

練習問題 2.1

1. or

A	B	$A \text{ or } B$
t	t	t
t	f	t
f	t	t
f	f	f

2. if A , then B

A	B	if A , then B
t	t	t
t	f	f
f	t	t
f	f	t

練習問題 3.1

1. $A \leftrightarrow (\neg\neg A)$

$$\frac{\frac{\frac{[A]^2 \quad [\neg A]^1}{\neg - E}}{\perp}{\neg - I, 1}}{\neg\neg A}{\neg\neg I}}{A \rightarrow \neg\neg A}{\rightarrow - I, 2}$$

$$\frac{\frac{[\neg\neg A]^1}{A}{\neg\neg - E}}{\neg\neg A \rightarrow A}{\rightarrow - I, 1}$$

2. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

$$\frac{\frac{\frac{[A]^1}{A \vee B} \vee - I \quad [\neg(A \vee B)]^3}{\perp}{\neg - E}}{\frac{\perp}{\neg A} \rightarrow - I, 1} \quad \frac{\frac{\frac{[B]^2}{A \vee B} \vee - I \quad [\neg(A \vee B)]^3}{\perp}{\neg - E}}{\frac{\perp}{\neg B} \rightarrow I, 2}$$

$$\frac{\frac{\neg A \wedge \neg B}{\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)} \rightarrow - I, 3}{\neg(A \vee B) \rightarrow (\neg A \wedge \neg B)} \rightarrow - I, 3$$

$$\frac{\frac{[A \vee B]^2 \quad \frac{[A]^1 \quad \frac{[\neg A \wedge \neg B]^3}{\neg A} \wedge - E}{\perp}{\neg - E}}{\frac{\perp}{\neg(A \vee B)} \rightarrow - I, 2} \quad \frac{[B]^1 \quad \frac{[\neg A \wedge \neg B]^3}{\neg B} \wedge E}{\perp}{\neg - E}}{\frac{\perp}{\neg(A \vee B)} \rightarrow - I, 2} \vee - E, 1$$

$$\frac{\frac{\perp}{\neg(A \vee B)} \rightarrow - I, 2}{(\neg A \wedge \neg B) \rightarrow \neg(A \vee B)} \rightarrow - I, 3$$

3. $(\neg A \rightarrow (A \rightarrow B)) \vee A$

$$\begin{array}{c}
\frac{[A]^1 \quad [\neg A]^2}{\perp} \neg - E \\
\frac{\perp}{B} \perp - E \\
\frac{A \rightarrow B}{\neg A \rightarrow (A \rightarrow B)} \rightarrow - I, 1 \\
\frac{\neg A \rightarrow (A \rightarrow B)}{(\neg A \rightarrow (A \rightarrow B)) \vee A} \vee - I, 2
\end{array}$$

4. $(\neg A \rightarrow (A \wedge \neg B)) \rightarrow A$

$$\begin{array}{c}
\frac{[\neg A]^1 \quad [\neg A \rightarrow (A \wedge \neg B)]^2}{\frac{A \wedge \neg B}{A} \wedge - E} \rightarrow - E \\
\frac{\frac{A \wedge \neg B}{A} \wedge - E \quad [\neg A]^1}{\perp} \neg - E \\
\frac{\perp}{\neg \neg A} \neg - I, 1 \\
\frac{\neg \neg A}{A} \neg \neg - E \\
\frac{A}{(\neg A \rightarrow (A \wedge \neg B)) \rightarrow A} \rightarrow - I, 2
\end{array}$$

練習問題 3.2

1. $(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$

A	B	$A \rightarrow B$	$\neg B$	$\neg A$	$\neg B \rightarrow \neg A$	$(A \rightarrow B) \leftrightarrow (\neg B \rightarrow \neg A)$
t	t	t	f	f	t	t
t	f	f	t	f	f	t
f	t	t	f	t	t	t
f	f	t	t	t	t	t

2. $(A \wedge \neg A) \rightarrow B$

A	B	$\neg A$	$A \wedge \neg A$	$(A \wedge \neg A) \rightarrow B$
t	t	f	f	t
t	f	f	f	t
f	t	t	f	t
f	f	t	f	t

3. $(A \vee B) \rightarrow A$

A	B	$A \vee B$	$(A \vee B) \rightarrow A$
t	t	t	t
t	f	t	t
f	t	t	f
f	f	f	t

4. $B \rightarrow (A \wedge B)$

A	B	$A \wedge B$	$B \rightarrow (A \wedge B)$
t	t	t	t
t	f	f	t
f	t	f	f
f	f	f	t

5. $A \rightarrow (A \vee B)$

A	B	$A \vee B$	$A \rightarrow (A \vee B)$
t	t	t	t
t	f	t	t
t	t	t	t
t	f	f	t

6. $(\neg A \vee \neg B) \rightarrow \neg(A \vee B)$

A	B	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$A \vee B$
t	t	f	f	f	t
t	f	f	t	t	t
f	t	t	f	t	t
f	f	t	t	t	f
$\neg(A \vee B)$		$(\neg A \vee \neg B) \rightarrow \neg(A \vee B)$			
f		t			
f		f			
f		f			
t		t			

