Economic reasoning with demand and supply graphs

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Abstract

We investigate an interaction between mathematical logic and economics by exploring a mathematical model of economic reasoning. To this purpose, we focus on economic reasoning with linear demand and supply graphs. Linear graphs have the following advantages: Linear graphs are easily constructed and reasoning using such graphs can be effectively managed; linear graphs are qualitatively determinate with respect to our problems. We investigate the types of problems regarding general demand and supply functions that can be reduced to linear graphs and discuss abductive reasoning with demand and supply graphs. Then, we propose a model of economic reasoning by formalizing reasoning with linear graphs in the framework of natural deduction developed through proof theory of mathematical logic.

Keywords: mathematical logic, economic reasoning, demand and supply graph

1. Introduction

When one opens economics textbooks, one finds a number of graphs representing or illustrating various mathematical functions such as supply and demand functions (for example, [20, 30, 22, 13]). In fact, it is difficult to explain economics concepts such as law of demand and supply without any graphs or diagrams. Thus, in actual economic reasoning or problem solving using thus obtained economics concepts, it is natural to employ graphs and diagrams. However, models of human reasoning that are not restricted to economic reasoning that have been proposed in various fields have been restricted to models of reasoning with linguistic representations such as sentences and symbols. For example, models of economic reasoning proposed in qualitative reasoning studies are models with linguistic or symbolic representation using differential equations, algebraic calculations, causal relations, and so on ([27, 25, 8, 15]).

Considering actual human reasoning, we usually use not only linguistic or symbolic representations, but also, graphical or diagrammatic representations. Based on this observation, diagrammatic reasoning and heterogeneous reasoning employing combinations of linguistic and diagrammatic representations have

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been recently investigated from various viewpoints, such as mathematical logic, computer science, and cognitive science. See, for example, [21, 1, 28]. For economic reasoning, Tabachneck-Schijf, Leonardo, and Simon [29] proposed a cognitive model of heterogeneous reasoning called CaMeRa (computational model of multiple representations) based on cognitive experiments used to compare linguistic/sentential, mathematical symbolic, and graphical representations.

Among various graphs employed in economics, one of the most basic and frequently appearing graphs is the demand and supply graph, which also forms the basis of the CaMeRa model. Although laws of demand and supply generally hold for any demand and supply functions, it is not possible to draw a graph of *any* function. Thus, to illustrate laws of demand and supply by graphs, there are many different representatives of arbitrary demand and supply graphs (including linear graphs) in economics text books.

In this article, we focus on linear graphs, whose slopes are ± 1 in particular, and discuss the use of such linear graphs in economic reasoning. As it is often pointed out, a drawn graph is more or less specific, and it shows something not logically derived in general. Shimojima [28] called such property of diagrammatic representations *over-specificity*, and analyzed it from a viewpoint of mathematical logic and cognitive science. Thus, the use of graphs and diagrams in economics have been restricted to be apparatus of explanation. However, linear graphs have the following advantages: (1) linear graphs are easily constructed and reasoning using such graphs is effectively manageable; (2) they are qualitatively determinate, that is, which conclusion is qualitatively deduced from the given premises is uniquely determined with respect to our problems. We investigate the use of linear graphs in certain types of economic reasoning by revealing their range of applications, and we propose a model of economic reasoning with linear graphs.

One of the most effective techniques to construct models of reasoning is that developed through mathematical logic. Recently, interactions between mathematical logic and economics have been discussed. For example, Kaneko [17, 18] has investigated these from a rather wide viewpoint. On the one hand, economics can provide concrete and actual frameworks of reasoning problems, and on the other hand, mathematical logic can provide notions and techniques developed in traditional reasoning studies. One of the most remarkable examples of such interaction is the application of epistemic logic to game theory. [17] provides an introduction to economists on game theoretical applications of epistemic logic. Recent research has shown that epistemic logic can be applied to the analysis of common knowledge and an economic agent's rationality in game theory. Other than epistemic logic, in research on qualitative reasoning, for example in [16, 4], the logic programming PROLOG has been traditionally applied as an implementation of their reasoning models. Furthermore, there are more recent examples: [10] applies notions and techniques of set-theoretical model theory to integrate qualitative and quantitative frameworks; [19] surveys applications of theorem provers based on proof theory in mathematical logic to economics problems; [5] discusses market design from the viewpoint of computational complexity theory. In this article, by applying proof-theoretical techniques, we propose a model of economic reasoning with graphs. Such a model may be useful to investigate the notion of rationality in economics, actual problem solving, decision making, and for teaching economics.

The economic reasoning discussed in this article has been studied within the framework of qualitative reasoning, e.g., [8, 15]. Qualitative reasoning studies investigate reasoning based on qualitative information instead of precise quantitative information [14]. In qualitative reasoning studies, with the aim of implementation, economic reasoning "without graphs" is investigated. In some aspects, the economic reasoning we investigate here is an extension of previous research of qualitative reasoning, where either the demand or supply curve is allowed to shift just once. Such an analysis has been extended to a more complicated, multivariable setting in [8]. However, we concentrate on analyzing the basic demand and supply market, but allow simultaneous shifts of the demand and supply curves as in Example 2.1 below.

In Section 2, by using techniques of differentiation, we specify the range over which linear demand and supply functions can be effectively applied. Based on the analysis by differentiation, in Section 3, we discuss the use of linear demand and supply graphs, and propose a model of economic reasoning with graphs within the framework of proof theory of mathematical logic.

2. Reduction to linear demand and supply functions

In Section 2.1, we discuss our running example of economic reasoning with graphs. In Section 2.2, we analyze our reasoning problems by using techniques of differentiation. By formalizing our problems in Section 2.3, we discuss, in Sections 2.4 and 2.5, the consequences of the analysis in Section 2.2.

2.1. Example

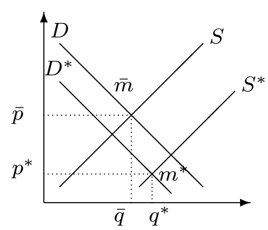
We begin by investigating the following example given by Krugman and Wells [20], which is slightly modified.

Example 2.1 ([20] p.94). When a new, faster computer chip is introduced, (1) demand for computers using the older, slower chips decreases. Simultaneously, (2) computer makers increase their production of computers containing the old chips in order to clear out their stocks of old chips. What happens to the equilibrium price and quantity of computers?

Relationships between quantity demanded and price, as well as between quantity supplied and price, are represented mathematically by functions of price, i.e., a demand function and a supply function, respectively. Then, they are frequently depicted by graphs. In economics, graphs of demand and supply functions are conventionally drawn on a two-dimensional plane, where the vertical axis represents price and the horizontal axis represents quantity demanded or supplied. (This is not consistent with standard mathematical convention, where the horizontal axis normally represents the independent variable, i.e., price.) The law of demand states that a higher price leads to a smaller quantity demanded; hence, demand curves generally slope downward. Similarly, the law of supply states that a higher price leads to a larger quantity supplied; hence, supply curves generally slope upward.

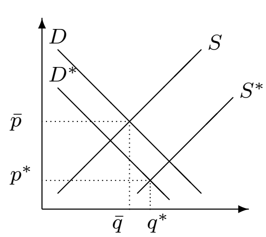
Condition (1) graphically corresponds to a leftward shift of the demand curve from the original D to D^*, which we denote as $D^* \leftarrow D$, and (2) graphically corresponds to a rightward shift of the supply curve from the original S to S^*, i.e., $S \rightarrow S^*$.

In the above example, no concrete demand and supply functions are given, and it is impossible to draw an "arbitrary" curve, as a drawn curve is more or less specific. There are various pseudo-arbitrary curves in economics textbooks. Among others, linear graphs are frequently applied. (In fact, the above example is explained by using linear graphs in [20].) However, careful attention is required in the use of linear graphs, as they may show something not logically derived in general. This phenomenon appears often in the use of diagrammatic/graphical representations, and is called *over-specificity* of diagrams by Shimojima [28]. Let us consider the above example by using the following linear graphs, where $\bar{m}$ is the original equilibrium between demand D and supply S, and m^* is the new equilibrium between shifted demand D^* and supply S^*.

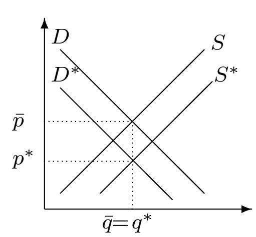


G

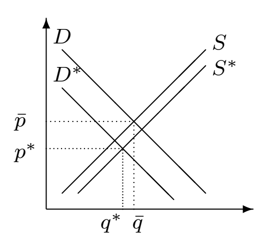
From the above graph G, on the one hand, we can read off $\bar{p} > p^*$, i.e., equilibrium price is decreased, which is a logical consequence of the given premises (1) and (2). On the other hand, we can also read off $\bar{q} < q^*$, i.e., equilibrium quantity is increased, although this is not a logically valid consequence. Whether equilibrium quantity increases or decreases depends on shift widths α and β (these are certain positive real numbers) of the demand and supply curves, respectively, and we need to examine the following three cases: $\alpha < \beta, \alpha = \beta$, and $\alpha > \beta$ from the qualitative viewpoint.



