A New Proof-Theoretical View on an Old "Dialogue Logic"

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1 Introduction

The development of modern logic in the second half of the twentieth century, especially after Tarski, was very much based on the framework of the syntax-semantics dualism, where a typical paradigm of logical researchers is that a proof-theoretic syntax and a Tarskian or Kripkian model theoretic semantics are set up first and the correspondence between syntactic provability and semantic validity is shown as a completeness theorem. This conservative paradigm of modern logic has been criticized by various authors, including intuitionist/constructivist logicians (such as [Martin-Löf 84], [Girard 06, 07]). In particular, new proof theoretic investigations to give a break-through to the understanding of logic in a deeper sense have been proposed from, among others, game semantics or dialogical logic view, where the dualist setting of the language-based syntax and the set-theoretic model-based semantics is replaced by a dynamic interactions of logical communication between players, Proponent-Opponent or player-environment; such new approaches then deal with proofs themselves and communicational or interactive meaning of proofs, instead of provability with which the traditional approaches mainly concerned. An extreme and successful usage of such an approach was Girard's Ludics' game approach [Girard 01]. There have also been various game theoretic approaches, such as the most traditional game/dialogical logic approach by Lorenzen-Lorenz (cf. [Lorenzen 61], [Lorenz-Lorenzen 78]), which are still limited only to the provability matter. Abramsky (eg. [Abramsky 94]) and Hyland-Ong (eg. [Hyland-Ong 00]) and their followers made important progresses to capture the meaning of proofs, instead of that of provability and a partial understanding of cut-elimination, in terms of compositionality.

In this paper we present a new view on the most traditional Lorenzen style game semantics/dialogical logics in order to capture the meaning of proof structures; we first show that the traditional game approach can be viewed from a more dynamic setting than the original one. In fact, Herbelin ([Herbelin 95]) gave an important step toward this: he showed that the Lorenzen game could be viewed at the level of proofs, instead of provability. We show in this paper that we can go beyond this and go to the level of proof-structures that are not necessarily completed proofs. We show a proof-search (or bottom-up proof construction) process and a Lorenzen-style game correspond to each other. This means that a proof structure (even if it is not a "provable" proof and may be infinite) correspond to a (not necessarily winning) strategy of the Lorenzen-style game. This means that any proof construction process can be understood as a Proponent-Opponent logical communication/dialogue process.

In the 1950s, Lorenzen introduced two-player games or dialogues with the proposal to give a dialogue-based understanding of intuitionistic logical reasoning. (Cf. [Felscher 85, 86].) In order to characterize the notion of provability in intuitionistic logic as the existence of a winning strategy for one player, Lorenzen and his followers formulated rules of logical dialogue. About thirty years later, [Felscher 85] gave an equivalence theorem between provability (the existence of a proof) of A in intuitionistic logic and the existence of a winning strategy for the first player in a logical dialogue for A.

[Herbelin 95] extended and made more precise the work of [Felscher 85], and he gave a one-to-one correspondence between the winning strategies of Lorenzen-Felscher Game and the cut-free (normal form) proofs of Kleene's sequent calculus, which is a variant of Gentzen's sequent calculus.

We show in this paper that this correspondence can be extended further to the correspondence between the set of (not necessarily winning) strategies and the set of (not necessarily completed) proof structures; i.e., a dynamic proof construction process (either it is finitely completed or infinitely expanding) and a game strategy (either it is a winning or not). We introduce a sequent calculus representation of Lorenzen-Felscher Game (called Sequent Calculus Game in this paper) in which not only a winning strategy but also any strategy can be identified with a cut-free proof structure. By ignoring certain decorations of the sequents (of our Sequent Calculus Game), the strategies can be identified with a certain set of cut-free proof structures of Kleene's sequent calculus. The winning strategies are characterized as a subset of the finite completed cut-free proofs.

2 Lorenzen-Felscher Game

In this section, we give an overview of Lorenzen-Felscher Game for classical propositional logic following [Felscher 85,86].

Lorenzen-Felscher Game is introduced as dialogues between two players called *Proponent* and *Opponent*. The rule of a dialogue is introduced based on the argumentation form, which describes how a composite formula may be attacked and how, if possible, this attack may be defended.

We use P (and O) for *Proponent* (and for *Opponent*). We use the meta variables X, Y ($X \neq Y$) to denote P or O.

Formulas are those of usual classical propositional logic.

We express an X's attack upon $A_1 \wedge A_2$ by choosing A_i (for $i = 1, 2$) as $(a^X; [A_i])$, an X's attack upon $A \vee B$ as $(a^X; [A \vee B])$, an X's attack upon $\neg A$ as $(a^Y; A)$, an X's defense to an attack by asserting A as $(d^X; A)$.

An X's assertion of a formula A is an X's attack upon A or an X's defense to an attack by A.