7. $(A \wedge (A \wedge B)) \leftrightarrow A$

A	B	$A \wedge B$	$A \wedge (A \wedge B)$	$(A \wedge (A \wedge B)) \leftrightarrow A$
t	t	t	t	t
t	f	f	f	f
f	t	f	f	t
t	f	f	f	t

8. $(A \rightarrow (B \rightarrow C)) \leftrightarrow ((A \wedge B) \rightarrow C)$

A	B	C	$B \leftrightarrow C$	$A \rightarrow (B \rightarrow C)$	$A \wedge B$	$(A \wedge (A \vee B)) \leftrightarrow A$
t	t	t	t	t	t	t
t	t	f	f	f	t	f
t	f	t	t	t	f	t
t	f	f	t	t	f	t
f	t	t	t	t	f	t
f	t	f	f	t	f	t
f	t	t	t	t	f	t
f	f	f	t	t	f	t

$$(A \rightarrow (B \rightarrow C)) \leftrightarrow ((A \wedge B) \rightarrow C)$$

 t t t t t t t t

9. $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

A	B	C	$B \vee C$	$A \wedge (B \vee C)$	$A \wedge B$	$A \wedge C$	$(A \wedge B) \vee (A \wedge C)$
-----	-----	-----	------------	-----------------------	--------------	--------------	----------------------------------

t	t	t	t	t	t	t	t
-----	-----	-----	-----	-----	-----	-----	-----

t	t	f	t	t	t	f	t
-----	-----	-----	-----	-----	-----	-----	-----

t	f	t	t	t	f	t	t
-----	-----	-----	-----	-----	-----	-----	-----

t	f	f	f	f	f	f	f
-----	-----	-----	-----	-----	-----	-----	-----

f	t	t	t	f	f	f	f
-----	-----	-----	-----	-----	-----	-----	-----

f	t	f	t	f	f	f	f
-----	-----	-----	-----	-----	-----	-----	-----

f	t	t	t	f	f	f	f
-----	-----	-----	-----	-----	-----	-----	-----

f	f	f	f	f	f	f	f
-----	-----	-----	-----	-----	-----	-----	-----

$$(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$$

 t t t t t t t t

練習問題 3.3

1. $(A \leftrightarrow B)$ と $((A \rightarrow B) \wedge (B \rightarrow A))$ が同じ真理値を持つことが下の真理表から分かる .

A	B	$A \leftrightarrow B$	$A \rightarrow B$	$B \rightarrow A$	$(A \rightarrow B) \wedge (B \rightarrow A)$
-----	-----	-----------------------	-------------------	-------------------	--

t	t	t	t	t	t
-----	-----	-----	-----	-----	-----

t	f	f	f	t	f
-----	-----	-----	-----	-----	-----

t	t	f	t	f	f
-----	-----	-----	-----	-----	-----

t	f	t	t	t	t
-----	-----	-----	-----	-----	-----

2. $(A \wedge B)$ と $\neg(\neg A \vee \neg B)$ が同じ真理値を持つことが下の真理表から分かる .

A	B	$A \wedge B$	$\neg A$	$\neg B$	$\neg A \vee \neg B$	$\neg(\neg A \vee \neg B)$
t	t	t	f	f	f	t
t	f	f	f	t	t	f
f	t	f	t	f	t	f
f	f	f	t	t	t	f

3. $(A \vee B)$ と $\neg(\neg A \wedge \neg B)$ が同じ真理値を持つことが下の真理表から分かる.

A	B	$A \vee B$	$\neg A$	$\neg B$	$\neg A \wedge \neg B$	$\neg(\neg A \wedge \neg B)$
t	t	t	f	f	f	t
t	f	t	f	t	f	t
f	t	t	t	f	f	t
f	f	f	t	t	t	f

練習問題 3.4

1. $(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)$

$$\begin{array}{c}
 \frac{[A]^1 \quad [A \rightarrow B]^3}{B} \rightarrow -E \quad \frac{[\neg B]^2}{\neg - E} \\
 \frac{\perp}{\neg A} \neg - I, 1 \\
 \frac{\neg B \rightarrow \neg A}{(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)} \rightarrow -I, 2 \\
 \frac{}{(A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A)} \rightarrow -I, 3
 \end{array}$$

2. $(A \wedge \neg A) \rightarrow C$

$$\begin{array}{c}
 \frac{[A \wedge \neg A]^1}{A} \wedge - E \quad \frac{[A \wedge \neg A]^1}{\neg A} \wedge - E \\
 \frac{\perp}{C} \perp - E \\
 \frac{}{(A \wedge \neg A) \rightarrow C} \rightarrow -I, 1
 \end{array}$$

5. $A \rightarrow (A \vee B)$

$$\begin{array}{c}
 \frac{[A]^1}{A \vee B} \vee - I \\
 \frac{}{A \rightarrow (A \vee B)} \rightarrow -I, 1
 \end{array}$$

