation possible. This is why you cannot explain number by means of correlation (equinumerosity). You must not explain number by means of correlation; you can explain it by means of possible correlation, and this precisely presupposes number.
>
> You cannot rest the concept of number upon correlation. [...] When Frege and Russell attempt to define number through correlation, the following has to be said:
>
> A correlation only obtains if it has been *produced*. Frege thought that if two sets have equally many members, then there is already a correlation too... Nothing of the sort! A correlation is there only when I actually correlate the sets, i.e. as soon as I specify a definitive relation. But if in this whole chain of reasoning the *possibility* of correlation is meant, then it presupposes precisely the concept of number. Thus there is nothing at all to be gained by the attempt to base number on correlation.[5]

We have to keep in mind when interpreting Wittgenstein that the Frege-Russell definition of number is in terms of logic, that is in terms of 'classes' and 'objects' that belong to them. A one-one correlation is thus meant to be a pairing of these objects.

With this point kept in mind, the modality argument is as follows: it is not the case that *there always is* a one-one correlation, as defined above, between the objects belonging to any two classes with the same number (of objects belonging to them). Of course, there *could* be such a correlation between any two classes with the same number of objects belonging to them. So one may claim that any such a correlation not yet

4. The minutes of the Trinity Mathematical Society, reproduced at Wittgenstein 2003, 373, show that Wittgenstein discussed this very topic during their meeting on May 28, 1930.

5. WVC, 164f.

established *can* always be established. One might counter this last move, however, by pointing out with Louis Goodstein that this 'can' is only a "logical possibility",[6] and that this possibility looks more like the *consequence of* the fact that the two classes have the same number of objects belonging to them, than a *condition for* them to have the same number of objects belonging to them. However, if one *already knows* that the two classes have the same number of objects belonging to them, then one surely *knows* that a one-one correlation *can* be established. So, the argument goes, Frege's philosophical claim for the priority of one-one correlation does not hold, and the definition is in danger of simply being circular. One could, however, point out that circular definitions abound in mathematics and that they are not necessarily vicious, so that the claim that the Frege-Russell procedure is in the end circular cannot be held against it without further justification. But, as we said, we do not wish to get side-tracked at this stage into issues pertaining to the evaluation of the argument.

The modality argument also occurs in the writings of Friedrich Waismann[7] and Louis Goodstein,[8] but in both cases one can show that the idea originates in Wittgenstein.[9] As Michael Dummett once pointed out, "very few objections […] have ever been raised" (Dummett 1991, 148) against the Frege-Russell definition, so the modality argument is for that reason of intrinsic importance, even if it is ultimately deemed a failure. But, apart from a short discussion of Waismann's version by Dummett,[10] it has attracted surprisingly little attention.[11] This fact might be explained

6. See Goodstein 1951, 19. This is also strongly implied in BT, 415; PG, 356.

7. See Waismann 1951, 108f.; and Waismann 1982, 45f.

8. See Goodstein 1951, 19.

9. In an 'Epilogue' to *Introduction to Mathematical Thinking*, Waismann identified a manuscript by Wittgenstein (possibly the manuscript now published as *Philosophical Remarks*) as the source for his argument (Waismann 1951, 245); we just saw that he knew the argument from a conversation with Schlick and Wittgenstein in 1931 that he recorded himself in Gabelsberger shorthand. As for Goodstein, he does not give any indication, but the fact that he had been a student of Wittgenstein in the early 1930s, who was largely inspired by him in his own work in mathematical logic, leads us to believe that Wittgenstein is again the source here.

10. See Dummett 1991, 148f.

11. While one of us was probably the first to attract attention to it (Marion 1998, 77–83) there is to our knowledge only a short abstract by Daniel Isaacson (Isaacson 1993), an interesting pair of papers by Boudewijn de Bruin (de Bruin 1999) and (de Bruin 2008), and a short discussion by Gregory Landini in *Wittgenstein's Apprenticeship with Russell* (Landini 2007, 168ff.). (Although Isaacson 1993 was in fact published earlier, it was prompted by Marion 1991, 81–88, which eventually found its way, in a revised form, in Marion 1998, 77–83.)

by the fact that Dummett's critique is generally taken as having put it to rest. Be this as it may, this is no reason to give up trying to understand the nature of the modality argument.

