Counter-Example Construction with Euler Diagrams

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Abstract

One of the traditional applications of Euler diagrams is as a representation or counterpart of the usual set-theoretical models of given sentences. However, Euler diagrams have recently been investigated as the counterparts of logical formulas, which constitute formal proofs. Euler diagrams are rigorously defined as syntactic objects, and their inference systems, which are equivalent to some symbolic logical systems, are formalized. Based on this observation, we investigate both counter-model construction and proof-construction in the framework of Euler diagrams. We introduce the notion of "counter-diagrammatic proof", which shows the invalidity of a given inference, and which is defined as a syntactic manipulation of diagrams of the same sort as inference rules to construct proofs. Thus, in our Euler diagrammatic framework, the completeness theorem can be formalized in terms of the existence of a diagrammatic proof or a counter-diagrammatic proof.

1 Introduction

Applications of diagrammatic or graphical representations to logical reasoning have recently been studied by logicians and mathematicians. Among various diagrams and applications thereof, one of the traditional applications regards certain diagrams as "models" of given sentences. Although it is difficult to clearly define what is a model, we broadly mean that a certain structure in that given sentences can be determined to be true or false. One of the latest examples is the Tarski's world of Barwise and Etchemendy [2]. Tarski's world consists of a grid containing small, medium, and large size blocks shaped as cubes, tetrahedra, and dodecahedra. We can regard Tarski's world as a particular kind of model for given sentences, and can determine the given sentences to be true or false in the world. Tarski's world was later developed to allow blocks of indeterminate size and shape [3], which makes it possible to treat Tarski's world more syntactically, and to develop a heterogeneous system called a *hyperproof*. However, inferences using Tarski's world are still designed model-theoretically. Another traditional example concerns Euler diagrams, which were originally introduced in the 18th century to illustrate syllogisms. A basic Euler diagram consists of circles and points, and syllogistic sentences are represented using inclusion and exclusion relations between the circles and points. Furthermore, a circle in an Euler diagram may be considered as a representation of a set, and a point may be considered as an entity of a given domain. Then, inclusion and exclusion relations between circles and points may be considered to represent set-theoretical notions such as subset and disjointness relations, respectively. Thus, Euler diagrams may also be considered as representations of the usual set-theoretical models of given sentences.

At the same time, Euler diagrams have recently been investigated as counterparts of logical formulas, which constitute formal proofs. Euler diagrams are rigorously defined as syntactic objects, allowing set-theoretical semantics to be defined. Inference systems are also formalized, and they are shown to be equivalent to some symbolic logical systems. Consequently, fundamental logical properties such as soundness and completeness have been investigated (e.g., [13, 9, 11]). In [11, 16, 17], we investigated our Euler diagrammatic system from a proof-theoretical viewpoint; we defined an Euler diagrammatic inference system, which proved to be sound and complete, and further investigated the normalization theorem with an appropriate notion of normal Euler diagrammatic proofs.

In this way, Euler diagrams are, on the one hand, regarded as models of given sentences, and on the other hand, they are exploited as syntactic objects that constitute formal proofs. Based on this fact, in this paper, we investigate both counter-model construction and proof-construction in the same framework of Euler diagrams. Contrasted to a proof, which shows the validity of a given inference, to disprove an inference, we usually construct a counter-model. A counter-model is a model in which all premises of a given inference are true, but its conclusion is false. The notions of models and proofs are traditionally defined in the fundamentally different frameworks of semantics and syntax, respectively. Based on the application of Euler diagrams as representations of set-theoretical models, we formalize the counter-model construction as a similar syntactic manipulation of Euler diagrams as syntactic inference rules to construct proofs. (In [4], Barwise and Etchemendy also suggested a possible application of Euler diagrams to construct counter-examples.) We then introduce the notion of *counter-diagrammatic proof*, which shows the invalidity of a given inference. In our Euler diagrammatic framework, then, the completeness theorem can be formalized as follows: *There exists a counter-diagrammatic proof or a diagrammatic proof for given Euler diagrams.* Our result may be compared with the usual proof-search or tableau methods in symbolic logic, where the leaves of a resulting proof-search tree define a model with little modification in the classical logical framework. One of the advantages of our Euler diagrams is that they provide an intuitive and visual representation of counter-models.

In Section 2, we review the syntax, semantics, and inference system of our Euler diagrams. We also introduce the completeness theorem of our system with respect to our set-theoretical semantics. The canonical construction of diagrammatic proofs, which is defined in the proof of completeness, is also applied to counter-diagrammatic proof construction with slight modifications. In Section 3, we define our counter-diagrams, which are counterparts of the usual set-theoretical counter-models, and define our counter-diagrammatic proofs. Then, in Section 4, we prove the appropriateness and sufficiency of our counter-diagrammatic proofs, which can be considered as counterparts of the usual soundness and completeness theorems. These theorems make it possible to formalize another completeness theorem for our Euler diagrammatic system. Our theorem is slightly restricted, although it is sufficient for the usual syllogistic or transitive inference. The theorem holds not for general Euler diagrams, but for diagrams in which points are located in appropriate regions.

2 System of Euler diagrams

We review the syntax and semantics of our Euler diagrams from [11] in Section 2.1, and our inference system in Section 2.2. In Section 2.3, we discuss the completeness theorem with respect to our set-theoretical semantics by focusing on the canonical construction of diagrammatic proofs.

2.1 Syntax and semantics of EUL

Our Euler diagram, called EUL-diagram, is defined as a plane with named circles and points. In this paper, our diagram is a slight extension of the most basic one given in [11]: As well as constant points, which correspond to constant symbols of the first-order language, we introduce existential points, which correspond to bound variables associated with the existential quantifier. Each diagram is specified by inclusion and exclusion relations maintained between circles and points. Based on the interpretation of circles (resp. points) as subsets (resp. elements) of a set-theoretical domain, each diagram is interpreted in terms of the relations that hold on it.

Definition 2.1. An **EUL-diagram** is a plane ($\mathbb{R}^2$) with a finite number, at least two, of **(named) simple closed curves** (simply called **(named) circles** and denoted by $A, B, C, \dots$), **constant points** (denoted by $a, b, c, \dots$), and **existential points** (denoted by $x, y, z, \dots$).
Constant points and existential points are collectively called **(named) points**, and are denoted by $p, q, p_1, p_2, \dots$.
No two named points are completely concurrent. No two named circles, or points, have the same name.
Named circles and points are collectively called **(diagrammatic) objects**, and denoted by $s, t, u, \dots$. We use a rectangle to represent the plane for a diagram. Diagrams are denoted by $\mathcal{D}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2, \dots$.
When $\mathcal{D}$ is a diagram, we denote the set of named points of $\mathcal{D}$ as $pt(\mathcal{D})$, the set of named circles of $\mathcal{D}$ as $cr(\mathcal{D})$, and the set of objects of $\mathcal{D}$ as $ob(\mathcal{D})$.
A diagram consisting of only two objects is called a **minimal diagram**, and is denoted by $\alpha, \beta, \gamma, \dots$.

Our diagrams are investigated in terms of the following topological relations between diagrammatic objects.

Definition 2.2. **EUL-relations** are the following reflexive asymmetric binary relation $\sqsubset$, and irreflexive symmetric binary relations $\vdash\!\dashv$ and $\bowtie$:

$A \sqsubset B$	"the interior of A is *inside of* the interior of B,"
$A \vdash\!\dashv B$	"the interior of A is *outside of* the interior of B,"
$A \bowtie B$	"there is at least one *crossing* point between A and B,"
$p \sqsubset A$	"p is *inside of* the interior of A,"
$p \vdash\!\dashv A$	"p is *outside of* the interior of A,"
$p \vdash\!\dashv q$	"p is *outside of* q (i.e., p is not located at point q)."

EUL-relations are denoted by $R, R_1, R_2, \dots$.

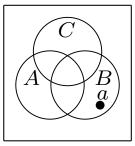
We sometimes express by $\sqsupset$ the inverse of $\sqsubset$. The set of EUL-relations that hold on a diagram $\mathcal{D}$ is uniquely determined, and we denote this set by $\mathsf{rel}(\mathcal{D})$. In the description of $\mathsf{rel}(\mathcal{D})$, we omit the reflexive relation $s \sqsubset s$ for each object s. Furthermore, we often omit relations of the form $p \vdash\!\dashv q$ for points p and q, which always hold by definition.

In the following, we consider the equivalence class of diagrams in terms of the EUL-relations.

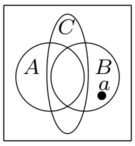
Definition 2.3 (Equivalence)**.** Any pair of EUL-diagrams $\mathcal{D}$ and $\mathcal{E}$ are **(syntactically) equivalent** if $\mathsf{rel}(\mathcal{D}) = \mathsf{rel}(\mathcal{E})$.

In what follows, we identify diagrams that are syntactically equivalent, and refer to them by a single name.

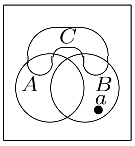
Remark 2.4 (Equivalence)**.** Our equation of diagrams is slightly rough in view of specific plane diagrams. For example, the following diagrams $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ are all equivalent, because $\mathsf{rel}(\mathcal{D}_1) = \mathsf{rel}(\mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_3) = \mathsf{rel}(\mathcal{D}_4) = \{a \vDash A, a \sqsubset B, a \vDash C, A \bowtie B, A \bowtie C, B \bowtie C\}$.



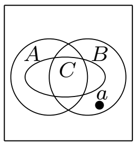
$\mathcal{D}_1$



$\mathcal{D}_2$



$\mathcal{D}_3$



$\mathcal{D}_4$

In [10], our system is extended by introducing intersection, union, and complement regions, respectively, as diagrammatic objects, allowing the above diagrams $\mathcal{D}_1, \mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ to be distinguished.

We also give a translation of diagrams into the usual first-order formulas. See [17] for a detailed description of our translation.

Definition 2.5 (Translation of diagrams)**.** Each named circle is translated into a unary predicate, and each constant (resp. existential) point is translated into a constant symbol (resp. variable). Then, each EUL-relation R is translated into a formula R° as follows:

$$(p \sqsubset A)^\circ := A(p); \qquad (p \vDash A)^\circ := \neg A(p);$$
$$(A \sqsubset B)^\circ := \forall x(A(x) \to B(x)); \qquad (A \vDash B)^\circ := \forall x(A(x) \to \neg B(x));$$
$$(A \bowtie B)^\circ := \forall x((A(x) \to A(x)) \wedge (B(x) \to B(x))).$$

Let $\mathcal{D}$ be an EUL-diagram with the following set of relations:

$$\mathsf{rel}(\mathcal{D}) = \{R_1, \ldots, R_i, x_1 \Box A_1, \ldots, x_1 \Box A_k, \ldots, x_l \Box A_1, \ldots, x_l \Box A_k\},$$

where $\Box$ is $\sqsubset$ or $\vDash$, and no existential point appears in $R_1, \ldots, R_i$. Then, the diagram $\mathcal{D}$ is translated into the following conjunctive formula:

$$\mathcal{D}^\circ := R_1^\circ \wedge \cdots \wedge R_i^\circ \wedge \exists x_1(\overline{A_1(x_1)} \wedge \cdots \wedge \overline{A_k(x_1)}) \wedge \cdots \wedge \exists x_l(\overline{A_1(x_l)} \wedge \cdots \wedge \overline{A_k(x_l)}),$$

where $\overline{A(x)}$ is $A(x)$ or $\neg A(x)$ depending on $\Box$.