7. $(A \wedge (A \wedge B)) \rightarrow A$

$$\begin{array}{c}
 \frac{[A \wedge (A \wedge B)]^1}{A} \wedge - E \\
 \frac{}{(A \wedge (A \wedge B)) \rightarrow A} \rightarrow -I, 1
 \end{array}$$

8. $(A \rightarrow (B \rightarrow C)) \leftrightarrow ((A \wedge B) \rightarrow C)$

$$\frac{\frac{[A \wedge B]^1}{B} \wedge - E \quad \frac{\frac{[A \wedge B]^1}{A} \wedge - E \quad [A \rightarrow (B \rightarrow C)]^2}{B \rightarrow C} \rightarrow - E}{\frac{C}{(A \wedge B) \rightarrow C} \rightarrow - I, 1} \rightarrow - E$$

$$\frac{}{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C)} \rightarrow - I, 2$$

$$\frac{\frac{[A]^2 \quad [B]^1}{A \wedge B} \wedge - I \quad [(A \wedge B) \rightarrow C]^3}{\frac{C}{B \rightarrow C} \rightarrow - I, 1} \rightarrow - E$$

$$\frac{\frac{B \rightarrow C}{A \rightarrow (B \rightarrow C)} \rightarrow - I, 2}{((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))} \rightarrow - I, 3$$

9. $(A \wedge (B \vee C)) \leftrightarrow ((A \wedge B) \vee (A \wedge C))$

$$\frac{\frac{[A \wedge (B \vee C)]^2}{B \vee C} \wedge - E \quad \frac{\frac{[A \wedge (B \vee C)]^2}{A} \wedge - E \quad [B]^1}{\frac{A \wedge B}{(A \wedge B) \vee (A \wedge C)} \vee - I} \wedge - I \quad \frac{\frac{[A \wedge (B \vee C)]^2}{A} \wedge - E \quad [C]^1}{\frac{A \wedge C}{(A \wedge B) \vee (A \wedge C)} \vee - I} \wedge - I}{\frac{(A \wedge B) \vee (A \wedge C)}{(A \wedge (B \vee C)) \rightarrow ((A \wedge B) \vee (A \wedge C))} \vee - E, 1} \rightarrow - I, 2$$

$$\frac{[(A \wedge B) \vee (A \wedge C)]^2 \quad \frac{\frac{[A \vee B]^1}{A} \wedge - E \quad \frac{[A \wedge B]^1}{B} \wedge - E}{\frac{A \wedge (B \vee C)}{B \vee C} \vee - I} \wedge - I \quad \frac{[A \wedge C]^1}{A} \wedge - E \quad \frac{[A \wedge C]^1}{C} \wedge - E}{\frac{A \wedge (B \vee C)}{B \vee C} \vee - I} \wedge - I \quad \frac{A \wedge (B \vee C)}{((A \wedge B) \vee (A \wedge C)) \rightarrow (A \wedge (B \vee C))} \vee - E, 1} \rightarrow - I, 2$$

練習問題 3.5

1. $((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))$

$$\frac{\frac{[A]^2 \quad [B]^1}{A \wedge B} (\wedge - I) \quad [(A \wedge B) \rightarrow C]^3}{\frac{C}{B \rightarrow C} (\rightarrow - I, 1)} (\rightarrow - E)$$

$$\frac{\frac{B \rightarrow C}{A \rightarrow (B \rightarrow C)} (\rightarrow - I, 2)}{((A \wedge B) \rightarrow C) \rightarrow (A \rightarrow (B \rightarrow C))} (\rightarrow - I, 3)$$

2. $(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C)$

$$\frac{\frac{[A \wedge B]^1}{B} (\wedge - E) \quad \frac{\frac{[A \wedge B]^1}{A} (\wedge - E) \quad [A \rightarrow (B \rightarrow C)]^2}{B \rightarrow C} (\rightarrow - E)}{\frac{C}{(A \wedge B) \rightarrow C} (\rightarrow - I, 1)} (\rightarrow - E)$$

$$\frac{(A \wedge B) \rightarrow C}{(A \rightarrow (B \rightarrow C)) \rightarrow ((A \wedge B) \rightarrow C)} (\rightarrow - I, 2)$$

3. $(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)$

$$\frac{[B]^2 \quad \frac{[\neg A]^1 \quad [\neg A \rightarrow \neg B]^3}{\neg B} \rightarrow - E}{\frac{\perp}{\neg \neg A} \neg - I, 1} \neg - E$$

$$\frac{A}{B \rightarrow A} \rightarrow - I, 2$$

$$\frac{B \rightarrow A}{(\neg A \rightarrow \neg B) \rightarrow (B \rightarrow A)} \rightarrow - I, 3$$

4. $(A \vee B \rightarrow C) \rightarrow (A \rightarrow C) \wedge (B \rightarrow C)$