On this score, two comments can be made at the outset. First, Dummett begins his defence of Frege thus:

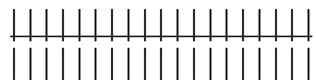
> The objection is readily answered. Frege invokes no modal notions: his definition is in terms of there *being* a suitable mapping. Waismann's objection can easily be reformulated as being that Frege owed us a criterion for the existence of relations, and that no such criterion can be framed without circularity.[12]

He then proceeds to show that such a criteria can be given without circularity, invoking in particular the axiom of choice for the (non-denumerably) infinite case, since one can *prove* with it the existence of one-one correlations between non-denumerably infinite sets. We have no qualms with this (at least for the moment), but one should note that Wittgenstein argues his point only for finite numbers: if one's wish is to understand the argument, it is better to restrict the discussion to this case, instead of attacking it in reference to a case it was not meant to cover. Secondly, this quotation shows that Dummett's objections are based on a reading of the modality argument as an 'ontological' argument about the *existence* of 'one-one correlations'; we think that this is incorrect and favour instead, following Boudewijn de Bruin, an 'epistemic' reading of it in terms of *knowledge* of 'one-one correlations'.[13]

The key to de Bruin's reformulation resides in noticing that Wittgenstein's own formulations are indeed in *epistemic* terms. It is not as if Wittgenstein was not wary of the ontological presuppositions of the Frege-Russell definition, i.e., about the *existence* of the 'one-one correlations' necessary for it to go through, as he frequently discusses them.[14] But his formulations of the modality argument are nearly always in terms of *knowledge* of 'one-one correlations', for example at the beginning of the following passage:

> Can I know there are as many apples as pears on this plate, without knowing how many? And what is meant by not knowing how many? And how can I find out how many? Surely by counting. It is obvious that you can discover that there are the same number by correlation, without counting the classes.

12. Dummett 1991, 148f.
13. See his de Bruin 1999; and de Bruin 2008.
14. For example, at AWL, 158, 161f., 164f.; PG 356; LFM, 162.



> In Russell's theory only an *actual* correlation can show the 'similarity' between the classes. Not the *possibility* of correlation, for this consists precisely in numerical equality. Indeed, the possibility must be an *internal* relation between the extensions of the concepts, but this internal relation is only given through the equality of the 2 numbers.[15]

With K standing for the usual operator from epistemic logic, de Bruin defines '*de re* knowledge' as knowledge that there is an object x such that one knows that it has the property P, or

$$\exists x\, K\mathrm{P}x$$

and '*de dicto* knowledge' as knowledge that there is an object x that has property P, or

$$K\exists x\, \mathrm{P}x$$

As is usually assumed, *de re* knowledge entails *de dicto* knowledge:

$$\exists x\, K\mathrm{P}x \rightarrow K\exists x\, \mathrm{P}x$$

So de Bruin introduces a notion of '*merely de dicto* knowledge', i.e., '*de dicto* but not *de re* knowledge':

$$K\exists x\, \mathrm{P}x \;\&\; \neg\exists x\, K\mathrm{P}x$$

These notions allow for the following reformulation of the modality argument. First, to draw a one-one correlation between the Fs and the Gs without any knowledge of 'how many' Fs and Gs there are gives us *merely de dicto* knowledge and can be reformulated as

$$K\exists n\, (\mathrm{N}x : \mathrm{F}x = n \;\&\; \mathrm{N}x : \mathrm{G}x = n) \;\&\; \neg\exists n\, K(\mathrm{N}x : \mathrm{F}x = n \;\&\; \mathrm{N}x : \mathrm{G}x = n)$$

This is the situation described above, where one can draw a one-one correlation for large numbers, and therefore know that there are equally many without knowing how many: one knows that there exists a number n which

15. PR, §118.

is the cardinality of F and G, but one does not know what number that is. Counting gives instead *de re* knowledge: one knows of some number n that it is the cardinality of the Fs and of the Gs, or

$$\exists n\, K(\mathrm{N}x : \mathrm{F}x = n \ \&\ \mathrm{N}x : \mathrm{G}x = n)$$

On the other hand, drawing a one-one correlation gives one *de re* knowledge of the one-one correlation, but this does not presuppose *de re* knowledge about sameness of cardinality. This notion of *de re* knowledge of the one-one correlation would also correspond to the notion of 'actual' one-one correlations in Wittgenstein's argument and it is the opposite to *merely de dicto* knowledge about a one-one correlation, which corresponds rather to the notion of 'possible' one-one correlations in Wittgenstein's argument.