When $\alpha < \beta$



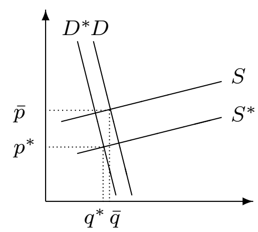
When $\alpha = \beta$



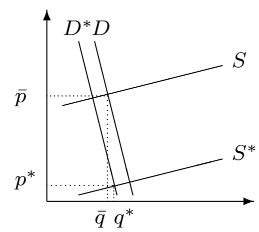
When $\alpha > \beta$

Then, according to the three cases, we find one of $\bar{q} < q^*, \bar{q} = q^*, \bar{q} > q^*$ holds depending on the relationship between shift widths α and β, and nothing is logically derived regarding the change of the equilibrium quantity in this example.

Furthermore, even if information on shift widths of demand and supply curves is given, what is derived for the change of the equilibrium quantity depends further on the magnitudes of the slopes of the demand and supply curves. For example, assume $\alpha < \beta$ in the above example. Then, $\bar{q} < q^*$ is derived in the above case where the slopes of the demand and supply curves are -1 and $+1$, respectively. However, as seen in the following graphs, we find that $\bar{q} < q^*$ is not a valid consequence when the slopes of the demand and supply curves are not fixed at -1 and $+1$. The following graphs depict the given premises $D^* \xleftarrow{\alpha} D, S \xrightarrow{\beta} S^*$, and $\alpha < \beta$, where $D^* \xleftarrow{\alpha} D$ expresses that the demand D shifts to D^* with shift width α, and $\bar{q} > q^*$ holds in G_2 on the left and $\bar{q} < q^*$ holds in G_3 on the right.



G_2



G_3

What is derived on an increase or decrease of the equilibrium quantity and price after shifts of demand and supply generally depends on (1) shift directions, (2) shift widths and (3) slopes of demand and supply curves (even if we restrict ourselves to linear graphs). Thus, for a rigorous analysis of problems such as the above example, the method of differentiation is applied to general demand and supply functions without specifying and depicting their graphs.

Remark 2.2. We can consider cases where demand and supply shift more than once. However, in such a case of multiple shifts, we are eventually required to compare the original and the final equilibriums. Thus, with some algebraic calculation on shift widths of demand and supply, cases of multiple shifts are reduced to cases of single simultaneous shifts.

2.2. Analysis by differentiation

We investigate Example 2.1 by using differentiation. To investigate the effect of a demand shift, a demand function is usually defined as $q = D(p, \alpha)$ by introducing an exogenous variable α, which represents a factor inducing the shift (see [13, 6] for example). However, because we are mainly interested in graphical shifts of demand and supply in this paper, we introduce exogenous variables α and β, directly representing the shift widths of demand and supply curves, respectively, without entering into the detail of the economic factors of the shifts.

Let $q = D(p)$ and $q = S(p)$ be demand and supply functions, respectively, where q is quantity and p is price. We assume that D, S as well as their inverses D^{-1}, S^{-1} are always differentiable. We further assume that demand and supply functions are strictly monotonically decreasing and increasing, respectively, that is, $\frac{dD}{dp} = D' < 0$ and $\frac{dS}{dp} = S' > 0$, which implies D^{-1} and S^{-1} are also strictly monotonically decreasing and increasing.

Let $\alpha, \beta \geq 0$ be shift widths along the p-axis of given demand and supply curves, respectively. After any shifts in demand and supply curves, we assume that there exists a new equilibrium, whose price is determined by α and β, and is denoted as $P^*(\alpha, \beta)$. The left shift of demand $D^* \xleftarrow{\alpha} D$ described by (1) in Example 2.1 is expressed in terms of a mathematical function as $D^*(p) = D(p + \alpha)$ for any p. In particular, when $p = P^*(\alpha, \beta)$, we have $D^*(P^*(\alpha, \beta)) = D(P^*(\alpha, \beta) + \alpha)$. Similarly for (2), $S \xrightarrow{\beta} S^*$ in Example 2.1 is functionally expressed as $S^*(p) = S(p + \beta)$ for any p, and in particular, we have $S^*(P^*(\alpha, \beta)) = S(P^*(\alpha, \beta) + \beta)$. As $D^*(P^*(\alpha, \beta)) = S^*(P^*(\alpha, \beta))$ at the new equilibrium, we have: $D(P^*(\alpha, \beta) + \alpha) = S(P^*(\alpha, \beta) + \beta)$. By similar consideration for pairs of directions of demand and supply shifts, we have the following four **basic equations** according to the combinations of $+$ and $-$ on both sides:

$$D(P^*(\alpha, \beta) \pm \alpha) = S(P^*(\alpha, \beta) \pm \beta)$$

where the $(+, +)$ combination of the above basic equations corresponds to the $(\xleftarrow{D}, \xrightarrow{S})$-pair of a left demand shift and a right supply shift and similarly; $(+, -)$ to $(\xleftarrow{D}, \xleftarrow{S})$; $(-, +)$ to $(\xrightarrow{D}, \xrightarrow{S})$; and $(-, -)$ to $(\xrightarrow{D}, \xleftarrow{S})$.

By partially differentiating both sides of our basic equations with respect to α and β, respectively, we have:

$$D' \cdot \left(\frac{\partial P^*}{\partial \alpha} \pm 1 \right) = S' \cdot \frac{\partial P^*}{\partial \alpha} \qquad D' \cdot \frac{\partial P^*}{\partial \beta} = S' \cdot \left(\frac{\partial P^*}{\partial \beta} \pm 1 \right)$$

$$D' \cdot \frac{\partial P^*}{\partial \alpha} \pm D' = S' \cdot \frac{\partial P^*}{\partial \alpha} \qquad D' \cdot \frac{\partial P^*}{\partial \beta} = S' \cdot \frac{\partial P^*}{\partial \beta} \pm S'$$

$$\frac{\partial P^*}{\partial \alpha} = \frac{\mp D'}{D' - S'} \qquad \frac{\partial P^*}{\partial \beta} = \frac{\pm S'}{D' - S'}$$

Here, $D' = \frac{dD}{dp}$ and $S' = \frac{dS}{dp}$. Note that the above four equations hold for any shift widths α, β. That is, whatever the magnitudes of α and β are, the positive or negative signs of $\frac{\partial P^*}{\partial \alpha}$ and $\frac{\partial P^*}{\partial \beta}$ are determined depending on the directions of the demand and supply shifts.

Then, we obtain the following total differential: $dP^* = \dfrac{\partial P^*}{\partial \alpha} d\alpha + \dfrac{\partial P^*}{\partial \beta} d\beta$. Based on the total differential, let us analyze the change ΔP^* in $P^*(\alpha, \beta)$. To compare the original equilibrium $P^*(0, 0)$, i.e., shift widths $\alpha = \beta = 0$, and the new equilibrium $P^*(\alpha, \beta)$, let $\Delta P^* = P^*(\alpha, \beta) - P^*(0, 0)$, $\Delta\alpha = \alpha - 0 = \alpha$ (≥ 0), and $\Delta\beta = \beta - 0 = \beta$ (≥ 0). Then, the change ΔP^* in $P^*(\alpha, \beta)$ is represented as follows:

$$\Delta P^* = \frac{\partial P^*}{\partial \alpha} \Delta\alpha + \frac{\partial P^*}{\partial \beta} \Delta\beta$$

We analyze the sign of ΔP^* by considering the following cases:

(i) When $\alpha \neq 0$ and $\beta \neq 0$, we further divide this case into the following four cases.

1. When $D(P^*(\alpha,\beta)+\alpha) = S(P^*(\alpha,\beta)+\beta)$, that is $D^* \xleftarrow{\alpha} D, S \xrightarrow{\beta} S^*$, we have:
$$\frac{\partial P^*}{\partial \alpha} = \frac{-D'}{D'-S'} < 0 \ \text{ and } \ \frac{\partial P^*}{\partial \beta} = \frac{S'}{D'-S'} < 0$$
Thus, $\Delta P^* < 0$.
2. When $D(P^*(\alpha,\beta)+\alpha) = S(P^*(\alpha,\beta)-\beta)$, that is $D^* \xleftarrow{\alpha} D, S^* \xleftarrow{\beta} S$, we have:
$$\frac{\partial P^*}{\partial \alpha} = \frac{-D'}{D'-S'} < 0 \ \text{ and } \ \frac{\partial P^*}{\partial \beta} = \frac{-S'}{D'-S'} > 0$$
Thus, the sign of ΔP^* is not determined and it depends on the slopes and shift widths of demand and supply in general.
3. When $D(P^*(\alpha,\beta)-\alpha) = S(P^*(\alpha,\beta)+\beta)$, that is $D \xrightarrow{\alpha} D^*, S \xrightarrow{\beta} S^*$, we have:
$$\frac{\partial P^*}{\partial \alpha} = \frac{D'}{D'-S'} > 0 \ \text{ and } \ \frac{\partial P^*}{\partial \beta} = \frac{S'}{D'-S'} < 0$$
Thus, the sign of ΔP^* is not determined.
4. When $D(P^*(\alpha,\beta)-\alpha) = S(P^*(\alpha,\beta)-\beta)$, that is $D \xrightarrow{\alpha} D^*, S^* \xleftarrow{\beta} S$, we have:
$$\frac{\partial P^*}{\partial \alpha} = \frac{D'}{D'-S'} > 0 \ \text{ and } \ \frac{\partial P^*}{\partial \beta} = \frac{-S'}{D'-S'} > 0$$
Thus, $\Delta P^* > 0$.