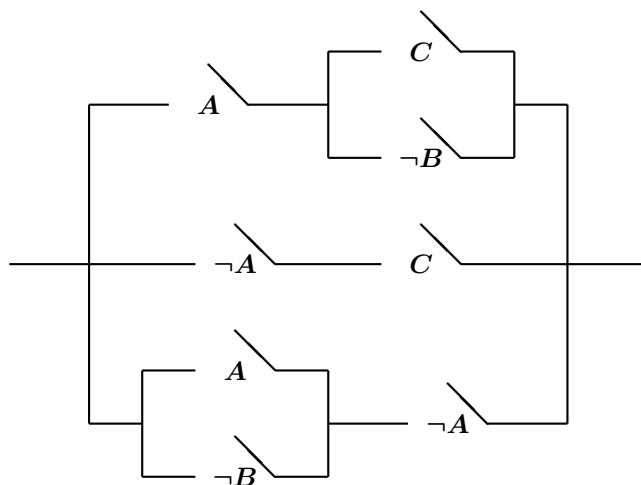
$$\frac{[A]^1}{A \vee B} (\vee - I) \quad \frac{[A \vee B \rightarrow C]^3}{A \rightarrow C} (\rightarrow - E) \quad \frac{[B]^2}{A \vee B} (\vee - I) \quad \frac{[A \vee B \rightarrow C]^3}{B \rightarrow C} (\rightarrow - E)$$

$$\frac{\frac{C}{A \rightarrow C} (\rightarrow - I, 1) \quad \frac{C}{B \rightarrow C} (\rightarrow - I, 2)}{(A \rightarrow C) \wedge (B \rightarrow C)} (\wedge - I)$$

$$\frac{(A \rightarrow C) \wedge (B \rightarrow C)}{(A \vee B \rightarrow C) \rightarrow (A \rightarrow C) \wedge (B \rightarrow C)} (\rightarrow - I, 3)$$

練習問題 3.6

まず，次の回路を論理式で表わせ．

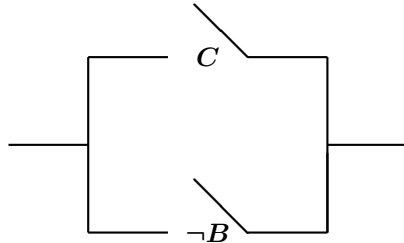


答え $(A \wedge (C \vee \neg B)) \vee (\neg A \wedge C) \vee ((A \vee \neg B) \wedge \neg A)$

次に，この回路と同じ役割を果たす最も単純な回路を与えよ．（そして，実際に，そ

の回路がもとのものと同値であることを論理的に証明せよ.)

答え



練習問題 3.7 次の前提より，定理を導き出せ．

1. 今日晴れているならば，僕はテニスをする．

晴れる $\rightarrow$ テニスをする

2. 私が勉強するならば，僕はテニスをしない．

勉強する $\rightarrow$ $\neg$ テニスをする

3. 私の気分がよければ，その日は必ず晴れている．

気分がよい $\rightarrow$ 晴れる

4. 私の気分が悪ければ，僕は勉強しない．

$\neg$ 気分がよい $\rightarrow$ $\neg$ 勉強する

(定理) 私は，今日勉強しない．

$\neg$ 勉強する

答え

“勉強する”を B ，“晴れる”を H ，“テニスをする”を T ，“気分がよい”を K とする．

$$\begin{array}{c}
 \frac{[K]^3 \quad K \rightarrow H}{H} \rightarrow -E \quad \frac{\frac{[H]^1 \quad H \rightarrow T}{T} \rightarrow E \quad \frac{[B]^2 \quad B \rightarrow \neg T}{\neg T} \rightarrow E}{\neg H} \neg - I, 1 \\
 \frac{\perp}{\neg H} \neg - E \\
 \frac{\perp}{\neg K} \neg - I, 3 \\
 \frac{[B]^2 \quad \neg B}{\neg B} \neg - E \quad \neg K \rightarrow \neg B \rightarrow E \\
 \frac{\perp}{\neg B} \neg - I, 2
 \end{array}$$

練習問題 3.8 次の前提より，定理を導き出せ．

1. 神が存在し，私が神を信じるならば，私は救われる．

神 $\wedge$ 信 $\rightarrow$ 救

2. 神が存在しなければ、悪魔が存在する。

$$\neg \text{神} \rightarrow \text{悪魔}$$

3. 私が救われるならば、神は存在する。

$$\text{救} \rightarrow \text{神}$$

4. 悪魔が存在するならば、私は神を信じる。

$$\text{悪魔} \rightarrow \text{信}$$

(定理 1) 私が救われることなく、しかも私が神を信じるならば、悪魔が存在する。

$$\neg \text{救} \wedge \text{信} \rightarrow \text{悪魔}$$

答え

$$\frac{\frac{\frac{[\neg \text{救} \wedge \text{信}]}{\text{信}}}{\text{神} \wedge \text{信}} \quad \text{神} \wedge \text{信} \rightarrow \text{救} \quad [\neg \text{救} \wedge \text{信}]}{\text{救} \quad \neg \text{救}} \quad \frac{\perp}{\neg \text{神}} \quad \neg \text{神} \rightarrow \text{悪}$$