With these notions at hand, one may indeed reformulate Wittgenstein's argument by simply pointing out that, according to him, *merely de dicto* knowledge of one-one correlation presupposes *de re* knowledge about sameness of cardinality; de Bruin has moreover argued that, under some constructivist principles about existence and knowledge, the modality argument is valid.[16] Again, we wish to steer clear of issues concerning the evaluation of the argument; we would like simply to ask for Wittgenstein's underlying arguments. Is there any reason why *merely de dicto* knowledge of one-one correlation would presuppose *de re* knowledge about sameness of cardinality? Why would one only *know* that a one-one correlation *can* be established only when one *already knows* that the two classes have the same number?

But asking this question is equivalent to asking: How could one establish a one-one correlation without counting? Let us thus suppose there are nine apples and nine oranges on a table. Of course, the purpose of a one-one

16. See de Bruin 2008, 365. These principles are: (1) for something to exist means that it be constructed; (2) every piece of knowledge must eventually rest on some constructive piece of knowledge; (3) there are precisely two independent ways to obtain knowledge about one-one correlation between two concepts, one involving one-one correlation, one involving cardinality. To assess the plausibility of attributing them to Wittgenstein would leave us far afield. At least this much shows that in order for his arguments to hold, Wittgenstein had to be committed to constructivist principles, a conclusion that the vast majority of his commentators have adamantly refused to draw; in despair they usually prefer to discount his philosophy of mathematics altogether. We should point out, however, that we assume here some equivalent to (3), which is that there are two independent ways to obtain knowledge about a one-one correlation, namely a direct way, by some form of subitization, and an indirect way, by counting.

correlation is *not* to find out 'how many' of these there are, it tells one only if there are 'as many' apples as there are oranges. Correlating them would mean something like putting an apple together with each orange, and when this procedure has come to an end, one can say that one now *knows* that there are as many apples as there are oranges. One can thus infer that 'The number of apples and oranges on this table is the same', without necessarily *knowing* what that number is. There is obviously no way one would know their number without resorting to counting, unless that number is small enough for one to take it in at a glance without any error.

It is a matter of human physiology, which is the topic of much research in psychology and neuroscience, that humans can recognize at a glance without failing numbers smaller than 4, and with occasional failure up to 7, but that, for higher numbers, they start counting. (One interesting point to make about subitizing is that the world's many abaci—Chinese, Japanese, Russian, etc.—are designed so that one can usually take in at a glance large numbers without subitizing numbers greater than five.) The process by which one immediately recognizes small numbers is called 'subitizing', from the Latin '*subitus*' or 'sudden'.[17] So, for very small numbers within the domain of subitization, one could recognize immediately the sameness of numbers. To circumvent such obvious limitations, one might arrange the sets of objects in a familiar pattern, e.g., two rows, so that one also immediately sees if they have the same numbers. Or one might in some cases, e.g., when these are figures on a sheet of paper, draw lines. One often finds the latter procedure in textbooks, to get the idea of a one-one correlation across to students.[18]

Wittgenstein was perfectly aware of these various criteria. For example, in section 115 of the *Big Typescript*, he wrote:

> Here incidentally there is a certain difficulty about the numerals (1), ((1) + 1), etc.: beyond a certain length we cannot distinguish them any further without counting the strokes, and so without translating the signs into different ones. "||||||||||" and "|||||||||||" cannot be distinguished in the same sense as 10 and 11, and so they aren't in the same sense distinct signs. The same thing