(ii) When $\beta = 0$, we have $\Delta P^* = \dfrac{\partial P^*}{\partial \alpha}\Delta\alpha$.

1. When $D(P^*(\alpha,\beta)+\alpha) = S(P^*(\alpha,\beta))$, that is $D^* \xleftarrow{\alpha} D$, we have $\dfrac{\partial P^*}{\partial \alpha} = \dfrac{-D'}{D'-S'} < 0$. Thus, $\Delta P^* < 0$.
2. When $D(P^*(\alpha,\beta)-\alpha) = S(P^*(\alpha,\beta))$, that is $D \xrightarrow{\alpha} D^*$, we have $\dfrac{\partial P^*}{\partial \alpha} = \dfrac{D'}{D'-S'} > 0$. Thus, $\Delta P^* > 0$.

(iii) When $\alpha = 0$, we have $\Delta P^* = \dfrac{\partial P^*}{\partial \beta}\Delta\beta$.

1. When $D(P^*(\alpha,\beta)) = S(P^*(\alpha,\beta)+\beta)$, that is $S \xrightarrow{\beta} S^*$, we have $\dfrac{\partial P^*}{\partial \beta} = \dfrac{S'}{D'-S'} < 0$. Thus, $\Delta P^* < 0$.
2. When $D(P^*(\alpha,\beta)) = S(P^*(\alpha,\beta)-\beta)$, that is $S^* \xleftarrow{\beta} S$, we have $\dfrac{\partial P^*}{\partial \alpha} = \dfrac{-S'}{D'-S'} > 0$. Thus, $\Delta P^* > 0$.

As observed in the above analysis, there exist two cases: whether ΔP^* is positive or negative is **determinate** or **indeterminate**, depending on combinations of shift directions.

By considering the inverse demand and supply functions $p = D^{-1}(q)$ and $p = S^{-1}(q)$, respectively, and by applying a similar calculation, we can investigate changes in equilibrium quantities. Let $Q^*(\alpha, \beta)$ be the new equilibrium quantity. Then, our basic equations about quantity are $D^{-1}(Q^*(\alpha, \beta)) \pm \alpha = S^{-1}(Q^*(\alpha, \beta)) \pm \beta$, where $(+,+)$ for $(\xrightarrow{D}, \xleftarrow{S})$; $(+,-)$ for $(\xrightarrow{D}, \xrightarrow{S})$; $(-,+)$ for $(\xleftarrow{D}, \xleftarrow{S})$; and $(-,-)$ for $(\xleftarrow{D}, \xrightarrow{S})$. Then, we obtain the following equation:

$$\Delta Q^* = \frac{\partial Q^*}{\partial \alpha}\Delta\alpha + \frac{\partial Q^*}{\partial \beta}\Delta\beta,$$

where $\dfrac{\partial Q^*}{\partial \alpha} = \dfrac{\mp 1}{D^{-1'} - S^{-1'}}$ and $\dfrac{\partial Q^*}{\partial \beta} = \dfrac{\pm 1}{D^{-1'} - S^{-1'}}$. Here, $D^{-1'} = \frac{dD^{-1}}{dq} < 0$ and $S^{-1'} = \frac{dS^{-1}}{dq} > 0$.

Let us investigate the case where $\alpha \neq 0, \beta \neq 0$ (the other cases are similar).

1. When $D \xrightarrow{\alpha} D^*, S^* \xleftarrow{\beta} S$, whether ΔQ^* is positive or negative is not determined.
2. When $D \xrightarrow{\alpha} D^*, S^* \xleftarrow{\beta} S$, we have $\Delta Q^* > 0$.
3. When $D^* \xleftarrow{\alpha} D, S^* \xleftarrow{\beta} S$, we have $\Delta Q^* < 0$.
4. When $D^* \xleftarrow{\alpha} D, S \xrightarrow{\beta} S^*$, whether ΔQ^* is positive or negative is not determined.

As the same calculation that is applied to price changes is also applied to the quantity changes by considering inverse functions D^{-1} and S^{-1}, we concentrate on the investigation of price changes in the following discussions.

2.3. Notation

To discuss the consequences of our analysis in Section 2.2, we fix our notation.

Real numbers are denoted by $a, b, c, \ldots$, and variables for real numbers are denoted by $p, q, x, y, \ldots$. The usual first-order terms are denoted by $t, t_1, \ldots$, and formulas are denoted by $\varphi, \psi, \theta, \varphi_1, \ldots$. By D and S, we denote given demand and supply functions, respectively. We assume that D, S as well as their inverses D^{-1}, S^{-1} are always differentiable. We further assume that demand and supply functions are strictly monotonically decreasing and increasing, respectively. When we mention a function without specifying demand or supply, we denote it by C. In particular, by $D^{(1)}$ and $S^{(1)}$, we denote linear demand and supply functions with slopes of -1 and $+1$, respectively. They are defined by the following formulas:

- $D^{(1)}(p) = -p + a$ for some real number a;
- $S^{(1)}(p) = p + a$ for some real number a

In our qualitative framework, exactly what number a represents is not important, and it is only used to distinguish the given linear functions in question. We are interested in these particular linear functions among others, and in what follows, we consider these $D^{(1)}$ and $S^{(1)}$ as "linear" functions.

We denote the **direction** of a shift $\rightarrow$ or $\leftarrow$ of a given function by σ. When a function C shifts to C^*, we introduce the term $w(C, C^*)$ to denote its **shift width**, which is a certain positive real number. We define $w(C, C^*)$ as the distance between p-intercepts: $w(C, C^*) = |C^{-1}(0) - C^{*-1}(0)|$. We usually denote a shift width $w(C, C^*)$ by $\alpha, \beta, \ldots$.

For a given function C, when the direction and width of its shift are specified, the function after the shift is uniquely determined, and we usually denote it by C^*. Then, by $\sigma(C)$, we denote a **shift formula** $C \xrightarrow{\alpha} C^*$ or $C^* \xleftarrow{\alpha} C$, where $w(C, C^*)$ is denoted by α. These shift formulas are defined as:

- $D \xrightarrow{\alpha} D^* \Leftrightarrow \forall p(D^*(p) = D(p - \alpha))$
- $S \xrightarrow{\alpha} S^* \Leftrightarrow \forall p(S^*(p) = S(p + \alpha))$

Similarly for left shifts $\leftarrow$. Thus, shift formulas are abbreviations of the above formulas on the right.

We assume that equilibrium always exists between given demand and supply functions, and furthermore, that a new equilibrium exists after shifts of the functions. This is possible because the magnitudes of p-, and q-values are not specified in our qualitative framework. For given demand and supply functions D and S, we introduce the term $D \cap S$ to denote the **equilibrium**, which is a pair consisting of a q-coordinate and a p-coordinate. When $D \cap S = (\bar{q}, \bar{p})$, we define $(D \cap S)_P = \bar{p}$ and $(D \cap S)_Q = \bar{q}$.

By using the above notation, the logical consequence relation described in Example 2.1 is expressed as follows (we assume $\alpha < \beta$, for shift widths α $(= w(D, D^*))$ of demand, and β $(= w(S, S^*))$ of supply, in addition to the given premises (1) and (2)):

$$D^* \xleftarrow{\alpha} D, S \xrightarrow{\beta} S^*, \alpha < \beta \ \models \ (D \cap S)_P > (D^* \cap S^*)_P$$

where the formulas on the left-hand side of $\models$ are premises, and each comma means a conjunction, and the formula on the right-hand side is the conclusion. $\models$ means that $(D \cap S)_P > (D^* \cap S^*)_P$ is a logically valid conclusion with differential calculus as in Section 2.2 for the given premises $D^* \xleftarrow{\alpha} D, S \xrightarrow{\beta} S^*$, and $\alpha < \beta$. The negation of the consequence relation $\models$ is expressed as $\not\models$.

When we consider our problems more abstractly without specifying the magnitude relationship between shift widths α and β, we denote the relation by w. Thus, w is one of $\alpha < \beta, \alpha = \beta, \alpha > \beta$. We also denote, by o, the magnitude relationship between equilibrium prices $(D \cap S)_P$ and $(D^* \cap S^*)_P$, as well as between equilibrium quantities $(D \cap S)_Q$ and $(D^* \cap S^*)_Q$. Then, the above logical consequence relation can be further abstracted to the following form:

$$\sigma_1(D), \sigma_2(S), w \ \models \ o$$

which is the most abstract description of our problems.