$$\frac{\text{悪}}{\neg \text{救} \wedge \text{信} \rightarrow \text{悪}}$$

練習問題 3.9

$$\frac{\frac{[2 \text{ 本足} \wedge \neg \text{飛}]^3}{2 \text{ 本足}} \wedge -E \quad \frac{2 \text{ 本足} \rightarrow \text{鳥} \vee \text{人間}}{\text{鳥} \vee \text{人間}} \rightarrow -E \quad \frac{[\text{鳥}]^1 \quad \text{鳥} \rightarrow \text{飛}}{\text{飛}} \rightarrow -E \quad \frac{[2 \text{ 本足} \wedge \neg \text{飛}]^3}{\neg \text{飛}} \wedge -E \quad \frac{[\text{人間}]^1 \quad [\neg \text{人間}]^2}{\perp} \vee -E, 1}{\frac{\perp}{\neg \neg \text{人間}} \neg -I, 2 \quad \frac{\neg \neg \text{人間}}{\text{人間}} \neg \neg -E \quad \frac{\text{人間}}{2 \text{ 本足} \wedge \neg \text{飛} \rightarrow \text{人間}} \rightarrow -I, 3}$$

練習問題 3.10

(与式)

↓

$$\frac{\frac{[A]}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \neg A \quad \frac{[\neg A]}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \neg \neg A}{\neg A \wedge \neg \neg A} \neg A \quad \frac{[A]}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \neg A \quad \frac{[\neg A]}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \neg \neg A}{\neg A \wedge \neg \neg A} \neg \neg A$$

$$\frac{\perp}{\neg(A \vee \neg A) \rightarrow \perp}$$

↓

$$\begin{array}{c}
\frac{[A]}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \quad \frac{[\neg A]}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \\
\frac{\perp}{\neg A} \quad \frac{\perp}{\neg \neg A} \\
\frac{\perp}{\neg(A \vee \neg A) \rightarrow \perp}
\end{array}$$

$\Downarrow$

$$\begin{array}{c}
\frac{[A]}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \\
\frac{\perp}{\neg A} \\
\frac{A \vee \neg A \quad \neg A}{A \vee \neg A} \quad \frac{[\neg(A \vee \neg A)]}{\perp} \\
\frac{\perp}{\neg(A \vee \neg A)} \\
\frac{\neg(A \vee \neg A)}{\neg(A \vee \neg A) \rightarrow \perp}
\end{array}$$

練習問題 4.2

例題とほぼ同様のため省略。

練習問題 4.3

2, 3 は $Nat \equiv (P \rightarrow P) \rightarrow (P \rightarrow P)$ であることを用いて以下のような証明図が作れる。

$$\begin{array}{c}
\frac{(P \rightarrow P) \rightarrow (P \rightarrow P) \quad [P \rightarrow P]^f}{P \rightarrow P} \rightarrow E \quad \frac{(P \rightarrow P) \rightarrow (P \rightarrow P) \quad [P \rightarrow P]^f}{P \rightarrow P} \rightarrow E \quad \frac{[P]^x}{P} \rightarrow E \\
\frac{P}{P \rightarrow P} \rightarrow I_x \quad \frac{P}{P \rightarrow P} \rightarrow E \\
\frac{P \rightarrow P \rightarrow I_x \quad P \rightarrow E}{(P \rightarrow P) \rightarrow (P \rightarrow P)} \rightarrow I_f
\end{array}$$

練習問題 4.4

省略。

練習問題 4.5

$$\begin{aligned}
If - then - else \ F \ s \ t &\equiv \lambda p^{Bool} \lambda x^{Nat} \lambda y^{Nat} ((px)y) F s t \\
&\triangleright \lambda x^{Nat} \lambda y^{Nat} ((Fx)y) s t \\
&\triangleright \lambda y^{Nat} ((Fs)y) t \\
&\triangleright (Fs) t \\
&\equiv ((\lambda a^{Nat} \lambda b^{Nat} b) s) t \\
&\triangleright ((\lambda b^{Nat} b) t) \\
&\triangleright t
\end{aligned}$$

練習問題 4.6

$$T \ AND \ T \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((T((Tm)n))((Tn)n))$$

$$\begin{aligned}
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((Tm)n) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} m \\
& \equiv T
\end{aligned}$$

$$\begin{aligned}
T \text{ AND } F & \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((T((Fm)n))((Fn)n)) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((Fm)n) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} n \\
& \equiv F
\end{aligned}$$

$$\begin{aligned}
F \text{ AND } T & \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((F((Tm)n))((Tn)n)) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((Tn)n) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} n
\end{aligned}$$

$$\begin{aligned}
F \text{ AND } F & \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((F((Fm)n))((Fn)n)) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((Fn)n) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} n
\end{aligned}$$

$$\begin{aligned}
T \text{ OR } T & \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((T((Tm)m))((Tm)n)) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((Tm)m) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} m
\end{aligned}$$

$$\begin{aligned}
F \text{ OR } F & \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((F((Fm)m))((Fm)n)) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} ((Fm)n) \\
& \triangleright \triangleright \lambda m^{Nat} \lambda n^{Nat} n
\end{aligned}$$