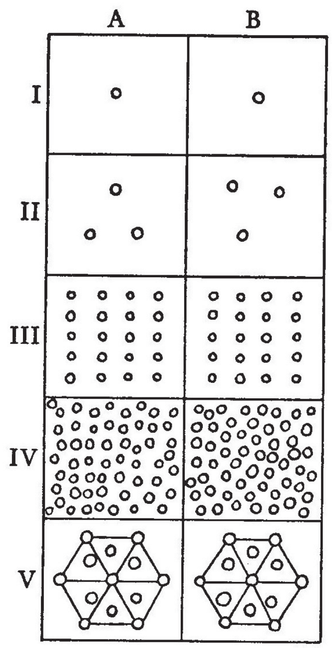
17. The term 'subitization' was introduced in Kaufman, Lord, Reese & Volkmann 1949. The idea that 'subitizing' and 'counting' are the result of two independent neural processes is still a matter of debate, but this is of no importance in the context of our discussion. For examples of contributions to this debate, see Simon & Vaishnavi 1996; Piazza *et al.* 2002; and Revkin *et al.* 2008.

18. See, e.g., Enderton 1977, 129 for one clear example.

> could also happen incidentally in the decimal system (think of the numbers 1111111111 and 11111111111), and that is not without significance.[19]

And, in section 118 of the same typescript, he draws a variety of criteria, I to V:

> Sameness of number, when it is a matter of a number of lines "that one can take it in a glance", is a different sameness from that which can only be established by counting the lines.



> Different criteria for sameness of number. In I and II the number that one immediately recognizes; in III the criterion of correlation, in IV we have to count both groups; in V we recognize the same pattern.[20]

One will have recognized subitization as involved in I and II.

These passages clearly show, therefore, that Wittgenstein was aware of the role played here by visual thinking.[21] He also saw that Russell & Whitehead implicitly rely on subitization:

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19. BT, 398; PG, 330.
20. BT, 414; PG, 354.
21. One objection here would be to rule out our discussion by claiming that it amount to 'psychologism'. To show that it isn't would leave us to far afield, so we would like simply to refer, in the case of the modality argument, to de Bruin's explanations in de Bruin 2008, 366ff., and for the surveyability argument, below, to Marion 2011, 150f.

> We actually say, "Well *this* is one and *this* is one." It is very important for the treatment of *Principia Mathematica* that there are classes whose numerical equality we can take in at a glance.[22]

And he did not condemn establishing one-one correlations by drawing lines either:

> [...] if asked whether *abc* and *def* could have different numbers, the answer is No, since these can be surveyed. Would you call it an experiment to correlate *abcd... w* and $\alpha\beta\gamma\ \dots\ \omega$ so as to *see* whether they have the same number? Would you say that you determine by experiment whether the number of numbers between 4 and 16 is the same as the number of those between 25 and 38? No, this is determined [...] using dashes or something similar.
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> It is a pernicious prejudice to think that using dashes is an experiment and substraction a calculation. This is comparable to supposing a Euclidean proof by using drawing is inexact whereas by using words it is not.[23]

The study of visual thinking in mathematics or logic has been considered a forbidden zone since Frege. One should note that Wittgenstein clearly objects in this passage to the formalist tendency, perhaps exacerbated in the Hilbert school, to denigrate it. As it turns out, the study of visual thinking has recently become more respectable. Marcus Giaquinto, who has been one of the main contributors to this field, concluded a recent survey stating that:

> Visual thinking can occur as a non-superfluous part of thinking through a proof and it can at the same time be irreplaceable, in the sense that one could not think through the same proof by a process of thought in which the visual thinking is replaced by some thinking of a different kind.[24]

As we can see, Wittgenstein could not but agree with this: the modality argument, properly understood, is rather in line with Giaquinto's comment.[25] This, of course, goes against the grain of much of Wittgenstein scholarship, where commentators often premise their interpretation on a formalist stance, which is not open to discussion, so to rule out the sort of things we say here. But these passages are clear: Wittgenstein recognizes the role of visual thinking and faults Russell for misunderstanding it.

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22. LFM, 164.
23. AWL, 158f.
24. Giaquinto 2008, 39f.; see also Giaquinto 2007.
25. One should note, however, that Giaquinto never discusses the point we are claiming Wittgenstein raised here, so we are not implying that there is a convergence between their ideas.