Definition 1 (Argumentation form) *Argumentation forms* of Lorenzen-Felscher Game ([Felscher 85]) for the connectives $\wedge, \vee, \neg$ are defined as follows (read from bottom up).

$\wedge$-argumentation form:

$$\begin{array}{c} d^X; A_i \\ a^Y; [A_i] \\ X\text{'s assertion of } A_1 \wedge A_2 \\ \text{(for } i = 1, 2) \end{array}$$

$\vee$-argumentation form:

$$\begin{array}{c} d^X; A_i \\ a^Y; [A_1 \vee A_2] \\ X\text{'s assertion of } A_1 \vee A_2 \\ \text{(for } i = 1, 2) \end{array}$$

¬-argumentation form:

$$\begin{array}{c}\text{No defense is possible}\\ a^Y; A\\ X\text{'s assertion of } \neg A\end{array}$$

A *move* is a pair of an X's assertion and a natural number called a reference number, which denotes for what number of assertion the move is attack/defense. We denote moves as $m, m_1, m_2, \ldots$. A move m is called an X's move if the assertion of m is an X's assertion.

The attack of i-th move will be said to be *open at k-th move* with $i < k$, if there is no j-th move with $i < j \leq k$ which carries a defense to the attack of i-th move according to the appropriate argumentation form.

Definition 2 (Dialogue) Let D be a sequence of moves $m_1, m_2, m_3, \ldots$.
D is a *dialogue for a formula A* if it satisfies the following conditions.

(D00) The first move m_1 is $(d^P; A, 0)$, and P's moves and O's moves appear alternately in D;

(D01) If i-th move m_i $(i > 1)$ is $(a^X; A, j)$ for $j < i$, then m_i is an X's attack upon the assertion in j-th move m_j according to the appropriate argumentation form;

(D02) If i-th move m_i $(i > 1)$ is $(d^X; A, j)$ for $j < i$, then m_i is an X's defense to the attack in j-th move m_j according to the appropriate argumentation form.

A dialogue D is a dialogue for classical logic if it satisfies the following conditions:

(D10) If m_i is a P's assertion of an atom α, then there is a move m_j for $j < i$ such that the assertion in m_j is O's assertion of α;

(D11) If, at $(i-1)$-th move, there are several open attacks suitable to be defended at i-th move of O, then only the *latest* of them may be defended at i-th move of O;

(D12) A P's attack may be defended by O at most once;

(D13) If the assertion in m_i is a P's assertion, then there is at most one m_j such that m_j is an attack for that assertion.

A dialogue for A is *won by P* if it is finite, ends with P's move and if the rules do not permit O to continue with another attack or defense.

Since a winning strategy is defined as a tree, a definition of a *tree* is given as follows. (See [Troelstra-Schwichtenberg 00].)

Definition 3 (Tree) A *tree* is a partially ordered set $(X, \leq)$ such that

- there is the lowest element (called *root*);
- all sets $\{y \mid y \leq x\}$ for $x \in X$ are linearly ordered and finite.

The elements of X are called the *nodes* of the tree. A *path* for x is a set $\{y \mid y \leq x\}$ which is a linearly ordered subset of X. *Branches* are maximal linearly ordered subsets of X. We define the height of a branch as the number of nodes in that branch. We also define the height of a tree as the height of a maximal branch.

In this paper, we consider trees whose height are less than ω.

If x and y are nodes of a tree with $x \leq y$ and if there are no nodes properly between x and y, then we call y a *successor node* of x.

We call X's move as X node. We define *strategies* as follows.

Definition 4 (Strategy) A *strategy for* A is a tree with moves as nodes, and it satisfies the following conditions.

(S0) Each branch is a dialogue for A;

(S1) each P node has at most one successor node;

(S2) each O node has all successor nodes from that node.

A *winning strategy for* A is a strategy for A, where each dialogue in it is won by P.

3 Sequent Calculus Game

We shall begin by introducing formulas and game sequents. Our *game sequent* is a modification of a usual sequent in Gentzen's sequent calculus. The idea of our modification of sequent originally comes from [Lâm 04]. (See [Gentzen 35] and [Takeuti 87] for sequent calculus.)

Definition 5 (Formula and game sequent) *Formulas* of Sequent Calculus Game are defined as follows.

1. $\alpha, \beta, \gamma, \ldots, \alpha_1, \alpha_2, \ldots$ are atomic formulas (atoms);
2. If A and B are formulas, then $\neg A, A \wedge B, A \vee B, \underline{A \vee B}$ are formulas.

A *game sequent* is of the form $[\Gamma], \Delta \vdash \Pi, [\Lambda]$ for any multisets of formulas Δ, Π and Λ, and any sequence (i.e. ordered multiset) of formulas Γ. We write $[\_\,], \Delta \vdash \_\,, [\Lambda]$ or simply $\Delta \vdash [\Lambda]$ when Γ and Π are empty.