練習問題 5.1

1. $\exists x \neg A(x) \rightarrow \neg \forall x A(x)$

$$\begin{array}{c}
\frac{\frac{\frac{[\exists x \neg A(x)]^3}{\perp} \quad \frac{\frac{A(t)}{\perp} \neg -E}{\neg \forall x A(x)} \neg -I,2}{\exists x \neg A(x) \rightarrow \neg \forall x A(x)} \rightarrow -I,3}{\frac{[\forall x A(x)]^2 \quad [\neg A(t)]^1}{\perp} \forall -E}
\end{array}$$

2. $\exists x \forall y \neg A(x, y) \rightarrow \neg x \exists y A(x, y)$

$$\begin{array}{c}
\frac{\frac{[\forall x \exists y A(x, y)]^3}{\exists y A(t, y)} \quad \forall -E \quad \frac{[\exists x \forall y \neg A(x, y)]^4}{\perp} \quad \exists -E, 1}{\perp} \quad \exists -E, 2 \\
\frac{\perp}{\neg \forall x \exists y A(x, y)} \quad \neg -I, 3 \\
\frac{\neg \forall x \exists y A(x, y)}{\exists x \forall y \neg A(x, y) \rightarrow \neg \forall x \exists y A(x, y)} \quad \rightarrow -I, 4
\end{array}$$

3. $\neg \forall x \neg A(x) \leftrightarrow \exists x A(x)$

$$\begin{array}{c}
\frac{[A(x)]^1}{\exists x A(x)} \quad \exists -I \quad \frac{[\neg \exists x A(x)]^2}{\perp} \quad \neg -E \\
\frac{\perp}{\neg A(x)} \quad \neg -I, 1 \\
\frac{\neg A(x)}{\forall x \neg A(x)} \quad \forall -I \quad \frac{[\neg \forall x \neg A(x)]^3}{\perp} \quad \neg -E \\
\frac{\perp}{\neg \neg \exists x A(x)} \quad \rightarrow -I, 2 \\
\frac{\neg \neg \exists x A(x)}{\exists x A(x)} \quad \neg \neg -E \\
\frac{\exists x A(x)}{\neg \forall x \neg A(x) \rightarrow \exists x A(x)} \quad \rightarrow -I, 3
\end{array}$$

3. 逆 $\exists x A(x) \rightarrow \neg \forall x \neg A(x)$

$$\begin{array}{c}
\frac{[\exists x A(x)]^3 \quad \frac{[A(x)]^1}{\neg A(x)} \quad \forall -E}{\perp} \quad \exists -E, 1 \\
\frac{\perp}{\neg \forall \neg A(x)} \quad \rightarrow -I, 2 \\
\frac{\neg \forall \neg A(x)}{\exists x A(x) \rightarrow \neg \forall x \neg A(x)} \quad \rightarrow -I, 3
\end{array}$$

4. $\forall x \exists y A(x, y) \leftrightarrow \neg \exists x \forall y \neg A(x, y)$

$$\begin{array}{c}
\frac{[\forall x \exists y A(x, y)]^4 \quad \frac{[A(u, v)]^1}{\neg A(u, v)} \quad \forall -E}{\exists y A(u, y)} \quad \forall -E \quad \frac{[\forall y \neg A(u, v)]^2}{\perp} \quad \forall -E \\
\frac{\perp}{\neg \exists x \forall y \neg A(x, y)} \quad \exists -E, 1 \\
\frac{\neg \exists x \forall y \neg A(x, y)}{\forall x \exists y A(x, y) \rightarrow \neg \exists x \forall y \neg A(x, y)} \quad \rightarrow -I, 4
\end{array}$$

4. 逆 $\neg \exists x \forall y \neg A(x, y) \rightarrow \forall x \exists y A(x, y)$

$$\begin{array}{c}
\frac{[A(x, y)]^1}{\exists y A(x, y)} \exists-I \quad \frac{[\neg \exists y A(x, y)]^2}{\neg \neg E} \\
\hline
\frac{\perp}{\neg A(x, y)} \neg-I, 1 \\
\frac{\neg A(x, y)}{\forall y \neg A(x, y)} \forall-I \\
\frac{\forall y \neg A(x, y)}{\exists x \forall y \neg A(x, y)} \exists-I \\
\frac{[\neg \exists x \forall y \neg A(x, y)]^3}{\neg \neg E} \\
\hline
\frac{\perp}{\neg \neg \exists y A(x, y)} \neg-I, 2 \\
\frac{\neg \neg \exists y A(x, y)}{\exists y A(x, y)} \neg \neg-E \\
\frac{\exists y A(x, y)}{\forall x \exists y A(x, y)} \forall-I \\
\hline
\frac{\neg \exists x \forall y \neg A(x, y) \rightarrow \forall x \exists y A(x, y)}{\rightarrow-I, 3}
\end{array}$$