Note that our diagram is abstractly a "conjunction of relations."

Note also that, based on the convention of Venn diagrams, we interpret the $\bowtie$-relation so that it does not convey any specific (inclusion or exclusion) information about the relationship between circles. (See [11] for this background.) Thus, $A \bowtie B$ is translated into a tautology as above. (Cf. Remark 2.8.)

Our semantics is based on the above translation, and hence, it is not faithful to plane diagrams. That is, our semantics is not the canonical interpretation of a diagram over a plane ($\mathbb{R}^2$), where named circles (and points) are interpreted by circles (resp. points) on the plane, but a more abstract set-theoretical interpretation.

Definition 2.6. A **model** M is a pair (U, I), where U is a non-empty set (the domain of M), and I is an interpretation function that assigns to each circle or constant point a *non-empty subset* of U such that $I(a)$ is a singleton for any constant point a, and $I(a) \neq I(b)$ for any constant points a, b with distinct names.

Note that the interpretation $I(A)$ of a circle A is assumed to be non-empty. (Cf. Remark 2.12.)

EUL-diagrams are interpreted in terms of the EUL-relations that hold on them. To avoid the complexity caused by existential points, we define our interpretation for *a set of diagrams*, instead of for a single diagram.

Definition 2.7. Let $\vec{\mathcal{D}}$ be a set of diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ such that $\mathsf{rel}(\vec{\mathcal{D}}) = \mathsf{rel}(\mathcal{D}_1) \cup \cdots \cup \mathsf{rel}(\mathcal{D}_n) = \{R_1, \ldots, R_i, x_1 \Box A_1, \ldots, x_1 \Box A_k, \ldots, x_l \Box A_1, \ldots, x_l \Box A_k\}$, where $\Box$ is $\sqsubset$ or $\vdash\!\dashv$, and no existential point appears in $R_1, \ldots, R_i$.
$M = (U, I)$ is a model of $\vec{\mathcal{D}}$, written as $M \models \vec{\mathcal{D}}$, if and only if:

- for every $R_j \in \mathsf{rel}(\vec{\mathcal{D}})$ $(1 \leq j \leq i)$,
  $I(s) \subseteq I(t)$ holds in M if R_j is $s \sqsubset t$; and
  $I(s) \cap I(t) = \emptyset$ holds in M if R_j is $s \vdash\!\dashv t$;
- for every x_j $(1 \leq j \leq l)$, there exists an $m_j \in M$ such that $m_j \Box I(A_1)$ and $\cdots$ and $m_j \Box I(A_k)$ hold, where $\Box$ is $\in$ if $x_j \sqsubset A \in \mathsf{rel}(\vec{\mathcal{D}})$, and $\notin$ if $x_j \vdash\!\dashv A \in \mathsf{rel}(\vec{\mathcal{D}})$.

Remark 2.8 (Semantic interpretation of $\bowtie$-relation)**.** By Definition 2.7, the EUL-relation $\bowtie$ does not contribute to the truth condition of EUL-diagrams. $A \bowtie B$ might be understood as $I(A) \cap I(B) = \emptyset$ *or* $I(A) \cap I(B) \neq \emptyset$, which is true in any model.

The semantic consequence relation $\models$ between EUL-diagrams is defined as usual in symbolic logic. (See [11, 16] for a detailed description.)

Definition 2.9 (Consistency)**.** A set of diagrams $\vec{\mathcal{D}}$ is said to be **consistent** when it has a model.

Remark 2.10 (RCC)**.** Our system is essentially the same as the *region connection calculus RCC* [15]. The region connection calculus is a topological approach to qualitative spatial representation and reasoning, and it is applied, for example, to Geographic Information System (GIS). In RCC, eight basic relations between spatial regions (circles in our framework) including our inclusion, exclusion, and crossing (partially overlapping) relations are investigated. While RCC investigates general n-dimensional spaces of spatial regions, we concentrate on 2-dimensional diagrams. Thus, our system without any named points can be considered as a subsystem of RCC. See, for example, [6, 7, 5] for surveys of RCC.

2.2 Inference system GDS

Mineshima-Okada-Takemura [11] introduced an Euler diagrammatic inference system called the Generalized Diagrammatic Syllogistic inference system, or GDS. This mainly consists of two kinds of inference rules: Deletion and Unification. Deletion allows us to delete a diagrammatic object from a given diagram. Unification (rules U1–U10, PI) allows us to unify two diagrams into one whose semantic information is equivalent to the conjunction of the two original diagrams. Two kinds of constraint are imposed on unification. One is the *constraint* *for determinacy*, which blocks disjunctive ambiguity with respect to the location of points, and the other is the *constraint for consistency*, which blocks representing inconsistent information in a single diagram. Thus, unification can be applied only when the constraint is satisfied.

Each unification rule is described by specifying (i) two premise diagrams, one of which is required to be minimal (except for PI); (ii) diagrammatic operations to introduce a new object into, or to rearrange a configuration of objects of, one of the premise diagrams. We also give schematic illustrations and concrete examples of the application of rules. We further specify the set of EUL-relations $\mathsf{rel}(\mathcal{D}+\alpha)$ of the unified diagram.

We have another kind of rule called Ren, which is used for renaming a named point in a given diagram. This corresponds to the renaming of a bound variable, and the introduction of the existential quantifier in view of the usual symbolic logic.

We also have three axioms. In particular, axiom A1 is based on our interpretation of the $\bowtie$-relation (cf. Remark 2.8), and A3 is based on our interpretation of circles being non-empty sets.

Of the eleven unification rules (U1–U10 and PI), we describe only U7, U10, and PI. The full list of inference rules of GDS can be found in Appendix A and [11, 16].

In the following, we avoid notational complexity in a diagram by expressing each named point, say $\overset{a}{\bullet}$, simply by its name, a.

Definition 2.11. The inference rules of GDS consist of the following Axioms (A1–A3), Unifications (U1–U10, PI), Ren, and Del.

Axiom:

A1 For any pair of circles A and B, any minimal diagram where $A \bowtie B$ holds is an axiom.

A2 Any EUL-diagram that consists only of points is an axiom.

A3 For any existential point x and any circle A, any minimal diagram where $x \sqsubset A$ holds is an axiom.

Unification:

U7 Premises: A minimal diagram α on which $A \vdash\!\dashv B$ holds; and a diagram $\mathcal{D}$ such that $B \in cr(\mathcal{D})$ (but $A \notin cr(\mathcal{D})$).

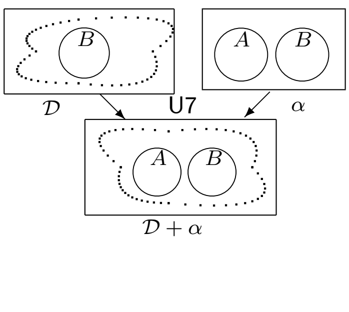
Constraint for determinacy: $p \sqsubset B$ holds for all $p \in pt(\mathcal{D})$.

Operation: Add the circle A to $\mathcal{D}$ (with preservation of all relations on $\mathcal{D}$) so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $A \vdash\!\dashv B$ holds; (2) $A \bowtie X$ holds for all circles X ($\not\equiv B$) such that $B \sqsubset X$ or $B \vdash\!\dashv X$ or $B \bowtie X$ holds on $\mathcal{D}$.

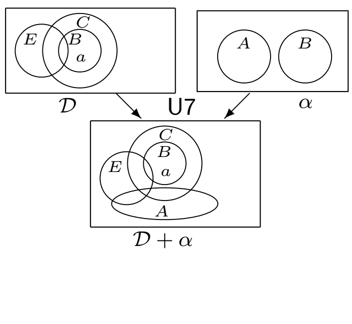
The set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ of the unified diagram is specified as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid B \sqsubset X \text{ or } B \vdash\!\dashv X \text{ or } B \bowtie X \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\}$$
$$\cup \{X \vdash\!\dashv A \mid X \sqsubset B \in \mathsf{rel}(\mathcal{D})\} \cup \{p \vdash\!\dashv A \mid p \in pt(\mathcal{D})\}$$

Schema of U7



Example of U7



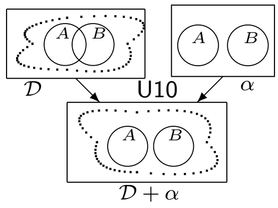
U10 Premises: $A \vdash\!\dashv B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.

Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$.

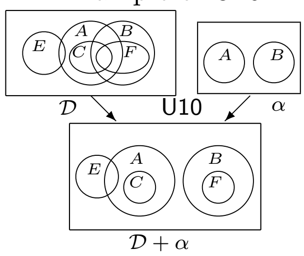
Operation: Modify all circles X (including A) and Y (including B) of $\mathcal{D}$ such that $X \sqsubset A$ and $Y \sqsubset B$, respectively, hold on $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D} + \alpha$: (1) $X \vdash\!\dashv B$ holds; (2) $X \Box t$ holds with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$ for all objects t of $\mathcal{D}$ such that $t \sqsubset A, X \Box t \in \mathsf{rel}(\mathcal{D})$; (3) $Y \vdash\!\dashv A$ holds; (4) $Y \Box s$ holds with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$ for all objects s of $\mathcal{D}$ such that $s \sqsubset B, Y \Box s \in \mathsf{rel}(\mathcal{D})$.

$$\mathsf{rel}(\mathcal{D} + \alpha) = \big(\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\big)$$
$$\cup \{X \vdash\!\dashv Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}$$

Schema of U10



Example of U10

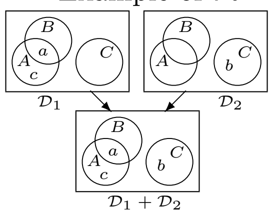


PI (Point Insertion) Premises: $X \Box Y \in \mathsf{rel}(\mathcal{D}_1)$ *iff* $X \Box Y \in \mathsf{rel}(\mathcal{D}_2)$ holds for any circles X, Y with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$, and $pt(\mathcal{D}_2) = \{p\}$ such that $p \notin pt(\mathcal{D}_1)$.