Game sequents are denoted by $S, S', S_1, S_2, \ldots$.

We introduce *game trees* as special trees.

Definition 6 (Game tree) A *game tree* is a tree with game sequents as nodes, and it satisfies the following properties 1, 2 and 3.

1. For any two nodes S and S', S' is a successor node of S (written as $\frac{S'}{S}$) iff they satisfy the following forms:

$$\frac{[\Gamma], \Delta \vdash \Pi, [A_i, \Lambda, A_1 \wedge A_2]}{[\Gamma], \Delta \vdash A_1 \wedge A_2, \Pi, [\Lambda]}\ a^O_{\wedge_i}\ {\scriptstyle (i\,=\,1,\,2)} \qquad \frac{[\Gamma, A_i], \Delta, A_1 \wedge A_2 \vdash \Pi, [\Lambda]}{[\Gamma], \Delta, A_1 \wedge A_2 \vdash \Pi, [\Lambda]}\ a^P_{\wedge_i}\ {\scriptstyle (i\,=\,1,\,2)}$$

$$\frac{[\Gamma], \Delta \vdash \Pi, [A, B, \Lambda, A \vee B]}{[\Gamma], \Delta \vdash A \vee B, \Pi, [\Lambda]}\ a^O_\vee \qquad \frac{[\Gamma, \underline{A \vee B}], \Delta, A \vee B \vdash \Pi, [\Lambda]}{[\Gamma], \Delta, A \vee B \vdash \Pi, [\Lambda]}\ a^P_\vee$$

$$\frac{[\Gamma], \Delta, A \vdash \Pi, [\Lambda, \neg A]}{[\Gamma], \Delta \vdash \neg A, \Pi, [\Lambda]}\ a^O_\neg \qquad \frac{[\Gamma], \Delta, \neg A \vdash A, \Pi, [\Lambda]}{[\Gamma], \Delta, \neg A \vdash \Pi, [\Lambda]}\ a^P_\neg$$

$$\frac{[\Gamma], \Delta, A \vdash \Pi, [\Lambda]}{[\Gamma, A], \Delta \vdash \Pi, [\Lambda]}\ d^O \qquad \frac{[\Gamma], \Delta \vdash A, \Pi, [\Lambda]}{[\Gamma], \Delta \vdash \Pi, [A, \Lambda]}\ d^P$$

$$\frac{[\Gamma], A_i, \Delta \vdash \Pi, [\Lambda]}{[\Gamma, \underline{A_1 \vee A_2}], \Delta \vdash \Pi, [\Lambda]}\ d^O_{\vee_i}\ {\scriptstyle (i = 1, 2)}$$

Here, note that since Γ is a sequence of formulas, with the above rule, Γ is, in fact, a stack.

As above, each rule has a label $a^O_{\wedge_1}, a^O_\vee, a^P_\neg \dots$ etc. and we call instances of the above rule *moves*. And if a label has a superscript O (resp. P), then we call it O's move (resp. P's move). We shall use $m, m', m_1, m_2, \dots$ as meta variables for the labels of moves.

In a branch, we shall describe S' as a O *sequent* (resp. P *sequent*) if $\frac{S'}{S}\ m^O$ ($\frac{S'}{S}\ m^P$) is a move. In particular, the root is O sequent.

2. **(Alternation)** In each branch, O sequents and P sequents appear alternately;

3. **(Restriction on atoms)** If, in a branch, there is a move such that $\frac{[\Gamma], \Delta \vdash \alpha, \Pi, [\Lambda]}{[\Gamma], \Delta \vdash \Pi, [\alpha, \Lambda]}\ d^P$ or $\frac{[\Gamma], \Delta, \neg\alpha \vdash \alpha, \Pi, [\Lambda]}{[\Gamma], \Delta, \neg\alpha \vdash \Pi, [\Lambda]}\ a^P_\neg$, then α should appear in Δ.

We introduce a *play* which is a branch in a game tree.

Definition 7 (Play) A *play for a formula* A is a branch in a game tree where

- the root is O sequent of the form $[\_], \_ \vdash \_, [A]$ where any formula of the form $\underline{B \vee C}$ does not occur as a subformula of A;
- for the formula A of the root, there is no move of the form $\frac{[\Gamma], \Delta \vdash \Pi, A, [\Lambda]}{[\Gamma], \Delta \vdash \Pi, [A, \Lambda]}\ d^P$ other than the first move $\frac{[\_], \_ \vdash A, [\_]}{[\_], \_ \vdash \_, [A]}\ d^P$;
- after a move of the form $\frac{[\Gamma], \Delta, \neg A \vdash A, \Pi, [\Lambda]}{[\Gamma], \Delta, \neg A \vdash \Pi, [\Lambda]}\ a^P_\neg$ for non atomic A, there is no move of the form $\frac{[\Gamma'], \Delta' \vdash A, \Pi', [\Lambda']}{[\Gamma'], \Delta' \vdash \Pi', [\Lambda', A]}\ d^P$ for this A.