5. $\forall x A(x) \wedge \neg \exists y B(y) \leftrightarrow \forall z (A(z) \wedge \neg B(z))$

$$\begin{array}{c}
\frac{[\forall x A(x) \wedge \neg \exists y B(y)]^2}{\forall x A(x)} \wedge-E \quad \frac{[B(a)]^1}{\exists y B(y)} \exists-I \quad \frac{[\forall x A(x) \wedge \neg \exists y B(y)]^2}{\neg \exists y B(y)} \wedge-E \\
\hline
\frac{\forall x A(x)}{A(a)} \forall-E \quad \frac{\perp}{\neg B(a)} \neg-I, 1 \\
\hline
\frac{A(a) \wedge \neg B(a)}{\forall z (A(z) \wedge \neg B(z))} \wedge-I \\
\frac{\forall z (A(z) \wedge \neg B(z))}{\forall x A(x) \wedge \neg \exists y B(y) \rightarrow \forall z (A(z) \wedge \neg B(z))} \forall-I \\
\hline
\frac{}{\rightarrow-I}
\end{array}$$

5. 逆 $\forall z (A(z) \wedge \neg B(z)) \rightarrow \forall x A(x) \wedge \neg \exists y B(y)$

$$\begin{array}{c}
\frac{[\forall z (A(z) \wedge \neg B(z))]^3}{A(a) \wedge \neg B(a)} \forall-E \quad \frac{[\forall z (A(z) \wedge \neg B(z))]^3}{A(a) \wedge \neg B(a)} \forall-E \\
\hline
\frac{A(a) \wedge \neg B(a)}{A(a)} \wedge-E \quad \frac{[B(a)]^1}{\neg B(a)} \wedge-E \\
\hline
\frac{A(a)}{\forall x A(x)} \forall-I \quad \frac{\perp}{\neg \exists y B(y)} \neg-I \\
\hline
\frac{\forall x A(x) \wedge \neg \exists y B(y)}{\forall z (A(z) \wedge \neg B(z)) \rightarrow \forall x A(x) \wedge \neg \exists y B(y)} \wedge-I \\
\hline
\frac{}{\rightarrow-I}
\end{array}$$

6. $\forall x (A(x) \wedge B(x)) \leftrightarrow \forall x A(x) \wedge \forall x B(x)$

$$\begin{array}{c}
\frac{[\forall x (A(x) \wedge B(x))]^1}{A(a) \wedge B(a)} \forall-E \quad \frac{[\forall x (A(x) \wedge B(x))]^1}{A(a) \wedge B(a)} \forall-I \\
\hline
\frac{A(a) \wedge B(a)}{A(a)} \wedge-E \quad \frac{A(a) \wedge B(a)}{B(a)} \wedge-E \\
\hline
\frac{A(a)}{\forall x A(x)} \forall-I \quad \frac{B(a)}{\forall x B(x)} \forall-I \\
\hline
\frac{\forall x A(x) \wedge \forall x B(x)}{\forall x (A(x) \wedge B(x)) \rightarrow \forall x A(x) \wedge \forall x B(x)} \wedge-I \\
\hline
\frac{}{\rightarrow-I}
\end{array}$$

6. 逆 $\forall x A(x) \wedge \forall x B(x) \rightarrow \forall x (A(x) \wedge B(x))$

$$\begin{array}{c}
\frac{[\forall x A(x) \wedge \forall x B(x)]^1}{\forall x A(x)} \wedge -E \quad \frac{[\forall x A(x) \wedge \forall x B(x)]^1}{\forall x B(x)} \wedge -E \\
\frac{\frac{\forall x A(x)}{A(a)} \forall -E \quad \frac{\forall x B(x)}{B(a)} \forall -I}{A(a) \wedge B(a)} \wedge -I \\
\frac{A(a) \wedge B(a)}{\forall x(A(x) \wedge B(x))} \forall -I \\
\frac{\forall x(A(x) \wedge B(x))}{\forall x A(x) \wedge \forall x B(x) \rightarrow \forall x(A(x) \wedge B(x))} \rightarrow -I
\end{array}$$

$$7. \exists x(A(x) \vee B(x)) \rightarrow \exists x A(x) \vee \exists x B(x)$$

$$\begin{array}{c}
\frac{[A(a)]^1}{\exists x A(x)} \exists -I \quad \frac{[B(a)]^1}{\exists x B(x)} \exists -I \\
\frac{[A(a) \vee B(a)]^2 \quad \frac{\exists x A(x) \vee \exists x B(x)}{\exists x A(x) \vee \exists x B(x)} \vee -I \quad \frac{\exists x A(x) \vee \exists x B(x)}{\exists x A(x) \vee \exists x B(x)} \vee -I}{\exists x(A(x) \vee B(x))} \vee -E,2 \\
\frac{[\exists x(A(x) \vee B(x))]^3 \quad \frac{\exists x A(x) \vee \exists x B(x)}{\exists x A(x) \vee \exists x B(x)} \exists -E}{\exists x(A(x) \vee B(x)) \rightarrow \exists x A(x) \vee \exists x B(x)} \rightarrow -I,3
\end{array}$$