Operation: Add the point p to $\mathcal{D}_1$ so that the following conditions are satisfied on $\mathcal{D}_1 + \mathcal{D}_2$: (1) $p \Box t$ of $\mathsf{rel}(\mathcal{D}_2)$ holds for all objects t; (2) $p \vdash\!\dashv q$ holds for all $q \in pt(\mathcal{D}_1)$.

$$\mathsf{rel}(\mathcal{D}_1 + \mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{p \vdash\!\dashv q \mid q \in pt(\mathcal{D}_1)\}$$

Example of PI



The notion of **diagrammatic proof**, or **d-proof** for short, is defined inductively as tree structures consisting of diagrams and the above inference rules. (See Fig. 1 in Example 2.15 below.) The **provability** relation $\vdash$ is defined as usual in terms of the existence of a d-proof. See [16] for details.

Remark 2.12 (Existential import)**.** In our system, the interpretation $I(A)$ of a circle A is postulated to be non-empty (Definition 2.6). According to this definition, in our inference rules, we adopt any minimal diagram such that $x \sqsubset A$ holds as an axiom (A3). Our definition corresponds to the existential import in the literature of syllogisms. (With this postulate, for example, *Some A are B* is a valid conclusion of *All A are B*.) Without this postulate, two minimal diagrams, say α in which $A \sqsubset B$ holds and β in which $A \vdash\!\dashv B$ holds, are consistent when A denotes the empty set. However, it is difficult to express α and β in a single diagram, as our system lacks a device to express the emptiness of circles. The introduction of another device such as shading, which expresses the emptiness of the corresponding region (cf. [13]), may partly cope with this problem. However, the question remains as to whether we draw the shaded circle A inside or outside B (whichever is legal in an abstract sense). Thus, in our basic Euler diagrammatic system, we assume the existential import.

2.3 Completeness of GDS

GDS is shown to be sound and complete with respect to our set-theoretical semantics. The completeness theorem of GDS is proved using the following particular case, in which the given conclusion is restricted to a minimal diagram.

Proposition 2.13 (Atomic completeness)**.** *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be a set of* EUL*-diagrams which is consistent. Let β be a minimal diagram. If β is a semantic consequence of $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (i.e., $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \beta$), then β is provable from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (i.e., $\mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \beta$) in* GDS*.*

Note that, to avoid introducing the diagrammatic counterpart of the so-called absurdity rule in our system, we impose, in the formulation of our completeness, a consistency condition on the set of premise diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, i.e., it has a model. See [11] for a discussion.

The above atomic completeness is proved by constructing syntactic models, called *canonical models*, in a similar way as for the construction of Lindenbaum algebras in the literature of algebraic semantics for various propositional logics. See [16, 11] for a detailed proof.

To show the general completeness theorem, we define a canonical method to construct d-proofs using the atomic completeness. The **canonical d-proofs** are defined in a similar way as for normal proofs of natural deduction: we first decompose the given premise diagrams into minimal diagrams, and then, from the minimal diagrams, construct the conclusion diagram in consideration of the constraint of each rule. We give a rather detailed account of the canonical construction below, because it is also applied to construct our counter-diagrammatic proofs in Section 4.

Proposition 2.14 (Completeness)**.** *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be consistent. If $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$, then $\mathcal{D}_1, \ldots, \mathcal{D}_n \vdash \mathcal{E}$ in* GDS*.*

Proof (sketch). We define a canonical method to construct a d-proof of $\mathcal{E}$ from the given premise diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (see also Example 2.15 below):

(I) Minimal part: Using the atomic completeness, we derive all: (1) point-free minimal diagrams whose relation holds on the conclusion $\mathcal{E}$; (2) pointed minimal diagrams consisting of every point contained in a premise; (3) pointed minimal diagrams obtained by combining the point-free minimal diagrams of (1) with axiom A3 of the form $x_A \sqsubset A$ for a fresh point x_A and for a circle A contained in a premise or the conclusion.

(II) Venn part: By applying U1 and U2 for the above pointed minimal diagrams of (I-2) and (I-3), we construct, for every point p of the Minimal part, a "Venn-like diagram", in which $A \bowtie B$ holds for any pair of circles in it, and which consists of p and all circles of the conclusion $\mathcal{E}$. If it is not possible to construct a Venn-like diagram for a point because of the constraint for determinacy, then it is not constructed.

Applying U8 and axiom A1, we also construct a point-free Venn-like diagram consisting of all circles of $\mathcal{E}$.

(III) Modification part: Using U9 and U10, and the point-free minimal diagrams obtained in the Minimal part (I-1), we modify the forms and positions of the circles in each Venn-like diagram so that they correspond to those of the conclusion $\mathcal{E}$.

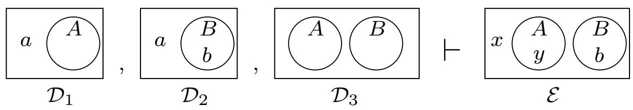
(IV) PI part: By applying PI, we unify all diagrams obtained in the Modification part.

(V) Renaming part: We finally obtain $\mathcal{E}$ by applying Ren, Del (and a derived rule called the copy operation, see Appendix A and [16]).

In the **Venn part**, we are able to ignore points for which we cannot construct a Venn-like diagram because, for any point contained in the conclusion $\mathcal{E}$, the relations between the point and all circles of $\mathcal{E}$ are determined. In the **Modification part**, we are able to apply rules U9 and U10 without obstruction. This is because diagrams $\mathcal{E}_1$ and $\mathcal{E}_2$ are both derived from the premises; if $\mathcal{E}_1$ and $\mathcal{E}_2$ could not be unified because of the constraint for consistency of rules U9, U10, it would imply that the premises are inconsistent. In the **PI part**, we are able to unify all diagrams obtained thus far, because all circles and points of these diagrams are essentially the same as those of the conclusion, except for their names.

The correctness of the above canonical construction is shown in [11, 16]. ■

Example 2.15 (Canonical d-proof of GDS)**.** As an illustration of canonical d-proofs, let us consider the following diagrams $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{E}$:



Note that these premises cannot be unified directly because of the constraint for determinacy. We have a canonical d-proof of $\mathcal{E}$ from $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$ as in Fig. 1, where the derivations of $\mathcal{D}_{13}$ and $\mathcal{D}_{14}$ are omitted because they are similar to that of $\mathcal{D}_{12}$.

We first derive pointed minimal diagrams $\mathcal{D}_1, \mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6$ consisting of a point contained in premise (I-2). Although we omit them in Fig. 1, we also derive minimal diagrams consisting of a fresh point (z for A and w for B) obtained from axiom A3 (I-3). Next, in the Venn part, with A1, U1 and U2, we construct Venn-like diagrams $\mathcal{D}_7$ (without any point), and $\mathcal{D}_8$ and $\mathcal{D}_9$, each of which consists of the above point (a, b, z, w) and all circles A and B of $\mathcal{E}$. Then, in the Modification part, with rule U10, we modify the above Venn-like diagrams so that the positions of the circles fit those of the conclusion $\mathcal{E}$. Then, in the PI part, we unify $\mathcal{D}_{10}, \mathcal{D}_{11}, \mathcal{D}_{12}, \mathcal{D}_{13}, \mathcal{D}_{14}$ to obtain $\mathcal{D}_{18}$. Finally, by renaming points a and z, and by deleting w, we obtain the conclusion $\mathcal{E}$.

3 Counter-diagram construction

Based on the application of Euler diagrams as representations of set-theoretical models, we investigate the construction of counter-models using Euler diagrams. In Section 3.1, we illustrate, through some examples, our idea to construct counter-models using Euler diagrams. In Section 3.2, we define our counter-diagrams and counter-diagrammatic proofs.

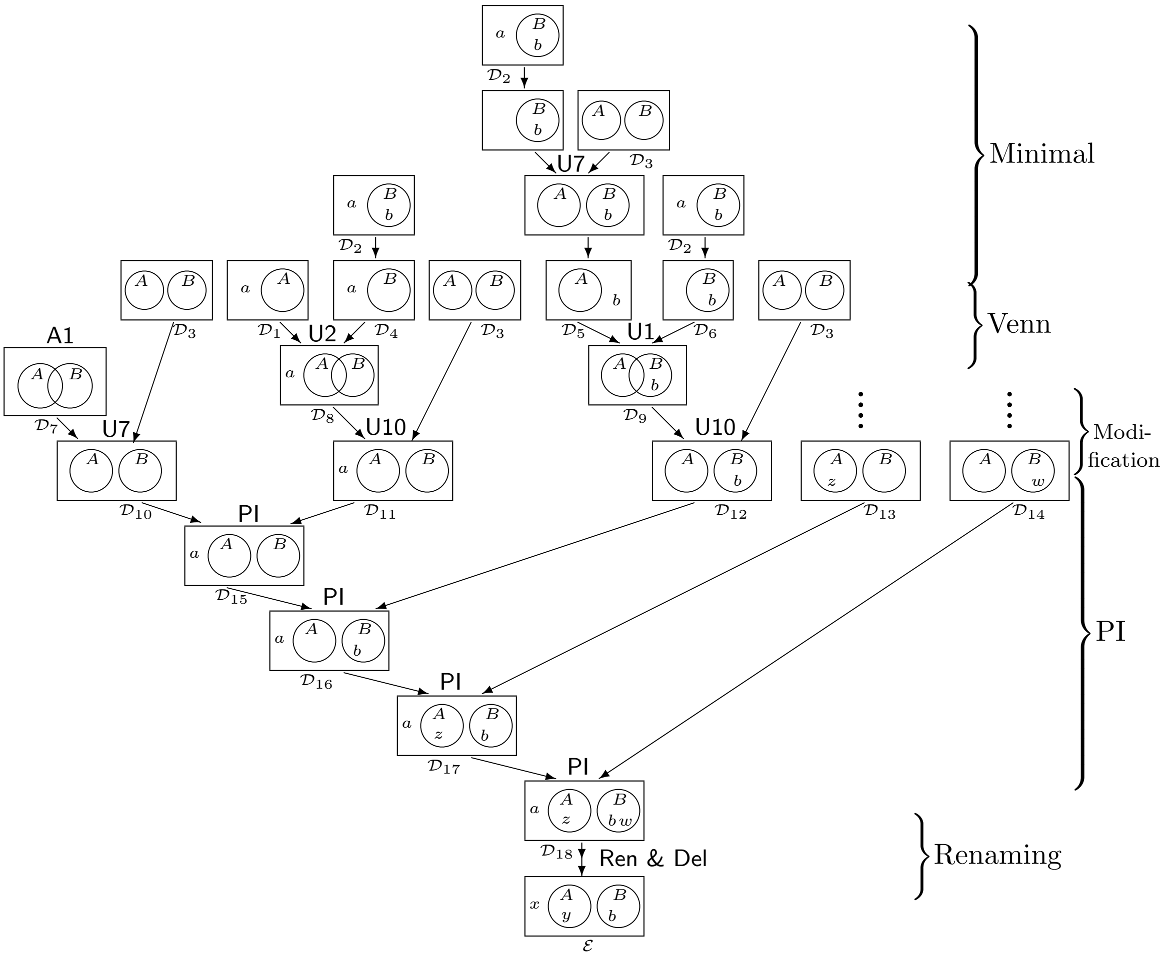
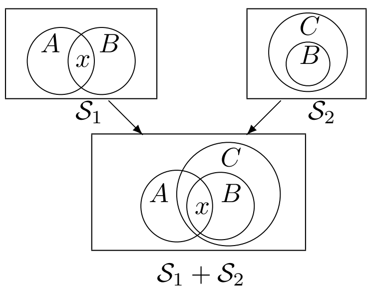


Fig. 1 Canonical d-proof

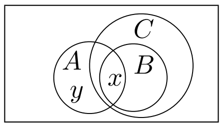
3.1 Introduction to counter-diagram construction

Let us consider how we disprove a given inference using Euler diagrams.