We define the *length of a play* be the number of nodes in it. We denote plays as $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2, \dots$.

A strategy for Proponent is a special game tree defined as follows:

Definition 8 (Strategy) A *strategy for* A is a game tree where

- each branch is a play for A;
- each O sequent has at most one successor node;
- each P sequent has all successor nodes from that node.

We denote strategies by $\sigma, \sigma_1, \sigma_2, \ldots$.

A play for A is said to be won by P if it is finite, and its leaf is P node of the form $[\,\_\,], \Delta, \alpha \vdash \alpha, [\Lambda]$.

A *winning strategy* for A is a strategy for A where each play in it is won by P.

We have the following lemma, which has an important corollary.

Lemma 1 *In a play $\mathcal{P}$,*

- *each O sequent is of the form $[\,\_\,], \Delta \vdash \,\_\,, [\Lambda]$;*
- *each P sequent is of the form either $[A], \Delta \vdash \,\_\,, [\Lambda]$ or $[\,\_\,], \Delta \vdash A, [\Lambda]$.*

Proof. By induction on the length of $\mathcal{P}$. ■

As an immediate consequence of this lemma, we have the following corollary called *restriction on O's move.*

Corollary 1 (Restriction on O's move) *Let σ be a strategy for $\vdash [C]$, then O's moves in σ are one of the following forms.*

$$\frac{\Delta \vdash [A, \Lambda, A \wedge B] \quad \Delta \vdash [B, \Lambda, A \wedge B]}{\Delta \vdash A \wedge B, [\Lambda]}\, a^O_{\wedge_1}, a^O_{\wedge_2} \qquad \frac{\Delta \vdash [A, B, \Lambda, A \vee B]}{\Delta \vdash A \vee B, [\Lambda]}\, a^O_{\vee}$$

$$\frac{\Delta, A \vdash [\Lambda, \neg A]}{\Delta \vdash \neg A, [\Lambda]}\, a^O_{\neg} \qquad \frac{\Delta, A \vdash [\Lambda]}{[A], \Delta \vdash [\Lambda]}\, d^O \qquad \frac{A, \Delta \vdash [\Lambda] \quad B, \Delta \vdash [\Lambda]}{[\underline{A \vee B}], \Delta \vdash [\Lambda]}\, d^O_{\vee_1}, d^O_{\vee_2}$$

Thus, in a strategy, we shall consider the above two moves $a^O_{\wedge_1}$ and $a^O_{\wedge_2}$ as one move $a^O_{\wedge}$; and similarly we consider $d^O_{\vee_1}$ and $d^O_{\vee_2}$ as one move $d^O_{\vee}$.

4 The relationship between Lorenzen-Felscher Game and our Sequent Calculus Game

In Lorenzen-Felscher Game, a move is defined as a pair of an assertion (an attack or a defense) and a natural number which indicates to which assertion that attack or defense is made. In our Sequent Calculus Game, contexts play a role of such reference numbers.

Here, for any game sequent $[\Gamma], \Delta \vdash \Pi, [\Lambda]$, we shall describe $[\Gamma]$ as *Undefended O's assertions*; Δ as *O's assertions*; Π as *P's assertions*; $[\Lambda]$ as *Undefended P's assertions.*

We shall explain the correspondence between Lorenzen-Felscher Game and our Sequent Calculus Game by the following example.

A dialogue for $\alpha \vee \neg\alpha$ is as follows:

$$
\begin{array}{c}
(d^P; \alpha, 2)\\
(a^O; \alpha, 3)\\
(d^P; \neg\alpha, 2)\\
(a^O; [\alpha \vee \neg\alpha], 1)\\
(d^P; \alpha \vee \neg\alpha, 0)
\end{array}
$$

We interpret each move in this dialogue by a game sequent as follows:

$(d^P; \alpha \vee \neg\alpha, 0)$ is interpreted as $\vdash \alpha \vee \neg\alpha$.

O's attack upon $\alpha \vee \neg\alpha$ $(a^O; [\alpha \vee \neg\alpha], 1)$ is interpreted as $\vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]$. Note that this game sequent is obtained from the lower sequent $\vdash \alpha \vee \neg\alpha$ by moving the attacked formula $\alpha \vee \neg\alpha$ of P's assertions to P's undefended assertions, and adding α and $\neg\alpha$ to P's undefended assertions.

P's defense $(d^P; \neg\alpha, 2)$ is interpreted as $\vdash \neg\alpha, [\alpha, \alpha \vee \neg\alpha]$. Note that this game sequent is obtained from the lower sequent $\vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]$ by moving the defended formula $\neg\alpha$ of P's undefended assertions to P's assertions.