$$7. \text{逆} \exists x A(x) \vee \exists x B(x) \rightarrow \exists x(A(x) \vee B(x))$$

$$\begin{array}{c}
\frac{[A(a)]^1}{A(a) \vee B(a)} \vee -I \\
\frac{[\exists x A(x)]^2 \quad \frac{\exists x(A(x) \vee B(x))}{\exists x(A(x) \vee B(x))} \exists -I \quad \frac{[B(a)]^2}{A(a) \vee B(a)} \vee -I}{\exists x(A(x) \vee B(x))} \exists -E,1 \\
\frac{[\exists x A(x) \vee \exists x B(x)]^3 \quad \frac{\exists x(A(x) \vee B(x))}{\exists x(A(x) \vee B(x))} \exists -E,1 \quad \frac{[\exists x B(x)]^2 \quad \frac{\exists x(A(x) \vee B(x))}{\exists x(A(x) \vee B(x))} \exists -I}{\exists x(A(x) \vee B(x))} \vee -E,2}{\exists x A(x) \vee \exists x B(x) \rightarrow \exists x(A(x) \vee B(x))} \rightarrow -I,3
\end{array}$$

$$8. \forall x \forall y(A(x) \rightarrow B(y)) \leftrightarrow (\exists x A(x) \rightarrow \forall y B(y))$$

$$\begin{array}{c}
\frac{[\forall x \forall y(A(x) \rightarrow B(y))]^3}{\forall y(A(a) \rightarrow B(y))} \forall -E \\
\frac{[A(a)]^1 \quad \frac{\forall y(A(a) \rightarrow B(y))}{A(a) \rightarrow B(b)} \forall -E}{B(b)} \rightarrow -E \\
\frac{[\exists x A(x)]^2 \quad \frac{B(b)}{\exists x A(x)} \exists -E,1}{\forall y B(y)} \forall -I \\
\frac{\forall y B(y)}{\exists x A(x) \rightarrow \forall y B(y)} \rightarrow -I,2 \\
\frac{\exists x A(x) \rightarrow \forall y B(y)}{\forall x \forall y(A(x) \rightarrow B(y)) \rightarrow (\exists x A(x) \rightarrow \forall y B(y))} \rightarrow -I,3
\end{array}$$

$$8. \text{逆} (\exists x A(x) \rightarrow \forall y B(y)) \rightarrow \forall x \forall y(A(x) \rightarrow B(y))$$

$$\begin{array}{c}
\frac{[A(a)]^1}{\exists x A(x)} \exists-I \quad \frac{[\exists x A(x) \rightarrow \forall y B(y)]^2}{\forall y A(y)} \rightarrow-E \\
\frac{\forall y A(y)}{B(b)} \forall-E \\
\frac{A(a) \rightarrow B(b)}{\forall y (A(a) \rightarrow B(y))} \rightarrow-I,1 \\
\frac{\forall y (A(a) \rightarrow B(y))}{\forall x \forall y (A(x) \rightarrow B(y))} \forall-I \\
\frac{\forall x \forall y (A(x) \rightarrow B(y))}{(\exists x A(x) \rightarrow \forall y B(y)) \rightarrow \forall x \forall y (A(x) \rightarrow B(y))} \rightarrow-I,2
\end{array}$$

9. $\exists x \exists y (A(x) \rightarrow B(y)) \leftrightarrow (\forall x A(x) \rightarrow \exists y B(y))$

$$\begin{array}{c}
\frac{[\forall x A(x)]^1}{A(x)} \forall-E \quad \frac{[A(x) \rightarrow B(y)]^2}{B(y)} \rightarrow-E \\
\frac{B(y)}{\exists y B(y)} \exists-I \\
\frac{[\exists y (A(x) \rightarrow B(y))]^3}{\forall x A(x) \rightarrow \exists y B(y)} \rightarrow-I,1 \\
\frac{[\exists x \exists y (A(x) \rightarrow B(y))]^4}{\forall x A(x) \rightarrow \exists y B(y)} \exists-E,2 \\
\frac{\forall x A(x) \rightarrow \exists y B(y)}{\exists x \exists y (A(x) \rightarrow B(y)) \rightarrow (\forall x A(x) \rightarrow \exists y B(y))} \exists-E,3 \\
\frac{\exists x \exists y (A(x) \rightarrow B(y)) \rightarrow (\forall x A(x) \rightarrow \exists y B(y))}{\exists x \exists y (A(x) \rightarrow B(y)) \rightarrow (\forall x A(x) \rightarrow \exists y B(y))} \rightarrow-I,4
\end{array}$$

9. 逆 $(\forall x A(x) \rightarrow \exists y B(y)) \rightarrow \exists x \exists y (A(x) \rightarrow B(y))$