Example 3.1. *Some A are B, All B are C* $\not\models$ *All A are C*
We first represent the two premises by diagrams $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, and then unify these diagrams to obtain $\mathcal{S}_1 + \mathcal{S}_2$ as follows:



From this unified diagram $\mathcal{S}_1 + \mathcal{S}_2$, we can read off the valid conclusion *Some A are C*.
To construct a diagram to falsify the given conclusion *All A are C*, we add a fresh existential point y to $\mathcal{S}_1 + \mathcal{S}_2$ as follows:

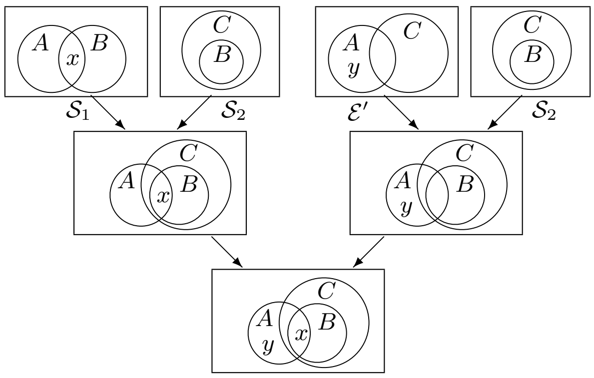


$\mathcal{S}_1 + \mathcal{S}_2 + y$

The existential point y represents that "there exists something that is A but not C" (*Some A are not C*), i.e., the negation of *All A are C*. Note that even after the addition of y, all of the $\sqsubset, \vdash\!\dashv$ relations holding on the premises ($x \sqsubset A$ and $x \sqsubset B$ on $\mathcal{S}_1$, and $B \sqsubset C$ on $\mathcal{S}_2$) are maintained. Hence, the above $\mathcal{S}_1 + \mathcal{S}_2 + y$ neatly illustrates a counter-model of the given inference, in which all premises are true but the conclusion is false.

We can list all of the operations to construct counter-diagrams, such as the addition of a new point as illustrated in the above example. However, let us consider another way to construct counter-diagrams. Theoretically speaking, $\varphi_1, \ldots, \varphi_n \not\models \psi$ means that there exists a model M such that $M \models \varphi_1 \wedge \cdots \wedge \varphi_n \wedge \neg\psi$. Based on this fact, we construct a diagram that falsifies the given conclusion, and, by unifying the diagram with all premises, we construct a counter-diagram for the given inference. Let us consider the same inference as Example 3.1.

Example 3.2. *Some A are B*, *All B are C* $\not\models$ *All A are C*
Let $\mathcal{E}'$ be a diagram such that $\mathsf{rel}(\mathcal{E}') = \{y \sqsubset A, y \vdash\!\dashv C, A \bowtie C\}$, which represents *Some A are not C*, and which falsifies the given conclusion *All A are C*. Then, we unify this $\mathcal{E}'$ with $\mathcal{S}_1$ and $\mathcal{S}_2$ as follows:

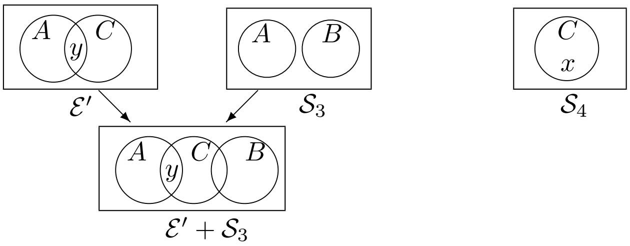


$\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{E}'$

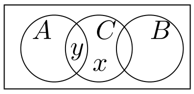
Thus, the resulting diagram $\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{E}'$ is the same as $\mathcal{S}_1 + \mathcal{S}_2 + y$ in Example 3.1, and disproves the given inference.

Although the construction illustrated in Example 3.2 may be slightly unintuitive compared with the previous one in Example 3.1, it simplifies our definition of counter-diagrammatic proofs. Thus, we formalize our construction according to this idea. Let us examine another example.

Example 3.3. *No A are B*, *There is something C* $\not\models$ *No A are C*
Let $\mathcal{E}'$ be a diagram such that $\mathsf{rel}(\mathcal{E}') = \{y \sqsubset A, y \sqsubset C, A \bowtie C\}$, which represents *Some A are C*, and falsifies the given conclusion *No A are C*. We then unify this $\mathcal{E}'$ with the given premise $\mathcal{S}_3$, representing *No A are B*, as follows:



When we further attempt to unify diagram $\mathcal{S}_4$, representing *There is something* C, we find three possibilities for the position of x, that is, x is indeterminate with respect to circles A and B, and we cannot unify it from our rules. However, whichever of the three possibilities gives the position of x, we obtain a counter-diagram to disprove the given inference. For example, the following $\mathcal{E}''$ is a required counter-diagram:



$\mathcal{E}''$

Because we cannot obtain the above $\mathcal{E}''$ from $\mathcal{E}' + \mathcal{S}_3$ and $\mathcal{S}_4$ using our inference rules, we construct counter-diagrams by modifying rule PI and introducing another "invalid" rule called invalid Point Insertion (iPI). This arbitrarily fixes the position of a point from several possibilities. Then, we obtain $\mathcal{E}''$ from $\mathcal{E}' + \mathcal{S}_3$ and $\mathcal{S}_4$.

In this way, a given inference is shown to be invalid by: (1) constructing a diagram that falsifies the given conclusion; and (2) unifying the diagram of (1) with all premises. On the resulting diagram, the relation falsifies the given conclusion and all of the $\sqsubset$ and $\vdash\!\dashv$ relations of the given premises hold. Hence, it is a counter-diagram for the given inference.

Remark 3.4. Note that some $\bowtie$-relations between circles (cf. Remark 2.8) remain unchanged in our counter-diagrams (say, $C \bowtie B$ on $\mathcal{E}''$ in Example 3.3). In view of the usual model theory, it is more appropriate, in the construction of counter-diagrams, to fix every $\bowtie$-relation to either an $\sqsubset$- or $\vdash\!\dashv$-relation. This is possible, and if we include such operations, our counter-diagrams become closer to the usual counter-models. However, such operations make our counter-diagram construction slightly more cumbersome, and without them, our purpose to give counter-examples is accomplished.

3.2 Counter-diagrams and counter-diagrammatic proofs

By formalizing the examples of the previous section, we define our counter-diagram construction. We first define our counter-diagrams.

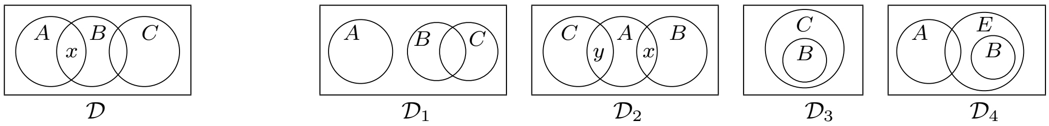
Definition 3.5 (Counter-diagrams)**.** A diagram $\mathcal{E}$ is a **counter-diagram** of $\mathcal{D}$, and vice versa, when one of the following holds between $\mathcal{D}$ and $\mathcal{E}$:

- $a \sqsubset B$ holds on $\mathcal{D}$, and $a \vdash\!\dashv B$ holds on $\mathcal{E}$;
- $A \sqsubset B$ holds on $\mathcal{D}$, and, for some x, $x \sqsubset A$ and $x \vdash\!\dashv B$ hold on $\mathcal{E}$;
- $A \vdash\!\dashv B$ holds on $\mathcal{D}$, and, for some x, $x \sqsubset A$ and $x \sqsubset B$ hold on $\mathcal{E}$.

Note that our diagram is abstractly a conjunction of relations, and the above definition lists the so-called "counter-relation" for every relation.

Not all diagrams have a counter-diagram. Diagrams of axioms A1, A2, and A3 do not have any counter-diagrams. Furthermore, as illustrated in the following Example 3.6, there may be a number of counter-diagrams for a given diagram $\mathcal{D}$, because a counter-diagram of $\mathcal{D}$ may contain circles and points that are irrelevant to $\mathcal{D}$.

Example 3.6. Let $\mathcal{D}$ be the following diagram on the left:



Then, all of $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3$, and $\mathcal{D}_4$ above are counter-diagrams of $\mathcal{D}$. Thus, a diagram and its counter-diagram do not necessarily share the same circles and points, and it is sufficient that a relation that holds on a counter-diagram falsifies a relation of the given diagram.

Remark 3.7 (Negation)**.** The notion of a counter-diagram of $\mathcal{D}$ and of the negation of $\mathcal{D}$ are, in general, different. Whereas there are a number of counter-diagrams of $\mathcal{D}$, the negation of $\mathcal{D}$ should be uniquely determined. To represent the negation of $\mathcal{D}$ in Example 3.6, we need a certain device to represent disjunction, such as the linking between points seen in [13, 9]. This is because, as described in the translation of diagrams (Definition 2.5), our diagrams are the conjunction of relations, and $\mathcal{D}$ is translated as $\forall y(A(y) \to \neg C(y)) \land \exists x(A(x) \land B(x) \land \neg C(x))$. Thus, the negation of $\mathcal{D}$ should be a representation of the disjunctive formula $\exists y(A(y) \land C(y)) \lor \neg\exists x(A(x) \land B(x) \land \neg C(x))$, which means "some A are C, *or* the region inside A, B and outside C is missing, i.e., empty." See also Remark 3.10 below.

Lemma 3.8 (Inconsistency)**.** *If $\mathcal{D}$ is a diagram, and $\mathcal{E}$ is one of its counter-diagram, then $\mathcal{D}$ and $\mathcal{E}$ are inconsistent, that is, there is no model M such that $M \models \mathcal{D}$ and $M \models \mathcal{E}$.*

Proof. Assume $M \models \mathcal{D}$. We show $M \not\models \mathcal{E}$. Based on the definition of counter-diagrams, we divide the cases according to relations holding on $\mathcal{D}$ and $\mathcal{E}$.