We can generalize our logical consequence relation by allowing any first-order formulas to be premises and conclusions. However, in this paper, we concentrate on the above restricted logical consequence relation for simplicity.

2.4. Determinacy of linear demand and supply functions

As a special case of the previous analysis on general functions in Section 2.2, we examine the case of linear demand and supply functions to verify that the linear functions are qualitatively determinate. That is, when shift directions and the magnitude relationship between shift widths are fixed, which conclusion is qualitatively deduced is uniquely determined. As seen in Theorem 2.7, the case of general demand and supply functions is not qualitatively determinate (i.e., the conclusion is not uniquely determined), even if their shift directions, shift widths, and slopes are specified.

Theorem 2.3 (Linear determinacy). *For any σ_1, σ_2, for any w, there exists unique o with respect to prices as well as quantities such that*

$$\sigma_1(D^{(1)}),\ \sigma_2(S^{(1)}),\ w \models o$$

Proof. As a special case of our previous analysis in Section 2.2, we investigate the case where $D' = -1$ and $S' = 1$. In this case, $\dfrac{\partial P^*}{\partial \alpha} = \dfrac{\mp D'}{D' - S'} = \dfrac{\mp 1}{2}$ and $\dfrac{\partial P^*}{\partial \beta} = \dfrac{\pm S'}{D' - S'} = \dfrac{\mp 1}{2}$. Thus, $\Delta P^* = \dfrac{\mp\alpha \mp \beta}{2}$.

We show the case where $\alpha \neq 0$ and $\beta \neq 0$ for this proof. The other cases are the same as those in Section 2.2.

1. When $D^{(1)}(P^*(\alpha,\beta) + \alpha) = S^{(1)}(P^*(\alpha,\beta) + \beta)$, that is $D^{(1)*} \xleftarrow{\alpha} D^{(1)}, S^{(1)} \xrightarrow{\beta} S^{(1)*}$, we have:

   $$\Delta P^* = \frac{-\alpha - \beta}{2} < 0$$

   Thus, in this case, whatever w is, ΔP^* is always decreased.
2. When $D^{(1)}(P^*(\alpha,\beta) + \alpha) = S^{(1)}(P^*(\alpha,\beta) - \beta)$, that is $D^{(1)*} \xleftarrow{\alpha} D^{(1)}, S^{(1)*} \xleftarrow{\beta} S^{(1)}$, we have:

   $$\Delta P^* = \frac{-\alpha + \beta}{2}$$

   We further divide this case into the following three cases according to the magnitude relationship between shift widths α and β.
   (a) When $\alpha < \beta$, we have $\Delta P^* > 0$.
   (b) When $\alpha > \beta$, we have $\Delta P^* < 0$.
   (c) When $\alpha = \beta$, we have $\Delta P^* = 0$.

3. When $D^{(1)}(P^*(\alpha,\beta) - \alpha) = S^{(1)}(P^*(\alpha,\beta) + \beta)$, that is $D^{(1)} \xrightarrow{\alpha} D^{(1)*}, S^{(1)} \xrightarrow{\beta} S^{(1)*}$, we have:

$$\Delta P^* = \frac{\alpha - \beta}{2}$$

   We further divide this case into the following three cases according to the magnitude relationship between α and β.
   (a) When $\alpha < \beta$, we have $\Delta P^* < 0$.
   (b) When $\alpha > \beta$, we have $\Delta P^* > 0$.
   (c) When $\alpha = \beta$, we have $\Delta P^* = 0$.
4. When $D^{(1)}(P^*(\alpha,\beta) - \alpha) = S^{(1)}(P^*(\alpha,\beta) - \beta)$, that is $D^{(1)} \xrightarrow{\alpha} D^{(1)*}, S^{(1)*} \xleftarrow{\beta} S^{(1)}$, we have:

$$\Delta P^* = \frac{\alpha + \beta}{2} > 0$$

■

Note that if the information w on the magnitude relationship between shift widths is not given, there is no valid conclusion o in the above cases (2) and (3) as it is in the analysis in Section 2.2.

Remark 2.4. In the case of linear demand and supply functions, we obtain the same results through a simple algebraic calculation without using differentiation techniques as follows. Let $D(p) = -p + a$ and $S(p) = p + b$ for some a, b. Then, we obtain the following basic equations: $-P^*(\alpha,\beta) \mp \alpha + a = P^*(\alpha,\beta) \pm \beta + b$. Thus, we have $P^*(\alpha,\beta) = \dfrac{\mp\alpha \mp \beta + a - b}{2}$. Then, $\Delta P^* = P^*(\alpha,\beta) - P(0,0) = \dfrac{\mp\alpha \mp \beta}{2}$, which is exactly the same result as the formula in the above analysis by differentiation.

2.5. Reduction to linear functions

By checking the analyses in Sections 2.2 and 2.4, we find that when information on the magnitude relationship w between shift widths is "not given," exactly the same conclusion is derived in both cases of general and linear demand and supply functions.

Theorem 2.5. *For any $D, S, \sigma_1, \sigma_2, o$,*

$$\sigma_1(D), \sigma_2(S) \models o \;\; \textit{if and only if} \;\; \sigma_1(D^{(1)}), \sigma_2(S^{(1)}) \models o$$

Thus, we also have the following contrapositive form:

$$\sigma_1(D^{(1)}), \sigma_2(S^{(1)}) \not\models o \;\; \textit{if and only if} \;\; \sigma_1(D), \sigma_2(S) \not\models o$$

Thus, without information on the magnitude relationship between shift widths, problems involving general demand and supply functions are reduced to problems involving linear functions. However, when information on the magnitude relationship w between shift widths is given, the above theorem does not hold any longer, which is shown by Example 2.1.

Proposition 2.6. *There exists some D and S such that $D^{(1)*} \xleftarrow{\alpha} D^{(1)}, S^{(1)} \xrightarrow{\beta} S^{(1)*}, \alpha < \beta \models (D^{(1)} \cap S^{(1)})_Q < (D^{(1)*} \cap S^{(1)*})_Q$ but $D^* \xleftarrow{\alpha} D, S \xrightarrow{\beta} S^*, \alpha < \beta \not\models (D \cap S)_Q < (D^* \cap S^*)_Q$.*
This means; it is NOT the case that for any D, S, for any σ_1, σ_2, for any w, o, "$\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$ implies $\sigma_1(D), \sigma_2(S), w \models o$."

In what follows we show that the $\Rightarrow$-direction of Theorem 2.5 holds even if w is given. Based on the analysis presented in Section 2.2, we find that the positive or negative sign of the change ΔP^* is determinate or indeterminate depending on combinations of shift directions σ_1 and σ_2. We say that the given shift directions σ_1 and σ_2 are **determinate** with respect to price (or quantity) change when the magnitude relationship o between equilibrium prices (resp. quantities) is uniquely determined for given σ_1 and σ_2. That is, there exists a unique o with respect to price (resp. quantity) such that $\sigma_1(D), \sigma_2(S) \models o$ holds. Otherwise, we say that the given σ_1 and σ_2 are **indeterminate** with respect to price (or quantity) change. By examining the indeterminate cases of our analysis in Section 2.2 in more detail, we obtain the following theorem. It is trivial that what is logically derivable by any functions D, S is also logically derivable by linear $D^{(1)}, S^{(1)}$ in particular. The following theorem is different from this trivial fact, and claims that what is logically derivable by "some" functions (not restricted to be linear) is also logically derivable by linear $D^{(1)}, S^{(1)}$.

Theorem 2.7 (Linear reduction). *For any D, S, for any σ_1, σ_2, for any w, o,*

$$\textit{if } \sigma_1(D), \sigma_2(S), w \models o \;\; \textit{then} \;\; \sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$$

In other words, for any σ_1, σ_2, for any w, o,

$$\textit{if } \exists DS(\sigma_1(D), \sigma_2(S), w \models o) \;\; \textit{then} \;\; \sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$$

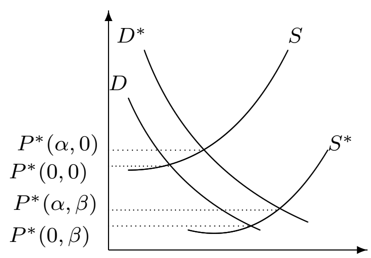
Proof. Assume that $\sigma_1(D), \sigma_2(S), w \models o$ for some D and S. We first divide the case according to whether or not the given σ_1, σ_2 are determinate.

(i) When σ_1, σ_2 are determinate with respect to price change, its conclusion is obtained independently of w. Hence, we have $\sigma_1(D), \sigma_2(S), w \models o$ iff $\sigma_1(D), \sigma_2(S) \models o$ in this case. Then, by Theorem 2.5, we have $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}) \models o$. Again, because σ_1, σ_2 are determinate, we have $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}) \models o$ iff $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$, and hence, $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$.