O's attack upon $\neg\alpha$ $(a^O; \alpha, 3)$ is interpreted as $\alpha \vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]$. Note that this is obtained from the lower sequent $\vdash \neg\alpha, [\alpha, \alpha \vee \neg\alpha]$ by moving the attacked $\neg\alpha$ of P's assertions to P's undefended assertions, and adding α to O's assertions.

P's defense $(d^P; \alpha, 2)$ is interpreted as $\alpha \vdash \alpha, [\neg\alpha, \alpha \vee \neg\alpha]$. Note that this is obtained from the lower sequent $\alpha \vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]$ by moving the defended α of P's undefended assertions to P's assertions.

Finally, we add $\vdash [\alpha \vee \neg\alpha]$ as the root.

Then the resulting interpretation is as follows:

$$
\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\alpha \vdash \alpha, [\neg\alpha, \alpha \vee \neg\alpha]}{\alpha \vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]}}{\vdash \neg\alpha, [\alpha, \alpha \vee \neg\alpha]}}{\vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]}}{\vdash \alpha \vee \neg\alpha}}{\vdash [\alpha \vee \neg\alpha]}
$$

It is clear that this tree satisfies the definition of play.

Conversely, the above play for $\vdash [\alpha \vee \neg\alpha]$ can be seen as a dialogue by adding reference numbers instead of contexts. We shall interpret each move in the play by a move of Lorenzen-Felscher Game.

At the move $\dfrac{\vdash \alpha \vee \neg\alpha}{\vdash [\alpha \vee \neg\alpha]}\ d^P$, since $\alpha \vee \neg\alpha$ is the main formula of this move, it is interpreted as $(d^P; \alpha \vee \neg\alpha, 0)$.

At the move $\dfrac{\vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]}{\vdash \alpha \vee \neg\alpha}\ a^O_\vee$, since $\alpha \vee \neg\alpha$ is the main formula which is decomposed, it is interpreted as $(a^O; [\alpha \vee \neg\alpha], 1)$. Here the reference number 1 is determined by checking where the main formula $\alpha \vee \neg\alpha$ of P's assertions occurs firstly.

At the move $\dfrac{\vdash \neg\alpha, [\alpha, \alpha \vee \neg\alpha]}{\vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]}\ d^P$, since $\neg\alpha$ is the main formula of this move, it is interpreted as $(d^P; \neg\alpha, 2)$. Here $\neg\alpha$ firstly occurred at the 2nd move labeled $a^O_\vee$.

At the move $\dfrac{\alpha \vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]}{\vdash \neg\alpha, [\alpha, \alpha \vee \neg\alpha]}\ a^O_\neg$, since $\neg\alpha$ is the main formula, it is interpreted as $(a^O; \alpha, 3)$. Note that $\neg\alpha$ firstly occurred in P's assertions at the 3rd move.

At the move $\dfrac{\alpha \vdash \alpha, [\neg\alpha, \alpha \vee \neg\alpha]}{\alpha \vdash [\alpha, \neg\alpha, \alpha \vee \neg\alpha]}\, d^P$, since the main formula is α, it is interpreted as $(d^P; \alpha, 2)$. Note that α firstly occurred in P's undefended assertions at the 2nd move.

Thus the resulting sequence of moves is the same as the above dialogue for $\alpha \vee \neg\alpha$.

Note that **(D10)** of Lorenzen-Felscher Game is the same as our rule of the restriction on atoms. **(D12)** and **(D13)** correspond to our definition of O's moves. **(D11)** corresponds to the definition that undefended O's assertions are a sequence of formulas.

In this way, any dialogue of Lorenzen-Felscher Game is identified with a play in our Sequent Calculus Game.

5 Kleene's sequent calculus for classical propositional logic

We introduce a variant of Gentzen's sequent calculus for classical propositional logic. The following system is introduced in [Troelstra-Schwichtenberg 00] and called Kleene's sequent calculus.

In the following system, the weakening rule and the contraction rule are hidden in the inference rules. Note that, recently, the importance of the structural rules of Gentzen's original sequent calculus has been pointed out. In particular, Girard's linear logic ([Girard 87]) is introduced by making these two structural rules explicit with the use of notion of modalities. (See [Okada 04].) From a viewpoint of Gentzen's sequent calculus, in the following system, the weakening rule is absorbed in the axiom rule; namely, we can consider the axiom rule of the form $\overline{\Delta, \alpha \vdash \alpha, \Lambda}\ ax$ as a combination of the original Gentzen's axiom rule of the form $\overline{\alpha \vdash \alpha}\ ax$ and some applications of the weakening rule. As for the contraction rule, it is absorbed in each inference rule of a logical connective; namely, each inference rule of a logical connective in Kleene's system can be considered as a combination of Gentzen's logical inference rule and the contraction rule. For example, $\neg L$-rule of the form: $\frac{\Delta, \neg A \vdash A, \Lambda}{\Delta, \neg A \vdash \Lambda}\,\neg L$ can be considered as a combination of Gentzen's $\neg L$-rule and the contraction rule of the form: $\dfrac{\dfrac{\Delta, \neg A \vdash A, \Lambda}{\Delta, \neg A, \neg A \vdash \Lambda}\,\neg L}{\Delta, \neg A \vdash \Lambda}\,contL$. Thus the weakening rule and the contraction rule are essential for Kleene's sequent calculus from a viewpoint of Gentzen's system.