When $A \sqsubset B$ holds on $\mathcal{D}$, and $\exists x(x \sqsubset A$ and $x \vdash\!\dashv B)$ holds on $\mathcal{E}$, we have $I(A) \subseteq I(B)$ in M because $M \models \mathcal{D}$. Hence, there is no m such that $m \in I(A)$ and $m \notin I(B)$, which implies $M \not\models \mathcal{E}$. The other cases are similar. ■

Next, we introduce an *invalid* rule called iPI (invalid Point Insertion), which arbitrarily fixes, from among several possibilities, the position of a point. In view of the symbolic logic, this rule corresponds to an invalid rule to conclude a disjunct from a disjunction of formulas expressing the possibilities. (See also Remark 3.10 below.) This rule has a constraint for *indeterminacy*, i.e., this rule can be applied when the position of a point is indeterminate. We give a slightly informal description of the constraint, rather than directly in terms of EUL-relations, as such a formal description is complicated. For the description, we tentatively regard each point in a diagram as a special circle that does not contain, nor cross, any other objects in a usual (determined) diagram. Then, we describe our constraint for indeterminacy using the (valid) unification of GDS.

Definition 3.9 (iPI)**.** The rule of invalid Point Insertion (iPI) is defined as follows.

Premises: $X \Box Y \in \mathsf{rel}(\mathcal{D}_2) \Rightarrow X \Box Y \in \mathsf{rel}(\mathcal{D}_1)$ holds for any circles X, Y with $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$, and $pt(\mathcal{D}_2) = \{p\}$ such that $p \notin pt(\mathcal{D}_1)$.

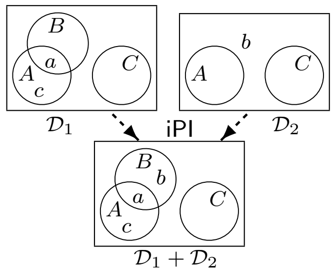
Constraint for indeterminacy: Let $\mathcal{D}'_2$ be the diagram obtained by regarding point p of $\mathcal{D}_2$ as a circle. Then, on the unified diagram $\mathcal{D}_1 + \mathcal{D}'_2$, there exists some circle X such that $p \bowtie X \in \mathsf{rel}(\mathcal{D}_1 + \mathcal{D}'_2)$.

Operation: Add point p to a region in $\mathcal{D}_1$ so that all relations $p\square Y$ of $\mathcal{D}_2$ hold.

$$\mathsf{rel}(\mathcal{D}_1 + \mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{p\square s \mid s \in ob(\mathcal{D}_1)\},$$

where $\square$ is either $\sqsubset$ or $\vdash\!\dashv$ and uniquely determined by the location of p in $\mathcal{D}_1 + \mathcal{D}_2$.

Example of iPI



Remark 3.10 (Disjunction)**.** iPI is an invalid rule, because we fix the position of a given point arbitrarily from among several possibilities. However, instead of fixing such a position arbitrarily, we are able to place the given point in all the possible positions and link them by introducing linking between points. This expresses the disjunctive information on the possible locations of the point. Then, such a rule with disjunction becomes valid. This idea of linking between points was introduced by Peirce [12], and has been adopted in recent Euler diagrammatic systems (e.g., [13, 9]). However, as is frequently pointed out (e.g., [8, 1]), such a device to represent disjunctive information violates the clarity and simplicity of diagrams. Thus, in our basic framework, we keep our diagrams free from such disjunctive ambiguity.

Let us define our counter-diagrammatic proofs.

Definition 3.11. A **counter-diagrammatic proof**, or **counter-d-proof** for short, of $\mathcal{E}$ under $\mathcal{D}_1, \dots, \mathcal{D}_n$ is a tree structure:

- consisting of GDS-rules (U1–U10, PI, Ren, Del) and iPI;
- whose premises are $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}'$, where $\mathcal{E}'$ is a counter-diagram of $\mathcal{E}$ such that no existential point of $\mathcal{E}'$ appears on $\mathcal{D}_1, \dots, \mathcal{D}_n$;
- in whose conclusion, all $\sqsubset, \vdash\!\dashv$-relations of $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}'$ hold.

We write $\mathcal{D}_1, \dots, \mathcal{D}_n \dashrightarrow \mathcal{E}$ when there exists a counter-d-proof of $\mathcal{E}$ under $\mathcal{D}_1, \dots, \mathcal{D}_n$.

Note that we may not use the iPI rule as illustrated in Example 3.2. Also, we can freely choose a counter-diagram $\mathcal{E}'$ that is added to the premises of a counter-d-proof. Hence, it may be the case that $\mathcal{E}'$ itself is a required conclusion of a counter-d-proof. In our proof of Theorem 4.6 below, we give a canonical method to choose $\mathcal{E}'$ and construct counter-d-proofs.

4 Completeness theorem

We now show the appropriateness (in Section 4.1) and sufficiency (in Section 4.2) of our counter-d-proofs, which, respectively, can be considered as counterparts of the usual soundness and completeness theorems. Then, we establish another formulation of completeness.

4.1 Appropriateness

We first show that our iPI rule is appropriately defined.

Lemma 4.1. *If* EUL*-diagrams* $\mathcal{D}_1, \mathcal{D}_2$ *have a model, then the diagram* $\mathcal{E}$ *obtained by applying* iPI *also has a model in which all* $\sqsubset, \vDash$*-relations of* $\mathcal{D}_1, \mathcal{D}_2$ *hold.*

Proof. Note that the reverse of iPI, i.e., a transformation of $\mathcal{E}$ into $\mathcal{D}_1$ or $\mathcal{D}_2$, is just the deletion of some objects, and hence, it is a sound transformation of a diagram. Thus, we have, for any model M', $M' \models \mathcal{E} \Rightarrow M' \models \mathcal{D}_1, \mathcal{D}_2$. Hence, to show the lemma, it is sufficient to construct a model M' of $\mathcal{E}$ from a given model M of $\mathcal{D}_1, \mathcal{D}_2$.

As explained in Remark 3.10, the possible positions of a given point express disjunctive information, and we can express the unified diagram of $\mathcal{D}_1$ and $\mathcal{D}_2$ using such disjunction. Hence, the given model M of $\mathcal{D}_1, \mathcal{D}_2$ is actually a model of one such disjunct, and by slightly modifying the interpretation of the given point, we can easily construct a model for another disjunct. This is formally defined as follows.

Let $M = (U, I)$ be a given model of $\mathcal{D}_1, \mathcal{D}_2$, where $p \in pt(\mathcal{D}_2)$ is the point to be inserted by iPI. We construct a model $M' = (U', I')$ as follows:

- $U' = U \cup \{m\} \quad (m \notin U)$
- $I'(a) = \begin{cases} m & \text{if } a = p \\ I(a) & \text{if } a \neq p \end{cases}$
- $I'(A) = \begin{cases} I(A) \cup \{m\} & \text{if } p \sqsubset A \text{ holds on } \mathcal{E} \\ I(A) & \text{if } p \vDash A \text{ holds on } \mathcal{E} \end{cases}$

M' is a model of $\mathcal{E}$ by the above definition. ■

By the above lemma, as well as the soundness of GDS, the conclusion of a counter-d-proof consisting of GDS-rules and iPI has a model. Furthermore, by the definition of counter-d-proofs, when a counter-diagram $\mathcal{E}$ has a model, we find that a given inference is invalid. Thus, we have the following theorem. The theorem is proved by induction on the length of a given counter-d-proof and in this sense, can be considered as the counterpart of the usual soundness theorem.

Theorem 4.2 (Invalidity)**.** *Let* $\mathcal{D}_1, \ldots, \mathcal{D}_n$ *be consistent. If there exists a counter-d-proof of* $\mathcal{E}$ *under* $\mathcal{D}_1, \ldots, \mathcal{D}_n$ *(i.e.,* $\mathcal{D}_1, \ldots, \mathcal{D}_n \dashrightarrow \mathcal{E}$*), then* $\mathcal{E}$ *is not a valid conclusion of* $\mathcal{D}_1, \ldots, \mathcal{D}_n$ *(i.e.,* $\mathcal{D}_1, \ldots, \mathcal{D}_n \not\models \mathcal{E}$*).*

Proof. We show that there exists a model M such that $M \models \vec{\mathcal{D}_n}$ and $M \not\models \mathcal{E}$ by induction on the length of a given counter-d-proof whose premises are $\vec{\mathcal{D}_n}, \mathcal{E}'$, and whose conclusion is $\mathcal{E}''$. $\vec{\mathcal{D}_n}, \mathcal{E}'$ is consistent, i.e., it has a model. This is because, if $\vec{\mathcal{D}_n}, \mathcal{E}'$ is inconsistent, we cannot obtain the conclusion $\mathcal{E}''$ of the given counter-d-proof, where all $\sqsubset, \vDash$-relations of $\vec{\mathcal{D}_n}, \mathcal{E}'$ should hold by definition. For the induction step, we divide the following cases according to how the diagram $\mathcal{E}''$ is obtained from $\vec{\mathcal{D}_n}, \mathcal{E}'$.

- When $\mathcal{E}''$ is obtained by applying a GDS-rule (U1–U10, PI, Ren, Del), by the soundness of GDS, any model of $\vec{\mathcal{D}_n}, \mathcal{E}'$ is also a model of $\mathcal{E}''$.
- When $\mathcal{E}''$ is obtained by applying iPI, by Lemma 4.1, there exists a model of both $\vec{\mathcal{D}_n}, \mathcal{E}'$ and $\mathcal{E}''$.

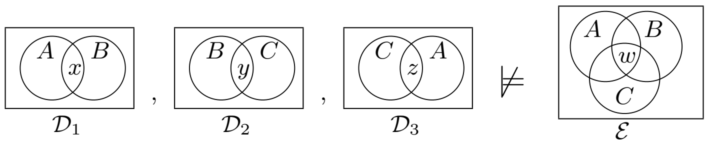
Therefore, there exists a model M such that $M \models \vec{\mathcal{D}_n}$ and $M \models \mathcal{E}''$. Because $\mathcal{E}''$ is a counter-diagram of $\mathcal{E}$ by the definition of counter-d-proofs, we have $M \not\models \mathcal{E}$ by Lemma 3.8. ■

4.2 Sufficiency

We investigate the sufficiency of our counter-d-proofs to characterize the invalidity of our inferences, and the resulting theorem can be considered as the counterpart of the usual completeness theorem.

Unfortunately, there exists an invalid inference, to which we cannot construct a counter-d-proof in our framework.

Example 4.3. Let us consider the following invalid inference $\mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3 \not\models \mathcal{E}$:



Whereas in any counter-diagram of $\mathcal{E}$, one of the relations $A \vdash\!\dashv B, A \vdash\!\dashv C$, and $B \vdash\!\dashv C$ should hold, any of these relations contradict one of the given premises. Hence, we cannot unify them. The diagram $\mathcal{E}'$ shown in the following Fig. 2, which is obtained by regarding the intersection of A, B, and C of $\mathcal{E}$ as "missing", may be considered as giving a counter-diagram of $\mathcal{E}$, because $\exists w(A(w) \wedge B(w) \wedge C(w))$ holds on $\mathcal{E}$ and $\neg\exists w(A(w) \wedge B(w) \wedge C(w))$ holds on $\mathcal{E}'$. However, in our basic framework, this $\mathcal{E}'$ is equivalent to $\mathcal{E}''$ on the right of Fig. 2, which cannot be considered as a counter-diagram of $\mathcal{E}$, since the intersection region appears, and the region may not be empty. We do not have any device (such as shading) to express the emptiness of the corresponding regions.