(ii) When the given σ_1, σ_2 are indeterminate with respect to price change (and hence $\alpha \neq 0, \beta \neq 0$), we examine the change $\Delta P^* = P^*(\alpha, \beta) - P^*(0, 0)$ in more detail. Let $\Delta P^*_\alpha = P^*(\alpha, 0) - P^*(0, 0)$, which is the change of the equilibrium price after a shift of D with its shift width α, and $\Delta P^*_\beta = P^*(0, \beta) - P^*(0, 0)$, which is the change of the equilibrium price after a shift of S with its shift width β (see the following graph G). Then, we have

$$\Delta P^* = \Delta P^*_\alpha + \Delta P^*_\beta = \frac{\Delta P^*_\alpha}{\Delta\alpha}\Delta\alpha + \frac{\Delta P^*_\beta}{\Delta\beta}\Delta\beta,$$

where $\Delta\alpha = \alpha\ (> 0)$ and $\Delta\beta = \beta\ (> 0)$, which is exactly the algebraic version of the total differential analyzed in Section 2.2.



G

Let $\left|\frac{\Delta P^*_\alpha}{\Delta\alpha}\right| = a$ and $\left|\frac{\Delta P^*_\beta}{\Delta\beta}\right| = b$. Then the indeterminate cases consist of the case $\Delta P^* = -a\alpha + b\beta$ (that is, $D^* \xleftarrow{\alpha} D, S^* \xleftarrow{\beta} S$), and the case $\Delta P^* = a\alpha - b\beta$ (that is, $D \xrightarrow{\alpha} D^*, S \xrightarrow{\beta} S^*$). We investigate the second case of $\Delta P^* = a\alpha - b\beta$ by dividing this case into the following three cases: $a < b, a > b$, and $a = b$.

1. When $a < b$, we further divide this case according to the magnitude relationship between α and β.
   (a) When $\alpha < \beta$, we have $a\alpha < b\beta$, and hence, we have $\Delta P^* = a\alpha - b\beta < 0$.
   In the case of linear functions, by Case 3-a of Theorem 2.3, we find that the same $\Delta P^* < 0$ holds when $\alpha < \beta$. Thus, we also have $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$ in this case.
   (b) Otherwise, when $\alpha \geq \beta$, we show that nothing is deduced qualitatively, that is, the premise of the theorem does not hold.
      i. Let $\alpha := \dfrac{\beta + \frac{b}{a}\beta}{2}$. Then, we have $a < b$ and $\alpha \geq \beta$, as well as $a\alpha < b\beta$, i.e., $\Delta P^* < 0$ in this case.
      ii. Let $\beta := \dfrac{a}{2b}\alpha$. Then, we have $a < b$ and $\alpha \geq \beta$, as well as $a\alpha > b\beta$, i.e., $\Delta P^* > 0$ in this case.
      iii. Let $\alpha := b, \beta := a$. Then, we have $a < b$ and $\alpha \geq \beta$, as well as $a\alpha = b\beta$, i.e., $\Delta P^* = 0$.

   Thus, in the case where $a < b$ and $\alpha \geq \beta$, by defining α, β appropriately, any result of $\Delta P^* < 0, \Delta P^* > 0$, and $\Delta P^* = 0$ holds, which means that there is no valid consequence o in this case.

Remark 2.8. The above analysis shows that the difference in magnitudes of slopes is reduced to the difference in magnitudes of shift widths.

2. When $a > b$, we further divide this case according to the magnitude relationship between α and β.
   (a) When $\alpha > \beta$, we have $a\alpha > b\beta$, and hence, we have $\Delta P^* = a\alpha - b\beta > 0$.
   In the case of linear functions, by Case 3-b of Theorem 2.3, we find that the same result $\Delta P^* > 0$ holds when $\alpha > \beta$. Thus, we also have $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$ in this case.
   (b) Otherwise, when $\alpha \leq \beta$, we show that the premise of the theorem does not hold.
      i. Let $\alpha := \dfrac{b}{2a}\beta$. Then, we have $a > b$ and $\alpha \leq \beta$, as well as $a\alpha < b\beta$, i.e., $\Delta P^* < 0$ in this case.
      ii. Let $\beta := \dfrac{\frac{a}{b}\alpha + \alpha}{2}$. Then, we have $a > b$ and $\alpha \leq \beta$, as well as $a\alpha > b\beta$, i.e., $\Delta P^* > 0$ in this case.
      iii. Let $\alpha := b, \beta := a$. Then, we have $a < b$ and $\alpha \geq \beta$, as well as $a\alpha = b\beta$, i.e., $\Delta P^* = 0$ in this case.

   Thus, in the case where $a > b$ and $\alpha \leq \beta$, there is no valid consequence o.
3. When $a = b$ (> 0), we have $\Delta P^* = a\alpha - b\beta = a\alpha - a\beta = a(\alpha - \beta)$.
   (a) When $\alpha < \beta$, we have $\Delta P^* < 0$, which corresponds to the linear case. (Theorem 2.3 Case 3-a.)
   (b) When $\alpha > \beta$, we have $\Delta P^* > 0$, which corresponds to the linear case. (Theorem 2.3 Case 3-b.)
   (c) When $\alpha = \beta$, we have $\Delta P^* = 0$, which corresponds to the linear case. (Theorem 2.3 Case 3-c.)

   Thus, in either case, we have $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$.

The other case of $\Delta P^* = -a\alpha + b\beta$ can be shown in the same way. ■

Let us consider the following contrapositive form of Theorem 2.7:
For any D, S, for any σ_1, σ_2, for any w, o,

$$\textit{if } \sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \not\models o \textit{ then } \sigma_1(D), \sigma_2(S), w \not\models o$$

This says that what does not hold with linear demand and supply functions also does not hold with any general functions. This means that the invalidity with linear demand and supply functions is generalized to the invalidity with general functions, although this is not so for the validity as seen in Proposition 2.6. Thus, problems related to the invalidity with general functions can be reduced to the invalidity with linear demand and supply functions.

3. Use of linear demand and supply graphs

The linear system, where demand and supply functions are restricted to be linear with slopes of -1 and $+1$, respectively, has the following special properties: (1) It is qualitatively determinate (Theorem 2.3 in Section 2.4); (2) Linear graphs are easily constructed, and reasoning using such graphs is effectively manageable. We discuss point (2) in the following Section 3.1. Unfortunately, Example 2.1 shows that we need to pay careful attention using linear graphs in deductive problems (when information on shift widths is given) because of the over-specificity. However, as discussed in the following Section 3.2, they are useful in abductive reasoning.

3.1. Inference with linear graphs

To make our strategy for solving given problems explicit, we formalize our operations on graphs as inference rules based on the natural deduction inference system. Natural deduction was introduced by Gentzen [9], and studied extensively by Prawitz [23], and is one of the major inference systems in proof theory. The inference rules of Gentzen's natural deduction are intended to be as close as possible to actual reasoning. Our formalization is simply a natural deduction style description of the usual operations on graphs, and it can be implemented to theorem proving systems such as [12].

We first define linear graphs in our system formally.

Definition 3.1 (Linear graph). A **linear graph** G consists of the following items:

- The first quadrant of the qp-coordinate space. (q-axis for the horizontal axis, and p-axis for the vertical axis.)
- At most two straight lines of slope -1, called **demand curves** and named $D^{(1)}, D^{(1)*}, D^{(1)}_1, \ldots$; at most two straight lines of slope 1, called **supply curves** and named $S^{(1)}, S^{(1)*}, S^{(1)}_1, \ldots$.
- Every pair of demand $D^{(1)}$ and supply $S^{(1)}$ curves in G has an intersection point, named $D^{(1)} \cap S^{(1)}$, which is accompanied by its coordinates; $(D^{(1)} \cap S^{(1)})_P$ for the p-coordinate, and $(D^{(1)} \cap S^{(1)})_Q$ for the q-coordinate.

In our graphical representation, we usually omit the name of an intersection point, as well as its coordinates, to avoid the visual complexity. Although our basic linear graph defined above consists of at most two demand and two supply curves (since these are sufficient for this paper), it can be generalized to include more curves.

We then define the *width* between two curves in a given graph.

Definition 3.2 (Width). Let C and C^* be a pair of demand or supply curves in a graph. We define the **width** $w(C, C^*)$ between C and C^* as the distance between the p-intercepts of C and C^*.

In our qualitative framework, we may rearrange every curve by retaining the magnitude relationship of widths among curves so that every p-intercept of the given curves appears in the first quadrant of the qp-space. Cf. Definition 3.5 and Example 3.6 of the equivalence of our graphs.

In addition to a graph drawn as a diagram, we consider the *type* of a graph, which is a symbolic specification thereof. The type of a graph also defines what kind of information we can extract from it; cf. our inference rule obs in Definition 3.7. As in graph theory, we usually do not distinguish between a drawn graph and its type, and denote both by G.