Since we intend to show the correspondence between Lorenzen-Felscher Game and Kleene's sequent calculus not only for winning strategies but also for strategies, we shall define, firstly, a *proof structure*, and then a *proof.*

We denote sequents as $S, S_1, S_2, \ldots$.

Definition 9 (Proof structure) A *proof structure* is a tree with sequents as nodes, and satisfies the following condition:

Each node S has at most two successor nodes S_1 and S_2 (written as $\frac{S_1}{S}$ or $\frac{S_1 \quad S_2}{S}$); and they satisfy the following inference rules.

Inference rules of Kleene's sequent calculus

$$\overline{\Delta, \alpha \vdash \alpha, \Lambda}\ ax$$

$$\frac{\Delta, \neg A \vdash A, \Lambda}{\Delta, \neg A \vdash \Lambda}\,\neg L \qquad \frac{\Delta, A \vdash \Lambda, \neg A}{\Delta \vdash \neg A, \Lambda}\,\neg R$$

$$\frac{A_i, \Delta, A_1 \wedge A_2 \vdash \Lambda}{\Delta, A_1 \wedge A_2 \vdash \Lambda} \wedge L_i \; {\scriptstyle (i=1,2)} \qquad \frac{\Delta \vdash A, \Lambda, A \wedge B \quad \Delta \vdash B, \Lambda, A \wedge B}{\Delta \vdash A \wedge B, \Lambda} \wedge R$$

$$\frac{A, \Delta, A \vee B \vdash \Lambda \quad B, \Delta, A \vee B \vdash \Lambda}{\Delta, A \vee B \vdash \Lambda} \vee L \qquad \frac{\Delta \vdash A_1 \vee A_2, \Lambda, A_i}{\Delta \vdash A_1 \vee A_2, \Lambda} \vee R_i \; {\scriptstyle (i=1,2)}$$

A proof structure for a sequent S is a proof structure whose root is S.

A *proof* for a sequent S is a proof structure where the root is S, and any branch is finite and its leaf is of the form $\overline{\Delta, \alpha \vdash \alpha, \Lambda}\ ax$.

Remark 1 Note that even if there are the same atoms in both side of a sequent in a branch, this does not mean that the sequent is a leaf. For example, the following proof structure is also a proof:

$$\cfrac{\cfrac{\overline{\alpha, \alpha, \alpha \vdash \alpha, \neg\alpha}\ ax}{\alpha, \alpha \vdash \alpha, \neg\alpha} \neg L}{\alpha \vdash \alpha, \neg\alpha} \neg L$$

6 Strategies and proof structures

Herbelin [Herbelin 95] established the correspondence between the winning strategies of Lorenzen-Felscher Game and a certain subset of the cut-free (normal form) proofs of Kleene's sequent calculus. We show that this correspondence can be extended to the correspondence between the set of (not necessarily winning) strategies and the set of (not necessarily completed finite) proof structures.

We consider a subsystem $\mathcal{K}$ of Kleene's sequent calculus where the set of proof structures is a subset of the set of proof structures of the original Kleene's seqeunt calculus. In the following subsystem $\mathcal{K}$, some inference rules of the original Kleene's system are combined into an inference rule. In particular, the $\neg L$-rule is introduced in combination with other R-rules of connectives or with the axiom rule.

This subsystem $\mathcal{K}$ is essentially the same as LKQ^* of [Herbelin 95].

Inference rules of $\mathcal{K}$

$$\overline{\Delta, \alpha \vdash \alpha, \Lambda}\ ax \qquad \frac{\Delta, A \vdash \Lambda, \neg A}{\Delta \vdash \neg A, \Lambda} \neg R$$

$$\frac{A_i, \Delta, A_1 \wedge A_2 \vdash \Lambda}{\Delta, A_1 \wedge A_2 \vdash \Lambda} \wedge L_i \; {\scriptstyle (i=1,2)} \qquad \frac{\Delta \vdash A, \Lambda, A \wedge B \quad \Delta \vdash B, \Lambda, A \wedge B}{\Delta \vdash A \wedge B, \Lambda} \wedge R$$