Fig. 2

This is because only relations between circles and points are considered, and some regions have no significance in our basic framework (cf. Remark 2.4), and hence, for some diagrams, we cannot construct an appropriate counter-diagram. If we take certain regions into account, such as the intersection of A, B, and C, so that $\mathcal{E}'$ and $\mathcal{E}''$ are distinguished, $\mathcal{E}'$ provides a counter-diagram to disprove the given inference.

Thus, to avoid the above example in our basic framework, we abandon the consideration of counter-diagrams for diagrams such as $\mathcal{E}$ in Example 4.3, where an existential point is located in a particular region.

Every point is located in a **minimal region**, which is part of the plane that lies inside some, at least one, of the circles and outside the remaining circles of the diagram. Thus, the position of a point is determined in general by the relations ($\sqsubset$ and $\vdash\!\dashv$) with all circles of the given diagram. However, some points are determined without specifying all of the relations with circles. For example, let us consider $\mathcal{S}_1 + \mathcal{S}_2 + \mathcal{E}'$ of Example 3.2, whose relations are $x \sqsubset A, x \sqsubset B, x \sqsubset C, y \sqsubset A, y \vdash\!\dashv B, y \vdash\!\dashv C, A \bowtie B, A \bowtie C, B \sqsubset C$. The position of x is uniquely determined by the relations with two circles A and B, i.e., $x \sqsubset A$ and $x \sqsubset B$. The remaining relation $x \sqsubset C$ is implied from the relation between circles $B \sqsubset C$ as well as $x \sqsubset B$. In the same way, the position of y is also determined by $y \sqsubset A$ and $y \vdash\!\dashv C$. We call a diagram "relational" if the position of every existential point in the diagram is determined by relations with at most two circles.

Definition 4.4 (Relational diagrams)**.** A diagram $\mathcal{D}$ is **relational** if, for every existential point x of $\mathcal{D}$, x is located inside at least one circle, and the relations of x with all circles of $\mathcal{D}$ is determined by relations with at most two circles.

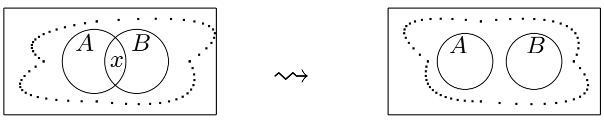
The diagram $\mathcal{E}' + \mathcal{S}_3$ of Example 3.3 is relational, since the position of y is determined by $y \sqsubset A$ and $y \sqsubset C$. Note that in such a case as $\mathcal{S}_4$ of Example 3.3, the position of x is determined by the relation with the single circle C. On the other hand, $\mathcal{E}''$ of Example 3.3 is not relational. This is because the position of x cannot be determined by a pair of circles from A, B, and C. For example, from $x \vdash\!\dashv A$ and $x \vdash\!\dashv B$, whether $x \sqsubset C$ or $x \vdash\!\dashv C$ is not determined. The same applies to $\mathcal{E}$ of Example 4.3, and it is not relational.

Relational diagrams are characterized by the fact that, if we regard the minimal region with an existential point as missing, a certain relation between circles is changed. For example, for a relational diagram such as $\{x \sqsubset A, x \sqsubset B, A \bowtie B\}$, when we regard the intersection of A and B to be missing, we obtain a counter-diagram $\{A \vdash\!\dashv B\}$, where the original relation $A \bowtie B$ is changed to $A \vdash\!\dashv B$. In contrast, for non-relational $\mathcal{E}$ in Example 4.3, where $A \bowtie B, A \bowtie C, B \bowtie C$ hold, even if we regard the intersection of A, B, and C with point w to be missing in $\mathcal{E}'$, the relations between circles remain unchanged.

Lemma 4.5 (Relational diagrams)**.** *Let $\mathcal{D}$ be a relational diagram, where the position of an existential point x is determined by the relations with circles A and B. If we erase the minimal region of $\mathcal{D}$, where x is located, then the relation between A and B is changed or otherwise, a new relation arises.*

Proof. Let $x \Box_1 A, x \Box_2 B, A \Box_3 B$ be relations between x, A, and B. The lemma is shown by examining all possible combinations of relations $\Box_1, \Box_2$, and $\Box_3$ from $\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie$. Since x is inside at least one circle by the definition of relational diagrams, we find that either $\Box_1$ or $\Box_2$ should be $\sqsubset$. Furthermore, it is not the case that all of $\Box_1, \Box_2, \Box_3$ are $\sqsubset$, since, in that case, the position of x is determined by only A. We show the following two cases, since the other cases are similar.

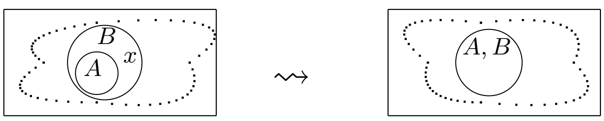
1. When $x \sqsubset A, x \sqsubset B, A \bowtie B$ hold, by erasing the minimal region of x, the diagram $\mathcal{D}$ is changed to $\mathcal{D}'$ as follows:



$\mathcal{D}$ $\mathcal{D}'$

In this case, $A \bowtie B$ is changed to $A \vdash\!\dashv B$.

2. When $x \vdash\!\dashv A, x \sqsubset B, A \sqsubset B$ hold, by erasing the minimal region of x, the diagram $\mathcal{D}$ is changed to $\mathcal{D}'$ as follows:



$\mathcal{D}$ $\mathcal{D}'$

In this case, a new relation $B \sqsubset A$ arises. ■

We consider counter-diagrams only for relational diagrams. Note that we also consider the equivalence classes of diagrams when we consider relational diagrams.

There is no need to restrict diagrams to be relational when we consider the usual syllogisms or transitive inference, where a conclusion diagram consists of only two circles. We need the restriction, as seen in the following proof of Theorem 4.6, when we consider a general diagram as a conclusion, in which only a particular existential point makes the given inference invalid.

Theorem 4.6 (Sufficiency)**.** *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be consistent, and $\mathcal{E}$ be relational. If $\mathcal{E}$ is not a valid conclusion of $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (i.e., $\mathcal{D}_1, \ldots, \mathcal{D}_n \not\models \mathcal{E}$), then there exists a counter-d-proof of $\mathcal{E}$ under $\mathcal{D}_1, \ldots, \mathcal{D}_n$ (i.e., $\mathcal{D}_1, \ldots, \mathcal{D}_n \dashrightarrow \mathcal{E}$).*

Proof. Let $\vec{\mathcal{D}_n} \not\models \mathcal{E}$, and let $\mathsf{rel}(\mathcal{E}) = \{R_1, \ldots, R_j, x_1 \Box A_1, \ldots, x_1 \Box A_k, \ldots, x_l \Box A_1, \ldots, x_l \Box A_k\}$, where $\Box$ is either $\sqsubset$ or $\vdash\!\dashv$, and no existential point appears in $R_1, \ldots, R_j$. We abbreviate the sequence $x_i \Box A_1, \ldots, x_i \Box A_k$ of point x_i as $\exists_i$. Then, there exists a model M such that $M \models \vec{\mathcal{D}_n}$ and $M \not\models \mathcal{E}$. Thus, there exists some i such that $M \not\models R_i$ or $M \not\models \exists_i$.

We first define a minimal counter-diagram $\mathcal{E}'$ from $\mathcal{E}$. We divide the following cases according to the form of false R_i or $\exists_i$ in M.

1. When R_i is $A \sqsubset B$, we take $\mathcal{E}'$ so that $\mathsf{rel}(\mathcal{E}')$ $=\{x \sqsubset A, x \vdash\!\dashv B, A \bowtie B\}$, where $x \notin pt(\vec{\mathcal{D}_n})$.
2. When R_i is $A \vdash\!\dashv B$, we take $\mathcal{E}'$ so that $\mathsf{rel}(\mathcal{E}')$ $=\{x \sqsubset A, x \sqsubset B, A \bowtie B\}$, where $x \notin pt(\vec{\mathcal{D}_n})$.
3. When R_i is $a \sqsubset B$, we take $\mathcal{E}'$ so that $\mathsf{rel}(\mathcal{E}') = \{a \vdash\!\dashv B\}$.
4. When R_i is $a \vdash\!\dashv B$, we take $\mathcal{E}'$ so that $\mathsf{rel}(\mathcal{E}') = \{a \sqsubset B\}$.
5. When $\exists_i$ is $\exists x_i(x_i \sqsubset A_1$ and $\cdots$ and $x_i \sqsubset A_h$ and $x_i \vdash\!\dashv B_1$ and $\cdots$ and $x_i \vdash\!\dashv B_m)$, we erase the minimal region in $\mathcal{E}$ that is inside $A_1, \ldots, A_h$ and outside $B_1, \ldots, B_m$, i.e., the minimal region where x_i is located. As $\mathcal{E}$ is relational, a relation between circles is changed or a new relation arises by Lemma 4.5, which is denoted by $X \Box Y$. We construct $\mathcal{E}'$ so that $\mathsf{rel}(\mathcal{E}') = \{X \Box Y\}$.

We claim that $\mathcal{E}'$ is a counter-diagram of $\mathcal{E}$, and $\vec{\mathcal{D}_n}, \mathcal{E}'$ is consistent. We prove case (5), as the others are immediate. Because $M \not\models \exists_i$, we have $M \models \neg\exists_i$. As $\mathcal{E}'$ consisting of $X \Box Y$ is obtained by erasing the corresponding region of $\exists_i$ from $\mathcal{E}$, $X \Box Y$ is implied by $\neg\exists_i$. Hence, $\mathcal{E}'$ is a counter-diagram of $\mathcal{E}$, and we have $M \models \mathcal{E}'$, which shows $\vec{\mathcal{D}_n}, \mathcal{E}'$ is consistent.

We next define a canonical way to construct a counter-d-proof from $\vec{\mathcal{D}_n}, \mathcal{E}'$ to $\mathcal{E}''$. All of the above cases are treated in the same way. The following construction is essentially the same as that defined in the proof of the completeness of GDS (Proposition 2.14). The only essential difference is that we apply the invalid rule iPI, as well as PI, in the canonical construction of counter-d-proofs. We describe the following construction by mentioning minor differences in each part. See also the example of a canonical counter-d-proof given in Fig. 3 of Example 4.7 below.