Definition 3.3 (Type). The **type of a graph** G is $(\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$, where:

- $\mathcal{C}$ is a sequence $(D_1^{(1)}, D_2^{(1)}; S_1^{(1)}, S_2^{(1)})$ of names of demand curves and supply curves in G, respectively, which are ordered from left to right as they are in the drawn graph G.
- l_w is the ordering of widths $w(D_1^{(1)}, D_2^{(1)})$ and $w(S_1^{(1)}, S_2^{(1)})$ in G.
- $\mathcal{E}$ is the set of points of intersection between demand and supply curves in G.
- l_p is the ordered set of all the p-coordinates of intersections.
- l_q is the ordered set of all the q-coordinates of intersections.

Example 3.4. The type of G in the following Example 3.6 is as follows:

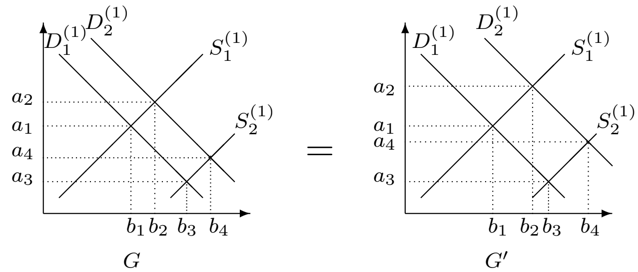
- $\mathcal{C} = (D_1^{(1)}, D_2^{(1)} \ ; \ S_1^{(1)}, S_2^{(1)})$
- $l_w = w(D_1^{(1)}, D_2^{(1)}) < w(S_1^{(1)}, S_2^{(1)})$
- $\mathcal{E} = \{D_1^{(1)} \cap S_1^{(1)}, D_2^{(1)} \cap S_1^{(1)}, D_1^{(1)} \cap S_2^{(1)}, D_2^{(1)} \cap S_2^{(1)}\}$
- $l_p = a_3 < a_4 < a_1 < a_2$
- $l_q = b_1 < b_2 < b_3 < b_4$

Based on types of graphs, we define the equivalence between graphs. Note that points of intersection are determined when curves are given. Furthermore, in our qualitative framework, the exact width values, as well as the qp-coordinates of the intersections, are not significant, whereas the ordering relation among them is important. Thus, we regard two graphs as being equivalent when their arrangement of curves and their ordering relation among widths are equivalent.

Definition 3.5. Two graphs $G = (\mathcal{C}, l_w, \mathcal{E}, l_p, l_q)$ and $G' = (\mathcal{C}', l'_w, \mathcal{E}', l'_p, l'_q)$ are **equivalent**, written as $G = G'$, if $\mathcal{C} = \mathcal{C}'$ and $l_w = l'_w$.

Note that if $\mathcal{C} = \mathcal{C}'$, then l_w and l'_w consist of the same widths $w(C_i, C_j)$, although the values of $w(C_i, C_j)$ of l_w and of l'_w may be different. Thus, $l_w = l'_w$ means that the ordering relation, not the exact value, is the same among the widths of l_w and l'_w.

Example 3.6 (Equivalence). The following graphs G and G' are equivalent because we have $\mathcal{C} = \mathcal{C}' = (D_1^{(1)}, D_2^{(1)}; S_1^{(1)}, S_2^{(1)})$, and $l_w = l'_w = w(D_1^{(1)}, D_2^{(1)}) < w(S_1^{(1)}, S_2^{(1)})$ even though the exact magnitude of $w(D_1^{(1)}, D_2^{(1)})$ in G is smaller than that in G'. Here, $(D_i^{(1)} \cap S_j^{(1)})_P$ (resp. $(D_i^{(1)} \cap S_j^{(1)})_Q$) is expressed by a_k (resp. b_k) for some k.



We now define our inference rules for our linear graphs in the style of natural deduction. Every inference rule of natural deduction has the following form:

$$\frac{\begin{array}{c}\vdots\\ \varphi_1\end{array} \quad \cdots \quad \begin{array}{c}\vdots\\ \varphi_n\end{array}}{\psi}\ \textsf{rule}$$

which states that from $\varphi_1, \ldots, \varphi_n$ (premises of the rule), we can infer ψ (the conclusion of the rule) by the rule. For example, the following $\wedge I$ states that from φ and ψ, we can infer $\varphi \wedge \psi$ by the rule $\wedge I$. A proof in natural deduction is structured as a tree consisting of formulas as its nodes and inference rules as its edges. The top formulas of the tree are the assumptions, and the other formulas of the tree are ones that follow from the formulas immediately above, using one of the inference rules. In the following rules, a formula written within square brackets (in rules of $\vee E, \Rightarrow I, \neg I, \exists E, RAA$) indicates that the assumptions of this form are closed at the inference. Closed assumptions are essentially tentative assumptions in the given proof, in contrast to not closed, i.e., open assumptions. For example, the following rule $\vee E$ is a formalization of the usual inference by case dividing: When we obtain a proof of θ in either case where φ or ψ is tentatively assumed, we can infer θ from $\varphi \vee \psi$ without assuming φ nor ψ. Each natural number n posed on the bracket and beside the name of a rule indicates that the assumption is closed at the inference rule. $\perp$ is the constant to denote the contradiction. See [7, 24] for a detailed introduction to natural deduction.

The natural deduction rules for first-order logic consist of the following dual pairs, each pair consisting of an *introduction (I)* and an *elimination (E)* rule for each connective $\wedge, \vee, \Rightarrow, \neg, \forall, \exists$, as well as $\perp E$ and RAA:

$$\frac{\begin{array}{c}\vdots\\ \varphi\end{array} \quad \begin{array}{c}\vdots\\ \psi\end{array}}{\varphi \wedge \psi}\ \wedge I \qquad \frac{\begin{array}{c}\vdots\\ \varphi_1 \wedge \varphi_2\end{array}}{\varphi_i}\ \wedge E_{(i\,=\,1,\,2)} \qquad \frac{\begin{array}{c}\vdots\\ \varphi_i\end{array}}{\varphi_1 \vee \varphi_2}\ \vee I_{(i\,=\,1,\,2)} \qquad \frac{\begin{array}{c}\vdots\\ \varphi \vee \psi\end{array} \quad \begin{array}{c}[\varphi]^n\\ \vdots\\ \theta\end{array} \quad \begin{array}{c}[\psi]^n\\ \vdots\\ \theta\end{array}}{\theta}\ \vee E, n$$

$$\frac{\begin{array}{c}[\varphi]^n\\ \vdots\\ \psi\end{array}}{\varphi \Rightarrow \psi} \Rightarrow I, n \qquad \frac{\begin{array}{c}\vdots\\ \varphi\end{array} \quad \begin{array}{c}\vdots\\ \varphi \Rightarrow \psi\end{array}}{\psi} \Rightarrow E \qquad \frac{\begin{array}{c}[\varphi]^n\\ \vdots\\ \bot\end{array}}{\neg\varphi} \neg I, n \qquad \frac{\begin{array}{c}\vdots\\ \varphi\end{array} \quad \begin{array}{c}\vdots\\ \neg\varphi\end{array}}{\bot} \neg E$$

$$\frac{\begin{array}{c}\vdots\\ \varphi(x)\end{array}}{\forall x \varphi(x)} \forall I \quad \frac{\begin{array}{c}\vdots\\ \forall x \varphi(x)\end{array}}{\varphi(t)} \forall E \quad \frac{\begin{array}{c}\vdots\\ \varphi(t)\end{array}}{\exists x \varphi(x)} \exists I \quad \frac{\begin{array}{c}\vdots\\ \exists x \varphi(x)\end{array} \qquad \begin{array}{c}[\varphi(x)]^n\\ \vdots\\ \psi\end{array}}{\psi} \exists E, n \quad \frac{\begin{array}{c}\vdots\\ \bot\end{array}}{\varphi} \bot E \quad \frac{\begin{array}{c}[\neg\varphi]^n\\ \vdots\\ \bot\end{array}}{\varphi} RAA, n$$

In $\forall E$ and $\exists I$, t is a first-order term. In $\forall I$, the variable x may not occur freely in any open assumption, on which $\varphi(x)$ depends; in $\exists E$, x may not occur freely in ψ nor in any open assumption on which ψ depends, except in $\varphi(x)$. (These variable conditions do not play an essential role in this paper. See [7] for a detailed explanation.)

Our inference rules for linear graphs are augmented to the above natural deduction system for the first-order logic. According to Plummer and Etchemendy [2], inference rules that are characteristic of heterogeneous systems combining formulas and diagrams are generally called *transfer rules*, and allow the transfer of information from one form of representation to another. Typical rules in Hyperproof [3, 1], which is one of the representative heterogeneous systems introduced by Barwise and Etchemendy, are apply (from formulas to a diagram) and observe (from a diagram to a formula). Our apply rule app has the following form:

$$\frac{G \quad \sigma(C) \quad w}{G^*} \ \mathsf{app}$$

where G^* is obtained from a given graph G by adding a new curve, say C^*, specified in the given shift formula $\sigma(C)$, and w specifies the ordering of widths in the resulting G^*. We interpret this rule as: "we apply $\sigma(C)$ and w to amplify G to G^*," or "we extend G to G^* by adding the new information of $\sigma(C)$ and w to G." Our app can only be applied when the ordering relation between shift widths in G^* is specified by the above w. Otherwise, the resulting G^* is not uniquely determined (under the equivalence of graphs).