$$\frac{A, \Delta, A \vee B \vdash \Lambda \quad B, \Delta, A \vee B \vdash \Lambda}{\Delta, A \vee B \vdash \Lambda} \vee L \qquad \frac{\Delta \vdash A \vee B, \Lambda, A, B}{\Delta \vdash A \vee B, \Lambda} \vee R$$

$$\overline{\Delta, \neg\alpha, \alpha \vdash \Lambda}\ \neg Lax \qquad \frac{\Delta, \neg\neg A, A \vdash \neg A, \Lambda}{\Delta, \neg\neg A \vdash \Lambda} \neg L \neg R$$

$$\frac{\Delta, \neg(A \wedge B) \vdash A, \Lambda, A \wedge B \quad \Delta, \neg(A \wedge B) \vdash B, \Lambda, A \wedge B}{\Delta, \neg(A \wedge B) \vdash \Lambda} \neg L \wedge R$$

$$\frac{\Delta, \neg(A \vee B) \vdash A, B, A \vee B, \Lambda}{\Delta, \neg(A \vee B) \vdash \Lambda}\ \neg L \vee R$$

Theorem 1 *For each strategy σ for A of Sequent Calculus Game, there is a unique proof structure π for A of $\mathcal{K}$ such that π is embeddable into σ.*

We shall give a sketch of our proof for this theorem. In order to prove this theorem, we introduce the notion of *PO-move*. Since, in a strategy, if a P's move is given, then the successive O's move is uniquely determined by Lemma 1, we consider a pair of a P's move and the O's successive move as a combined move in a strategy.

Definition 10 (*PO*-move) *A PO-move from $\Delta \vdash [\Lambda]$ to $\Delta_1 \vdash [\Lambda_1]$ (or from $\Delta \vdash [\Lambda]$ to $\Delta_1 \vdash [\Lambda_1]$ and $\Delta_2 \vdash [\Lambda_2]$)* is a pair of a P's move and the O's successive move of the form $\dfrac{\dfrac{\Delta_1 \vdash [\Lambda_1]}{[\Gamma'], \Delta' \vdash \Pi', [\Lambda']}\ m^O}{\Delta \vdash [\Lambda]}\ m^P$ $\left(\text{or } \dfrac{\dfrac{\Delta_1 \vdash [\Lambda_1] \quad \Delta_2 \vdash [\Lambda_2]}{[\Gamma'], \Delta' \vdash \Pi', [\Lambda']}\ m^O}{\Delta \vdash [\Lambda]}\ m^P\right)$ in a strategy.

As a special *PO*-move, we shall consider a pair of a P's move and the empty O's move of the form $\dfrac{\Delta, \alpha \vdash \alpha, [\Lambda]}{\Delta, \alpha \vdash [\alpha, \Lambda]}\ d^P$ or $\dfrac{\Delta, \neg\alpha, \alpha \vdash \alpha, [\Lambda]}{\Delta, \neg\alpha, \alpha \vdash [\Lambda]}\ a^P_{\neg}$.

Since a *PO*-move itself satisfies the definition of a strategy, we can consider a *PO*-move as a primitive component of a strategy.

The theorem follows from the following lemma:

Lemma 2 *There is a one-to-one function from the PO-moves to the inference rules of $\mathcal{K}$.*

This lemma is shown by defining the function as follows.

1. To $\dfrac{\Delta, \alpha \vdash \alpha, [\Lambda]}{\Delta, \alpha \vdash [\alpha, \Lambda]}\ d^P$, we assign ax-rule:

$$\frac{}{\Delta, \alpha \vdash \alpha, \Lambda}\ ax$$

2. To $\dfrac{\Delta, \neg\alpha, \alpha \vdash \alpha, [\Lambda]}{\Delta, \neg\alpha, \alpha \vdash [\Lambda]}\ a^P_{\neg}$, we assign $\neg Lax$-rule:

$$\frac{}{\Delta, \neg\alpha, \alpha \vdash \Lambda}\ \neg Lax$$

3. To $\dfrac{\dfrac{A_i, \Delta, A_1 \wedge A_2 \vdash [\Lambda]}{[A_i], \Delta, A_1 \wedge A_2 \vdash [\Lambda]}\ d^O}{\Delta, A_1 \wedge A_2 \vdash [\Lambda]}\ a^P_{\wedge_i}$, we assign $\wedge L_i$-rule:

$$\frac{A_i, \Delta, A_1 \wedge A_2 \vdash \Lambda}{\Delta, A_1 \wedge A_2 \vdash \Lambda}\ \wedge L_i$$

4. To $\dfrac{\dfrac{A, \Delta, A \vee B \vdash [\Lambda] \quad B, \Delta, A \vee B \vdash [\Lambda]}{[\underline{A \vee B}], \Delta, A \vee B \vdash [\Lambda]}\ d^O_{\vee 12}}{\Delta, A \vee B \vdash [\Lambda]}\ a^P_{\vee}$, we assign $\vee L$-rule:

$$\frac{A, \Delta, A \vee B \vdash \Lambda \quad B, \Delta, A \vee B \vdash \Lambda}{\Delta, A \vee B \vdash \Lambda}\ \vee L$$