In a nutshell, his point is as follows. One-one correlations can be divided into two classes: a first class will contain those that are actual, in the sense that they are produced by one of the above criteria, subitization, pattern recognition, drawing lines, etc. It is a fact, however, that all these criteria will eventually *peter out when numbers grow large enough*. (To take an obvious example, one would not be able to correlate with any amount of certainty two sets of 3 million elements by drawing lines.) So there must be a second class which will comprise all the possible, non-actual one-one correlations. Wittgenstein's argument is thus, simply, that it is illegitimate to assume that what is sufficient for the first class, namely some form of visual recognition, is also sufficient for the second class. Thus the different criteria for producing an 'actual' one-one correlation (subitization, drawing lines, etc.) eventually peter out, and, once they have effectively come to an end, one is left with no other choice but to count, which would give *de re* knowledge about sameness of (cardinal) number. So merely *de dicto* knowledge of one-one correlation will presuppose, once other criteria become ineffective, *de re* knowledge. This is the answer to our question: Is there any reason why *merely de dicto* knowledge of one-one correlation would presuppose *de re* knowledge about sameness of cardinality?

3. *The Surveyability Argument*

The same point, we contend, lies at the heart of Wittgenstein's surveyability argument. We won't spend too much time reconstructing the argument and its implications, as one of us did it elsewhere.[26] The target here is what may be called Russell's version of mathematical 'explicativism', in particular a pair of theses explicitly framed by Mark Steiner:[27]

i) it is sufficient to understand proofs written in the system of *Principia Mathematica* in order to know all the truths of arithmetic that we know; and
ii) it is possible for us actually to come to know arithmetical truths by constructing logical proofs of them.[28]

26. See Marion 2011.
27. See Steiner 1975, 25.
28. Steiner talks here in terms of proofs, but our discussion below, with formulas (a)–(c), does not involve proofs.

In a well-known passage from *Introduction to Mathematical Philosophy*, Russell pointed out that the "primitive concepts" contained in Peano's axioms, '0', 'number', and 'successor', are "capable of an infinite number of different interpretations, all of which will satisfy the five primitive propositions" (Russell 1919, 7). Given one such interpretation, one obtains a 'progression', which he defined as a series with a beginning but endless and containing no repetition and no terms that cannot be reached from the beginning in a finite number of steps. There is indeed an infinity of such 'progressions' which will, like the series of natural numbers, satisfy Peano's axioms—it suffices for example to start any given series with a natural number other than 0. So Russell argued that in Peano's arithmetic[29] "there is nothing to enable us to distinguish between […] different interpretations of his primitive ideas", while

> We want our numbers not merely to verify mathematical formulae, but to apply in the right way to common objects. We want to have ten fingers and two eyes and one nose. A system in which "1" meant 100, and "2" meant "101", and so on, might be all right for pure mathematics, but would not suit daily life. We want "0" and "number" and "successor" to have meanings which will give us the right allowance of fingers and eyes and nose. We have already some knowledge (though not sufficiently articulate or analytic) of what we mean by "1" and "2" and so on, and our use of numbers in arithmetic must conform to this knowledge.[30]

The idea here would be that an interpretation within the logical system of *Principia Mathematica* of Peano's axioms, provides a definite meaning to its basic number-theoretic concepts and that this interpretation would allow one to recover applications of arithmetic, i.e., that we have 'ten fingers and two eyes and one nose', etc. The very purpose of *Principia Mathematica* thus appears to be this:

iii) to set up an interpretation of Peano's axioms in order to provide a definite meaning to its primitive terms; and
iv) to recover ordinary applications of arithmetic.

The surveyability argument that Wittgenstein deploys against (i)–(iv) is easily stated by taking any ordinary number-theoretic equation, such as:[31]

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29. The expression 'Peano's arithmetic' occurs at Russell 1919, 5, and in this original sense, it differs from today's frequent use of it as a name for first-order arithmetic.

30. Russell 1919, 9.

31. One of Wittgenstein's examples, ad RFM, III, §11.

(a) 27 + 16 = 43

According to Wittgenstein, this equation must have a counterpart in Russell & Whitehead's *Principia Mathematica*, of the form: [32]

(b) $(\exists!_{27}x(Fx)\ \exists!_{16}x(Gx)\ \&\ «x\ \neg(Fx\ \&\ Gx)) \rightarrow\ (\exists!_{43}x(Fx \vee Gx))$

Now, Russell's stance in (i)-(iv) amounts to an 'explicativist' claim of the sort '(a), and (a) *because of* (b)'.[33] Against this, Wittgenstein first noted that (b) must merely be an abbreviation of a longer formula with a total of 43 variables on each side of the sign for the conditional, or a formula with iterated '!' such as this:

(c) (∃!!!!!!!!!!!!!!!!!!!!!!!!!!!*x* (F*x*) ∃!!!!!!!!!!!!!!!!!!!*x* (G*x*) & «*x* ¬(F*x* & G*x*)) →
(∃!!*x* (F*x* ∨ G*x*))[34]

He could then easily point out that this unabbreviated formula is 'unsurveyable' in the sense that one cannot tell the precise number of iterations of '!' unless one starts counting them. Wittgenstein is simply relying here on the fact that human beings cannot tell at a glance (without counting) that there are 27 exclamation marks following the first existential quantifier of (c). This is but the same point (subitization) made above. And one should note that this is not an appeal to 'vagueness' (as most 'anti-realist' readers of Wittgenstein assume); there is nothing vague at all about the fact that there are 27 exclamation marks.[35]

It is thus hard to see what value there would be for '(a) *because of* (b)' given that, visually, the strings of '!' in (c) provide no certainty. Moreover, even for the abbreviated formula (b) one has to calculate in order to know what to write on the right-hand side of the conditional. Doing this would

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32. One may wonder where Wittgenstein got formulas such as (b). The definition of addition in Part II, section B of *Principia Mathematica* is rather complicated because of the need to account for ambiguity of types and, as far as one can tell, there is no formula corresponding to (b). The closest is at *54.43:

$\vdash :.\ \alpha, \beta\ \varepsilon\ 1\ .\supset: \alpha \cap \beta = \Lambda\ .\equiv.\ \alpha \cup \beta\ \varepsilon\ 2.$

See Marion 2011, 142f. for a discussion.

33. Again, for a justification of this claim, see Marion 2011, 143ff. & 152–155.

34. This notation is even suggested from AWL, 148 quoted below.

35. This is the point made with help of (13) in Marion 2011, 150. One should note that, as explained in that paper, this reading of the surveyability argument goes against decades of misunderstanding it in terms of 'strict finitism'.

presuppose the very knowledge of the number-theoretic equation (a) which is supposedly certified by (b). Therefore, rather than (a) being grounded on (b), it is (b) which requires *knowledge* of (a) (to see that it is true is an application of (a)). There appears, therefore, to be a circularity in Russell's attempt to ground number-theoretical equations on logic. This is not to say, however, that it is devoid of any interest, since it draws links between addition of natural numbers in a number-theoretic calculus on the one hand and the union of disjoint classes in a logical calculus on the other. It is just that this does not mean that the latter stands as *foundation* for the former, in accordance with the 'explicativist' claim '(a), and (a) *because of* (b)'.[36] In *Philosophical Remarks*, Wittgenstein put the matter this way:

> How can I know that ||||||||| and ||||||||| are the *same* sign? It isn't enough that they look *alike*. For having roughly the same *Gestalt* can't be what constitutes the identity of signs, but just their being the same in number.
>
> If you write (E |||||) etc. (E |||||||) etc. . ⊃ (E ||||||||||||) --- A you may be in doubt as to how I obtained the numerical sign in the right-hand bracket if I don't know that it is the result of adding the two left-hand signs. I believe that makes it clear that this expression is only an application of 5 + 7 = 12 but doesn't represent this equation itself.[37]

Of course, Wittgenstein is writing sloppily, but one recognizes in his formula A here a variant of (a), where the strokes stand for the strings of '!' in its unabbreviated version (c).

The link with the modality argument should be obvious and can be seen immediately by considering a possible objection. The point Wittgenstein is making with respect to the unabbreviated version of (b) is relying on the limits of subitization. One could try and obviate these limitations, without reverting to counting, by drawing lines between the occurrences of '!' on the left-hand side and those on the right-hand side of the conditional, thus putting them into a one-one correlation that shows that both sides have the same number. As the argument goes, the problem is that, without counting, one would still *not know* which number that is, and, further, that any attempt at producing a one-one correlation by means of drawing lines will peter out with larger numbers, for which one could never be certain if the procedure has been applied correctly or if a mistake has crept in.