Definition 3.7. Inference rules for linear graphs consist of the following app and obs.

app: Let G be a graph, that contains a curve C but does not contain C^*. Let $C \xrightarrow{\alpha} C^*$ be a shift formula. Let w be an ordering condition that specifies the ordering of widths in G^*:

$$\frac{\begin{array}{c}\vdots\\ G\end{array} \quad \begin{array}{c}\vdots\\ C \xrightarrow{\alpha} C^*\end{array} \quad \begin{array}{c}\vdots\\ w\end{array}}{G^*} \ \mathsf{app}$$

where G^* is obtained from G by adding the curve C^* so that (1) C^* is parallel to C and located to the right of C; (2) C^* is orthogonal to every curve that is orthogonal to C; (3) The width between C^* and C is α; (4) All widths including α satisfy w.

Similarly for $C^* \xleftarrow{\alpha} C$.

obs: From a given graph G, we can extract, as a conclusion, any ordering relation o contained in l_p and l_q in the type of G:

$$\frac{\begin{array}{c}\vdots\\ G\end{array}}{o}\ \textsf{obs}$$

In an application of app, we assume that there always exists an intersection between the additional curve and every orthogonal curve already appearing in the given premise graph. This is feasible, as we can always arrange the positions of curves by retaining the ordering relation among widths under the equivalence of graphs in our qualitative framework.

When w is not given, we cannot apply our app straightforwardly. In such a case, we enumerate all possible linear orderings of widths in G^* and apply the slightly generalized $\vee E$-rule of natural deduction as follows:

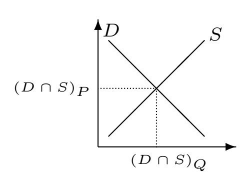
case:

$$\dfrac{\dfrac{}{(\alpha<\beta)\vee(\alpha=\beta)\vee(\alpha>\beta)}\,\textsf{ax} \quad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ G\end{array}\quad \begin{array}{c}\vdots\\ C \xrightarrow{\alpha} C'\end{array}\quad [\alpha<\beta]^m}{G_1}\,\textsf{app}\\ \vdots\\ G^*/\psi\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ G\end{array}\quad \begin{array}{c}\vdots\\ C \xrightarrow{\alpha} C'\end{array}\quad [\alpha=\beta]^m}{G_2}\,\textsf{app}\\ \vdots\\ G^*/\psi\end{array} \quad \begin{array}{c}\dfrac{\begin{array}{c}\vdots\\ G\end{array}\quad \begin{array}{c}\vdots\\ C \xrightarrow{\alpha} C'\end{array}\quad [\alpha>\beta]^m}{G_3}\,\textsf{app}\\ \vdots\\ G^*/\psi\end{array}}{G^*/\psi}\,\textsf{case}, m$$

where G^*/ψ denotes that either a graph G^* or a first-order formula ψ is obtained. Note that $(\alpha < \beta) \vee (\alpha = \beta) \vee (\alpha > \beta)$ is an axiom of real numbers, which is provable without any premise. By regarding the above part in a proof as an inference rule, we call it the rule of case. We sometimes omit the leftmost ax.

We usually omit the name of the inference rule in a proof to avoid visual complexity.

Remark 3.8.
In our system, we assume the following form of graphs as axioms, which is a graphical representation of the formula: there exist two linear functions $D(p) = -p + a$ and $S(p) = p + b$ for some a and b such that $(D \cap S)_P, (D \cap S)_Q > 0$.



Example 3.9. Fig. 1 is an example of a proof in our linear system, which describes Example 2.1.

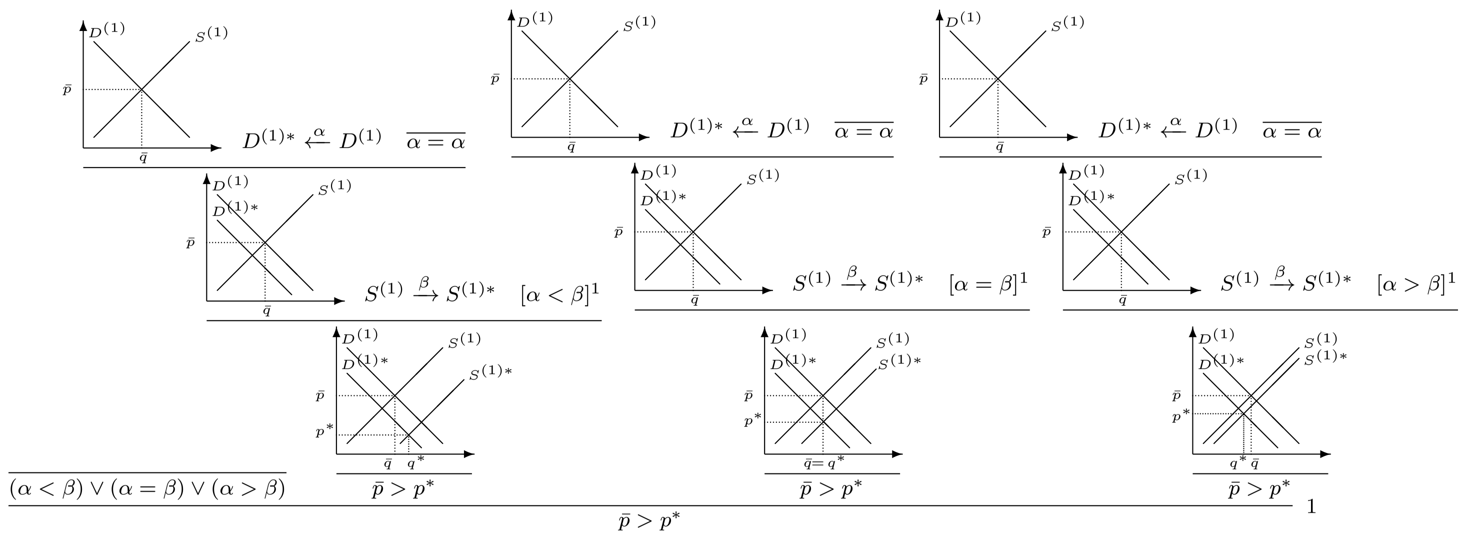


Fig. 1 A proof of $D^{(1)*} \xleftarrow{\alpha} D^{(1)}$, $S^{(1)} \xrightarrow{\beta} S^{(1)*} \models (D^{(1)} \cap S^{(1)})_P > (D^{(1)*} \cap S^{(1)*})_P$, where $(D^{(1)} \cap S^{(1)})_P$ (and $(D^{(1)*} \cap S^{(1)*})_P$) is denoted by $\bar{p}$ (resp. p^*).
Note that $(\alpha < \beta) \vee (\alpha = \beta) \vee (\alpha > \beta)$ and $\alpha = \alpha$ are axioms of real numbers.

Remark 3.10. One of the basic properties of formal logical systems is their completeness with respect to appropriate semantics, which states that the semantic notion of validity corresponds to the syntactic notion of validity, i.e., provability in the logical system. Our system is shown to be complete with respect to appropriate set-theoretical semantics for the real number system, such as an ordered real closed field, based on the type of a graph, i.e., symbolic specification of graphs. However, we do not explore the completeness property here, because it requires further tedious technical preparations, and it is not of primary concern in this article.

3.2. Abductive reasoning

In this section, we discuss the use of linear graphs in abductive reasoning, which is implied from Theorem 2.7. We first consider the following example similar to Example 2.1.

Example 3.11 (Abduction). When a new, faster computer chip is introduced, (1) demand for computers using the older, slower chips decreases (i.e., $D^* \xleftarrow{\alpha} D$). Simultaneously, (2) computer makers increase their production of computers containing the old chips in order to clear out their stocks of old chips (i.e., $S \xrightarrow{\beta} S^*$). When the equilibrium quantity falls in response to these events, what possible explanations are there for this change?

Let $(D \cap S)_Q = \bar{q}$ and $(D^* \cap S^*)_Q = q^*$. First, note that we cannot prove $\bar{q} > q^*$ under the given premises (1) and (2), as observed in Example 2.1. Thus, our task for this question is to find a possible explanation w, as well as particular curves D_1 and S_1, such that $D_1^* \xleftarrow{\alpha} D_1, S_1 \xrightarrow{\beta} S_1^*, w \models \bar{q} > q^*$ holds. When we consider linear graphs, the two given premises (1) and (2) provide the three linear graphs as depicted in Example 2.1 (and Fig. 1 in Example 3.9), according to whether $\alpha < \beta, \alpha = \beta$, or $\alpha > \beta$. Among those three graphs, we find a graph (the third one) in which $\bar{q} > q^*$ holds. Thus, we know that $\bar{q} > q^*$ holds when $\alpha > \beta$ holds for the shift widths of the linear demand and supply curves. Hence, we can propose $\alpha > \beta$ as a possible explanation w, as well as our linear demand and supply graph.