(I) Minimal part: Using the atomic completeness theorem (Proposition 2.13), we derive all: (1) point-free minimal diagrams; (2) pointed minimal diagrams consisting of every point that appears on $\vec{\mathcal{D}_n}, \mathcal{E}'$.

For the counter-diagram construction, we do not need axiom A3, as the necessary points for the required counter-diagram already appear on $\vec{\mathcal{D}_n}, \mathcal{E}'$.

(II) Venn part: By applying U1 and U2 for the above pointed minimal diagrams of (I-2), we construct, for every point p of the Minimal part, a Venn-like diagram consisting of p and all circles of $\vec{\mathcal{D}_n}, \mathcal{E}'$.

Contrary to the construction of the completeness proof, if it is not possible to construct a Venn-like diagram consisting of all required circles for a point because of the constraint for determinacy, we construct a partial Venn-like diagram consisting of as many circles (of $\vec{\mathcal{D}_n}, \mathcal{E}'$) as possible.

We also construct, by applying U8 and axiom A1, a point-free Venn-like diagram consisting of all circles of $\vec{\mathcal{D}_n}, \mathcal{E}'$.

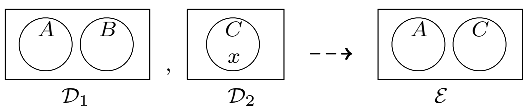
(III) Modification part: Using U9 and U10, and the point-free minimal diagrams obtained in the Minimal part (I-1), we modify the forms and positions of the circles in each Venn-like diagram.

(IV) PI/iPI part: By applying PI and iPI, we unify all diagrams obtained in the Modification part.

Observe that the resulting diagram $\mathcal{E}''$ is a counter-diagram of the given $\mathcal{E}$ in that all $\sqsubset, \vdash\!\dashv$-relations of premises $\mathcal{D}_1, \dots, \mathcal{D}_n$ hold, since a counter-diagram $\mathcal{E}'$ and all premises are unified to obtain $\mathcal{E}''$. Hence, we check that the above construction is well defined, that is, the constraint is satisfied for every rule applied in each part. The **Minimal part** is the same as the canonical construction for the completeness of GDS, and there is no problem. In the **Venn part**, only U1 and U2 are applied, and these rules have constraint for determinacy. As we construct the Venn-like diagrams to have as many circles as possible, there is no conflict with the constraint. (Note that we are not able to construct a Venn-like diagram if a given inference is valid. This is because $\vec{\mathcal{D}_n}, \mathcal{E}'$ is inconsistent by Lemma 3.8 in such a case.) In the **Modification part**, only U9 and U10 are applied, and these rules have the constraint for consistency. If diagrams $\mathcal{E}_1$ and $\mathcal{E}_2$ could not be unified because of this constraint, it would mean $\vec{\mathcal{D}_n}, \mathcal{E}'$ are inconsistent, which, as $\mathcal{E}_1$ and $\mathcal{E}_2$ are both derived form $\vec{\mathcal{D}_n}, \mathcal{E}'$, is a contradiction. Hence, we are able to apply U9 and U10 safely. In the **PI/iPI part**, rules PI and iPI are applied. If we ignore the point of these rules, one diagram is a sub-diagram of the other, and so we do not have any problem with this part.

Therefore, we are able to construct a counter-d-proof of $\mathcal{E}$ under $\mathcal{D}_1, \dots, \mathcal{D}_n$. ■

Example 4.7. As an illustration of the canonical construction of counter-d-proofs, let us consider the following diagrams $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{E}$ (cf. Example 3.3):



We have a canonical counter-d-proof of $\mathcal{E}$ under $\mathcal{D}_1, \mathcal{D}_2$ as in Fig. 3.

Let $\mathcal{E}'$ be the diagram such that $\{y \sqsubset A, y \sqsubset C, A \bowtie C\}$. Using atomic completeness, we first derive pointed minimal diagrams $\mathcal{D}_3, \mathcal{D}_4, \mathcal{D}_5$. Next, in the Venn part, with U8 and U1, we construct Venn-like diagrams $\mathcal{D}_6, \mathcal{D}_7$, and $\mathcal{D}_2$. (In fact, $\mathcal{D}_6$ is obtained by applying U8 to diagrams of the forms $A \bowtie B$ and $B \bowtie C$.) Although we are able to construct Venn-like diagrams $\mathcal{D}_6$ and $\mathcal{D}_7$ with all circles A, B, C of the given premises $\mathcal{D}_1, \mathcal{D}_2, \mathcal{E}'$, we cannot construct one for the point x, and hence we keep $\mathcal{D}_2$ unchanged as a partial Venn-like diagram consisting of the single circle C. Then, in the Modification part, U10 is used to modify the

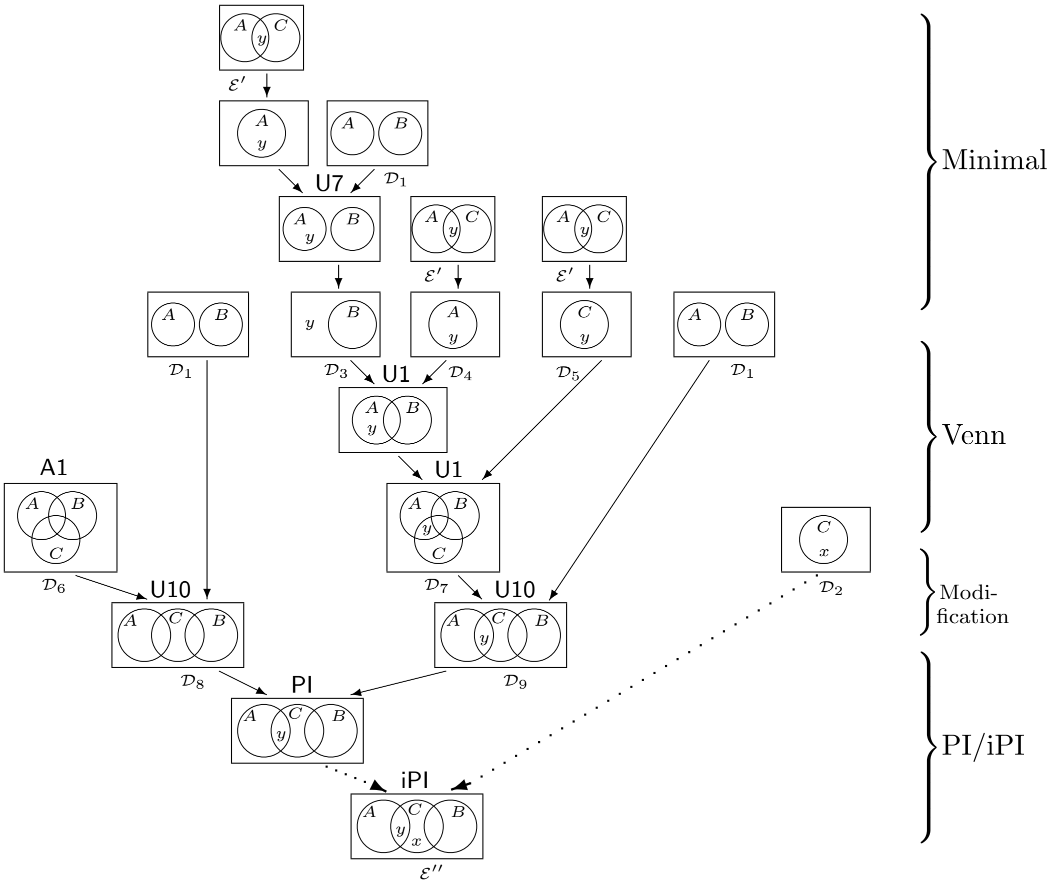


Fig. 3 Canonical counter-d-proof

above Venn-like diagrams to obtain $\mathcal{D}_8$ and $\mathcal{D}_9$. Here also, the partial Venn-like diagram $\mathcal{D}_2$ is left unchanged. Then, we unify the diagrams in the PI/iPI part. Because point x is indeterminate with respect to circles A and B, we arbitrarily fix the position of x so that $x \sqsubset C, x \vdash\!\dashv A, x \vdash\!\dashv B$ hold. Thus, the resulting $\mathcal{E}''$ is a counter-diagram of $\mathcal{E}$, because $y \sqsubset A, y \sqsubset C$ hold, and all relations of $\mathcal{D}_1$ and $\mathcal{D}_2$, i.e., $A \vdash\!\dashv B, x \sqsubset C$, hold. Therefore, $\mathcal{E}''$ is the required counter-diagram that shows $\mathcal{D}_1, \mathcal{D}_2 \dashrightarrow \mathcal{E}$.

By Theorems 4.2 and 4.6, we have: $\mathcal{D}_1, \ldots, \mathcal{D}_n \dashrightarrow \mathcal{E}$ *if and only if* $\mathcal{D}_1, \ldots, \mathcal{D}_n \not\models \mathcal{E}$. Then, combining this with the completeness of GDS (Proposition 2.14), we have the following theorem.

Theorem 4.8 (Completeness)**.** Let $\mathcal{D}_1, \ldots, \mathcal{D}_n$ be consistent and $\mathcal{E}$ be relational. There exists a d-proof of $\mathcal{E}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_n$, or a counter-d-proof of $\mathcal{E}$ under $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

5 Conclusion

We have investigated the construction of counter-models with Euler diagrams. A counter-diagram for a given diagram is defined so that they are inconsistent. Our counter-diagrammatic proofs are defined by a similar syntactic manipulation of Euler diagrams as syntactic inference rules to construct diagrammatic proofs. Our counter-diagrammatic proof, to show that $\mathcal{E}$ is not a valid conclusion of $\mathcal{D}_1, \dots, \mathcal{D}_n$, is a tree structure whose premises are $\mathcal{D}_1, \dots, \mathcal{D}_n$ and a counter-diagram $\mathcal{E}'$ of $\mathcal{E}$, and all meaningful relations of $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}'$ hold in the conclusion. We also defined a canonical construction of counter-diagrammatic proofs, which have essentially the same structures as the canonical diagrammatic proof.

We proved that our counter-diagrammatic proofs are appropriately and sufficiently defined for relational diagrams through Theorem 4.2 and Theorem 4.6, which can be considered as counterparts of the usual soundness and completeness theorems, respectively. These theorems, combined with the usual completeness theorem of our system with respect to set-theoretical semantics, enabled us to formulate another completeness theorem (Theorem 4.8), that asserts the existence of either a diagrammatic proof or a counter-diagrammatic proof for relational diagrams.

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A Inference rules of GDS

Unification rules are divided into three groups, Group (I), (II), and (III). The rules in Group (I) and (II) are classified according to the number and type of objects shared by a diagram $\mathcal{D}$ and a minimal diagram α. In Group (I), $\mathcal{D}$ and α share one object. The rules in this group are further divided into two types: those in which one point is shared (U1–U2) and those in which one circle is shared (U3–U8). Each rule is specified by the relation holding on α, and has a constraint for determinacy. In Group (II), $\mathcal{D}$ and α share two circles (hence α consists of two circles). We distinguish two rules in this group (U9 and U10), depending on whether $A \sqsubset B$ or $A \vdash\!\dashv B$ holds on α. The rule in Group (III) is PI (Point Insertion), whose premise diagrams are not restricted to be minimal.