On these points the surveyability argument bears more than a superficial resemblance with the modality argument. As a matter of fact it is

36 The idea is expressed, for example, at LFM, 260f.

37 PR, § 103.

so obvious that one wonders why it had remained hitherto noticed in the secondary literature. That Wittgenstein had the link between the two arguments always in mind can be seen from these following passages from, respectively, his 1933–34 and 1939 lectures:

> I shall now discuss the idea that "1+1=2" is an abbreviation of such statements as "If I have one apple in one hand, and another in the other, then I have two apples in both hands." In my notation this is: (E*1x*) *fx* (E*1x*) *gx* . (~$*x*) (*fx* . *gx*)) ⊃ (E*2x*) *fx* ∨ *gx*.[38] Now is it true that "1 + 1 = 2" is an abbreviation of the underlined? […] To use a simple example:
>
> $$(\mathrm{E}^{||}x)fx.(\mathrm{E}^{|||}x)gx.\sim(\exists x)fx.gx.\supset.(\mathrm{E}^{|||||}x)fx \ \vee\ gx.$$
>
> Whether this is a tautology or not I decide by adding. Now does it correspond to 2 + 3 = 5? This implication says nothing (as it is either a tautology or a contradiction). […] What is queer about the functional notation (E*15x*) *fx* (E*27x*) *gx* . (~$*x*) (*fx* . *gx*)) ⊃ (E4*2x*) *fx* ∨ *gx* is that we never use it when we are asked to reckon how many apples we have. One has to do an addition before one knows what to write after the quantifier in the consequent.
>
> This leads directly to examination of Russell's and Frege's theory of the cardinal numbers, of which the fundamental notion is correlation.[39]

> Russell puts down (*xy*) (*uv*) ⊃ (*xyuv*) and proves this is a tautology. But suppose you had a greater number of terms—ten million on each side—what would you do? You say you will have to correlate them. Here—(*xy*) (*uv*) ⊃ (*xyuv*)— it looks as if there were just one way of correlating. But with the huge number—would you correlate them in the same way?
>
> Is there only one way of correlating them? If there are more, which is the logical way?—You can do any damn thing you please. If you really wanted to prove by Russell's calculus the addition of two big numbers, you would already had to know how to add, count, etc.[40]

Such passages are particularly enlightening, since Wittgenstein discusses the surveyability argument using the very premises of his modality argument, thus bringing to the fore the common presuppositions of both arguments.

38. We keep here Wittgenstein's odd notation, where '(E*1x*) *fx*' is short for '∃*x* F*x* & ¬∃*x*,*y* (F*x* & F*y*)'. Here, it is equivalent to '$\exists!_1 x$ F*x*'.

39. AWL, 147f. Incidentally, this passage is followed immediately by a statement of the modality argument.

40. LFM, 159. Again, the formula '(*xy*) (*uv*) ⊃ (*xyuv*)' is only a rough version of our formula (b) above.

4. *Concluding remarks*

In this paper we aimed for a better understanding of Wittgenstein's modality argument, on the basis of an epistemic reading of it, emphasizing the central role played by a basic idea about visualization in his critique of the Frege-Russell use of one-one correlation in order to define sameness of number. We then pointed out that the same idea about visualization is also the key to his surveyability argument. We believe that this was a necessary step towards a proper understanding of the latter, as well as a number of other topics, such as his non-extensional view of mathematics as based on numerical calculations and his understanding of proofs by mathematical induction. We also avoided throughout any assessment of the value of these arguments and in closing we would simply point out, with respect to the modality argument, that it would be wrong to judge Wittgenstein's intentions merely on the basis of it; his considered view is not nearly as negative as it looks like from reading the above. It suffices to see this that one considers section 118 of the *Big Typescript* (also reproduced as Part II, section 21 of *Philosophical Grammar*). The modality argument occurs in that section, but it is used merely to criticize Russell and its occurrence is actually followed by some developments aiming at (partly) recovering the biconditional between 'one-one correlation' and 'sameness of number', except that this is done in such a way that the result cannot serve for a definition of natural numbers of the Frege-Russell kind. We hope to explain how Wittgenstein proceeds in a further paper, but for the moment it suffices to say that Wittgenstein's position was not as 'radical' as one usually makes it to be: he did not reject Hume's principle as such, but merely tried to understand it in his own non-extensional idiom.

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