This type of reasoning is called *abduction*, and frequently appears in scientific reasoning. Abduction has been extensively studied in the literature on artificial intelligence AI. (See, for example, [26, 11] for surveys of abduction in AI.) In the framework of AI, abduction is usually formalized as the task of finding a hypothesis (or explanation) H that explains a given observation O under a theory (or set of premises) T such that O is a valid consequence of T and H, i.e., $T, H \models O$, and T, H are consistent. Usually, it is assumed that without H, we cannot prove O, i.e., $T \not\models O$. Furthermore, restrictions such as "minimality" are imposed on H so that it represents "the best explanation" of the given observation O.

Our abductive problem is slightly generalized from the usual one, because we would like to find not only w but also specific demand and supply curves.

Thus, our abductive problem can be formalized as follows: *Given an observation o, as well as premises $\sigma_1(D')$ and $\sigma_2(S')$ for some D' and S', are there an explanation w and D, S such that $\sigma_1(D), \sigma_2(S), w \models o$ holds, i.e., $\exists w \exists DS(\sigma_1(D), \sigma_2(S), w \models o)$ holds?*

Theorem 2.7 implies that our abductive problem can be reduced to that with linear graphs.

Corollary 3.12 (Abduction). *For any σ_1, σ_2, for any o,*

$$\exists w'(\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w' \models o) \quad \textit{if and only if} \quad \exists w \exists DS(\sigma_1(D), \sigma_2(S), w \models o)$$

Proof. The $\Rightarrow$-direction is easily obtained by abstracting the given $D^{(1)}$ and $S^{(1)}$ to the existence of D and S.

To show the $\Leftarrow$-direction, assume that $\exists DS(\sigma_1(D), \sigma_2(S), w \models o)$ for some w. Then, by Theorem 2.7, we have $\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w \models o$. Therefore, we conclude that $\exists w'(\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w' \models o)$. ∎

The above Corollary 3.12 can be divided into the following two claims:

1. If $\exists w'(\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w' \models o)$ then $\exists w \exists DS(\sigma_1(D), \sigma_2(S), w \models o)$. That is, if we have an explanation in linear graphs, then we already have a general explanation, which is not surprising.
2. If $\neg \exists w'(\sigma_1(D^{(1)}), \sigma_2(S^{(1)}), w' \models o)$ then $\neg \exists w \exists DS(\sigma_1(D), \sigma_2(S), w \models o)$. That is, if there is no explanation with linear graphs, then there is no explanation in general.

Theorem 2.3 ensures that whether or not an explanation is found in the linear system is determinate. Furthermore, from the above claim (2), when we find there is no explanation in the linear system, we can conclude that there is no explanation in general. Thus, based on the determinacy of the linear system, our abductive problem in general is determinable. This is one of the advantages of linear graphs. Furthermore, linear graphs make graphical explanations possible. (Although whether linear graphs provide "good explanations" is another matter.)

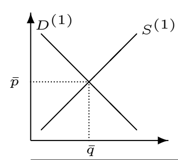
Let us investigate our strategy to solve abductive problems. For this purpose, resolution and proof-search to construct deductive proofs are usually applied. Our strategy in this paper is slightly different, and it can be considered as a kind of model enumeration. This is because our inference using graphs essentially corresponds to model construction by regarding our graph as a certain kind of representative model. When there is insufficient information on shift widths of supply and demand curves, we enumerate all possible cases (i.e., models). We can then determine the required explanation from among these cases, as seen in Example 3.11. To describe our abductive reasoning more formally, we modify our rule of `case` to `abcase` as follows.

abcase:

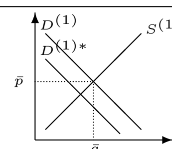
$$\dfrac{\overline{w_1 \vee w_2 \vee w_3}\ \mathsf{ax} \qquad \begin{array}{c} \dfrac{\begin{array}{c}\vdots\\ G\end{array}\quad \begin{array}{c}\vdots\\ \sigma(C)\end{array}\quad \underline{w_i}}{G_i}\ \mathsf{app} \\ \vdots \\ G^*/\psi \end{array}}{G^*/\psi}\ \mathsf{abcase}$$

where w_i $(i = 1, 2, 3)$ is one of $\alpha < \beta, \alpha = \beta, \alpha > \beta$, and the underlined $\underline{w_i}$ indicates a proposed explanation.

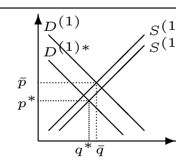
In contrast to case, which should be case exhaustive, it is not necessary to enumerate all cases of w_1, w_2, and w_3 by applying abcase, and it is sufficient to find a single case w_1, w_2, or w_3 in which the given conclusion (observation) holds. The above rule of abcase is not a sound rule, i.e., the conclusion G^*/ψ is not a logically valid conclusion of the given premises $G, \sigma(C)$, as we do not enumerate all possible cases of w_1, w_2, and w_3. However, abcase is useful to make our strategy for abductive problems clear by depicting our abductive reasoning in the usual proof tree form. Abductive reasoning in Example 3.11 is formally described by the following proof structure by using abcase.



$D^{(1)*} \xleftarrow{\alpha} D^{(1)}$ $\quad \overline{\alpha = \alpha}$ ax



$S^{(1)} \xrightarrow{\beta} S^{(1)*}$ $\quad \underline{\alpha > \beta}$



$$\dfrac{\overline{(\alpha < \beta) \vee (\alpha = \beta) \vee (\alpha > \beta)}\ \mathsf{ax} \qquad \overline{\bar{q} > q^*}}{\bar{q} > q^*}\ \mathsf{abcase}$$

Thus, we can propose $\alpha > \beta$, as well as the lowermost linear graph as our explanation. Note that if we regard the underlined $\alpha > \beta$ as a premise and delete the abcase-rule, we obtain a regal proof in our system.

We further examine another example, where the observation (conclusion) $\bar{q} > q^*$ of Example 3.11 is replaced by $(D \cap S)_P < (D^* \cap S^*)_P$, i.e., $\bar{p} < p^*$.

Example 3.13. Is there any explanation for given premises $D^* \xleftarrow{\alpha} D, S \xrightarrow{\beta} S^*$ and observation $(D \cap S)_P < (D^* \cap S^*)_P$?

As seen in the proof in Fig.1 in Example 3.9, $\neg((D \cap S)_P < (D^* \cap S^*)_P)$ is provable from the given premises. Thus, there is no explanation, and this is not restricted to the linear case, it is general by Corollary 3.12.

We formalize our procedure as follows.

1. By regarding given demand and supply functions as linear, we construct linear graphs by unifying given premises.
2. (a) When the given observation o is provable without abcase-rule, given premises are determinate. Thus, any w can be an explanation.
   (b) When the negation $\neg o$ of the given observation is provable without abcase-rule, the given premises are determinate. Thus, there is no explanation in general. (Cf. Example 3.13.)
   (c) When there is a proof structure of the given observation o with using abcase-rule, we can propose $\underline{w}, D^{(1)}, S^{(1)}$ in the proof structure as an explanation. (Cf. Example 3.11.)
   (There is no case where we cannot find w by allowing abcase-rue, because the linear system is determinate when w is given.)

4. Conclusion

We investigated an interaction between mathematical logic and economics. For this purpose, we focused on economic reasoning with linear demand and supply graphs. Linear graphs have the following advantages: (1) Linear graphs are easily constructed and reasoning using such graphs is effectively manageable (as discussed in Section 3.1); (2) Linear graphs are qualitatively determinate with respect to our problems, that is, which conclusion is qualitatively deduced from the given premises is uniquely determined (Theorem 2.3).

We showed in our qualitative framework that when information on the magnitude relationship between shift widths of given demand and supply functions is "not given," exactly the same logical consequences are derived in both cases of general and linear demand and supply functions (Theorem 2.5). Thus, linear demand and supply functions are useful in this fragment. However, when information on the magnitude relationship between shift widths is "given," something which is not derived in general is derived by linear functions because of the over-specificity (Proposition 2.6). In other words, the validity with linear demand and supply functions cannot be generalized to the validity with general functions. Thus, we need to pay careful attention in deductive reasoning with linear graphs, especially when information on the magnitude relationship between shift widths is given. However, we showed that the invalidity with linear demand and supply functions is generalized to the invalidity with general functions (Theorem 2.7).

Thus, based on Theorem 2.7, we investigated abductive reasoning in economics. Corollary 3.12 of the theorem shows that linear graphs are particularly effective in abductive reasoning. We also proposed a model of economic reasoning by formalizing economic reasoning with linear graphs in the framework of natural deduction developed in proof theory of mathematical logic.

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