(I) The case $\mathcal{D}$ and α share one object:

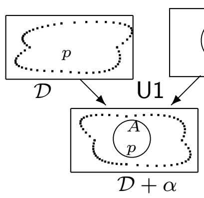
[U1] Premises: $p \sqsubset A$ holds on α, and $p \in pt(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \{p\}$.

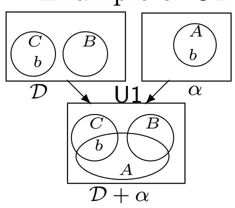
Operation: Add the circle A to $\mathcal{D}$ (with preservation of all relations on $\mathcal{D}$) so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \sqsubset A$ holds; (2) $A \bowtie X$ holds for all circles X of $\mathcal{D}$.

The set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ is specified as $\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$.

Schema of U1



Example of U1



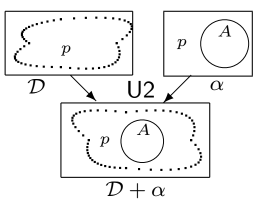
[U2] Premises: $p \vdash\!\dashv A$ holds on α, and $p \in pt(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \{p\}$.

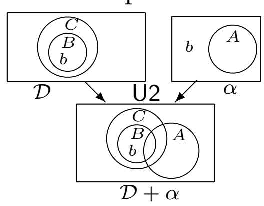
Operation: Add the circle A to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \vdash\!\dashv A$ holds; (2) $A \bowtie X$ holds for all circles X of $\mathcal{D}$.

$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$

Schema of U2



Example of U2

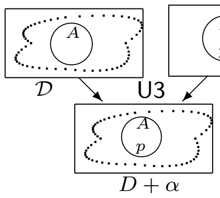


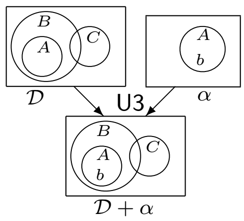
[U3] Premises: $p \sqsubset A$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $A \sqsubset X$ or $A \vdash\!\dashv X$ holds for all circles X of $\mathcal{D}$.

Operation: Add the point p to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \sqsubset A$ holds; (2) $p \vDash\!\dashv q$ holds for all points q such that $q \sqsubset A$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \;\cup\; \mathsf{rel}(\alpha) \cup$$
$$\{p \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \; \{p \vDash\!\dashv X \mid A \vDash\!\dashv X \in \mathsf{rel}(\mathcal{D})\} \cup \; \{p \vDash\!\dashv q \mid q \in pt(\mathcal{D})\}$$



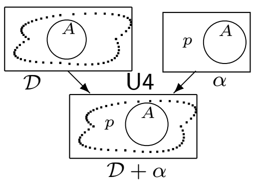


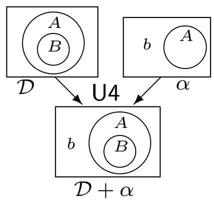
U4 Premises: $p \vDash\!\dashv A$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $X \sqsubset A$ holds for all circles X of $\mathcal{D}$.

Operation: Add the point p to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $p \vDash\!\dashv A$ holds; (2) $p \vDash\!\dashv q$ holds for all points q such that $q \vDash\!\dashv A$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{p \vDash\!\dashv X \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{p \vDash\!\dashv q \mid q \in pt(\mathcal{D})\}$$





U5 Premises: A minimal diagram α on which $A \sqsubset B$ holds; and a diagram $\mathcal{D}$ such that $B \in cr(\mathcal{D})$ (but $A \notin cr(\mathcal{D})$).

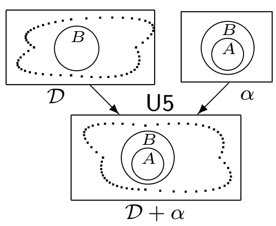
Constraint for determinacy: $p \vDash\!\dashv B$ holds for all $p \in pt(\mathcal{D})$.

Operation: Add the circle A to $\mathcal{D}$ (with preservation of all relations on $\mathcal{D}$) so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $A \sqsubset B$ holds; (2) $A \bowtie X$ holds for all circles X ($\not\equiv B$) such that $X \sqsubset B$ or $X \bowtie B$ holds on $\mathcal{D}$.

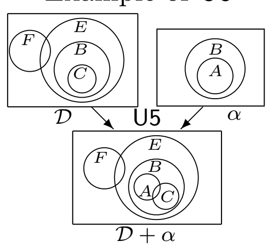
The set of relations $\mathsf{rel}(\mathcal{D}+\alpha)$ of the unified diagram is specified as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \sqsubset B \text{ or } X \bowtie B \in \mathsf{rel}(\mathcal{D}), X \not\equiv B\}$$
$$\cup \{A \sqsubset X \mid B \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vDash\!\dashv A \mid X \vDash\!\dashv B \in \mathsf{rel}(\mathcal{D})\} \cup \{p \vDash\!\dashv A \mid p \in pt(\mathcal{D})\}$$

Schema of U5



Example of U5

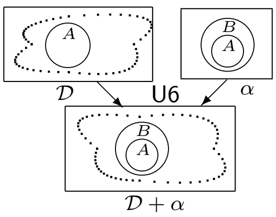


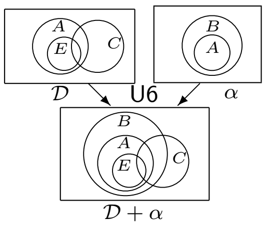
U6 Premises: $A \sqsubset B$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $p \sqsubset A$ holds for all $p \in pt(\mathcal{D})$.

Operation: Add the circle B to $\mathcal{D}$ so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $A \sqsubset B$ holds; (2) $B \bowtie X$ holds for all circles $X (\not\equiv A)$ such that $A \sqsubset X$ or $A \vdash\dashv X$ or $A \bowtie X$ holds on $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \vdash\dashv X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D}), X \not\equiv A\}$$
$$\cup \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{p \sqsubset B \mid p \in pt(\mathcal{D})\}$$



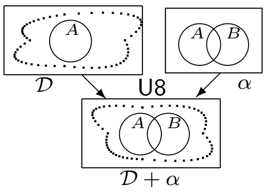


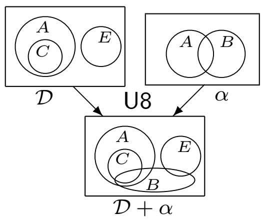
U8 Premises: $A \bowtie B$ holds on α, and $A \in cr(\mathcal{D})$.

Constraint for determinacy: $pt(\mathcal{D}) = \emptyset$.

Operation: Add the circle B to $\mathcal{D}$ so that $B \bowtie X$ holds for all circles X of $\mathcal{D}$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha) \cup \{B \bowtie X \mid X \in cr(\mathcal{D})\}$$





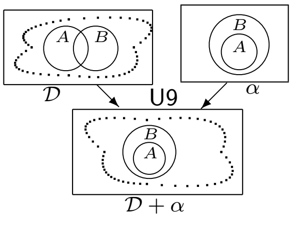
(II) The case $\mathcal{D}$ and α share two circles:

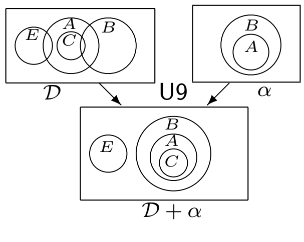
U9 Premises: $A \sqsubset B$ holds on α, and $A \bowtie B$ holds on $\mathcal{D}$.

Constraint for consistency: There is no object s such that $s \sqsubset A$ and $s \vdash\dashv B$ hold on $\mathcal{D}$.

Operation: Modify all circles X (including A) of $\mathcal{D}$ such that $X \sqsubset A$ holds so that the following conditions are satisfied on $\mathcal{D}+\alpha$: (1) $X \sqsubset B$ holds; (2) $X \square t$ holds with $\square \in \{\sqsubset, \sqsupset, \vdash\dashv, \bowtie\}$ for all object t of $\mathcal{D}$ such that $t \sqsubset A, X \square t \in \mathsf{rel}(\mathcal{D})$.

$$\mathsf{rel}(\mathcal{D}+\alpha) = \big(\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } B \sqsubset Y \in \mathsf{rel}(\mathcal{D})\} \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \vdash\dashv B \in \mathsf{rel}(\mathcal{D})\}\big)$$
$$\cup \{X \sqsubset Y \mid X \sqsubset A \text{ and } B \sqsubset Y \in \mathsf{rel}(\mathcal{D})\} \cup \{X \vdash\dashv Y \mid X \sqsubset A \text{ and } Y \vdash\dashv B \in \mathsf{rel}(\mathcal{D})\}$$





Other than unification, GDS has the following rules of Del and Ren.

Del (Deletion) Premise: $\mathcal{D}$ contains an object s.

Constraint: $\mathcal{D}$ is not minimal.

Operation: Delete the object s from $\mathcal{D}$.

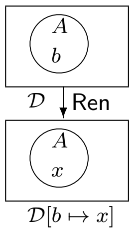
$\mathsf{rel}(\mathcal{D} - s) = \mathsf{rel}(\mathcal{D}) \setminus \{s \Box t \mid t \in ob(\mathcal{D}), \Box \in \{\sqsubset, \sqsupset, \mathrel{\vdash\!\dashv}, \bowtie\}\}$

Ren (Renaming) Premise: A diagram $\mathcal{D}$ containing a point p.

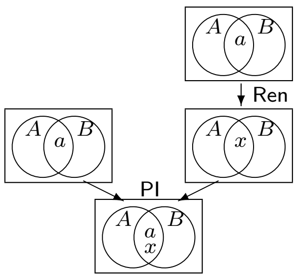
Operation: Replace the point p of $\mathcal{D}$ with an existential point x such that $x \not\equiv p$ and $x \notin pt(\mathcal{D})$.
The set of relations $\mathsf{rel}(\mathcal{D}[p \mapsto x])$ of the resulting diagram is specified as follows:

$$(\mathsf{rel}(\mathcal{D}) \setminus \{p \Box s \mid \Box \in \{\sqsubset, \mathrel{\vdash\!\dashv}\}, s \in ob(\mathcal{D})\}) \ \cup \ \{x \sqsubset s \mid p \sqsubset s \in \mathsf{rel}(\mathcal{D})\} \ \cup \ \{x \mathrel{\vdash\!\dashv} s \mid p \mathrel{\vdash\!\dashv} s \in \mathsf{rel}(\mathcal{D})\}$$

Example of Ren



Remark A.1 (Copy)**.** Combining Ren and PI rules, we are able to duplicate any point in a diagram as illustrated in the following:



Note that, in general, an additional point is restricted to be existential, and it is located in the same region as the original point.