5. To $\dfrac{\dfrac{\Delta \vdash [A, A \wedge B, \Lambda] \quad \Delta \vdash [B, A \wedge B, \Lambda]}{\dfrac{\Delta \vdash A \wedge B, [\Lambda]}{\Delta \vdash [A \wedge B, \Lambda]} d^P} a^O_{\wedge 12}}{}$, we assign $\wedge R$-rule:

$$\frac{\Delta \vdash A, A \wedge B, \Lambda \quad \Delta \vdash B, A \wedge B, \Lambda}{\Delta \vdash A \wedge B, \Lambda} \wedge R$$

6. To $\dfrac{\Delta, \neg(A \wedge B) \vdash [A, A \wedge B, \Lambda] \quad \Delta, \neg(A \wedge B) \vdash [B, A \wedge B, \Lambda]}{\dfrac{\Delta, \neg(A \wedge B) \vdash A \wedge B, [\Lambda]}{\Delta, \neg(A \wedge B) \vdash [\Lambda]} a^P_{\neg}} a^O_{\wedge 12}$, we assign $\neg L \wedge R$-rule:

$$\frac{\Delta, \neg(A \wedge B) \vdash A, \Lambda, A \wedge B \quad \Delta, \neg(A \wedge B) \vdash B, \Lambda, A \wedge B}{\Delta, \neg(A \wedge B) \vdash \Lambda} \neg L \wedge R$$

7. To $\dfrac{\Delta \vdash [A, B, A \vee B, \Lambda]}{\dfrac{\Delta \vdash A \vee B, [\Lambda]}{\Delta \vdash [A \vee B, \Lambda]} d^P} a^O_{\vee}$, we assign $\vee R$-rule:

$$\frac{\Delta \vdash A, B, A \vee B, \Lambda}{\Delta \vdash A \vee B, \Lambda} \vee R$$

8. To $\dfrac{\Delta, \neg(A \vee B) \vdash [A, B, A \vee B, \Lambda]}{\dfrac{\Delta, \neg(A \vee B) \vdash A \vee B, [\Lambda]}{\Delta, \neg(A \vee B) \vdash [\Lambda]} a^P_{\neg}} a^O_{\vee}$, we assign $\neg L \vee R$-rule:

$$\frac{\Delta, \neg(A \vee B) \vdash A, B, A \vee B, \Lambda}{\Delta, \neg(A \vee B) \vdash \Lambda} \neg L \vee R$$

9. To $\dfrac{\Delta, A \vdash [\neg A, \Lambda]}{\dfrac{\Delta \vdash \neg A, [\Lambda]}{\Delta \vdash [\neg A, \Lambda]} d^P} a^O_{\neg}$, we assign $\neg R$-rule:

$$\frac{\Delta, A \vdash \neg A, \Lambda}{\Delta \vdash \neg A, \Lambda} \neg R$$

10. To $\dfrac{\Delta, \neg\neg A, A \vdash [\neg A, \Lambda]}{\dfrac{\Delta, \neg\neg A \vdash \neg A, [\Lambda]}{\Delta, \neg\neg A \vdash [\Lambda]} a^P_{\neg}} a^O_{\neg}$, we assign $\neg L \neg R$-rule:

$$\frac{\Delta, \neg\neg A, A \vdash \neg A, \Lambda}{\Delta, \neg\neg A \vdash \Lambda} \neg L \neg R$$

7 Concluding remarks

We have shown in this paper that the one-to-one correspondence between winning strategies and the proofs of [Herbelin 95] can be extended to the correspondence between the set of (not necessarily winning) strategies and the set of (not necessarily completed) proof structures. We introduced a sequent calculus representation of Lorenzen-Felscher Game (called Sequent Calculus Game in this paper), in which not only a winning strategy but also any strategy can be identified with a cut-free proof structure. By ignoring certain decorations of the sequents (of our Sequent Calculus Game), the strategies can be identified with a certain set of cut-free proof structures of Kleene's sequent calculus. The winning strategies are characterized as a subset of the finite completed cut-free proofs.

We point out that, from a linear logical point of view, the combination of additive conjunction (&) and multiplicative disjunction ($\parr$) is essential for our result (and also for [Herbelin 95]). These connectives are not in de Morgan dual and consequently cause a difficulty in interpreting the cut-rule. [Blass 92], [Abramsky-Jagadeesan 94] and others overcome this difficulty by introducing the linear logical negation, and they interpret cut-rule as the composition of strategies.

For further work we plan to extend our study to the linear logic case and make clear the relationship between traditional Lorenzen-Felscher Game and Blass-Abramsky games.

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