Two Types of Diagrammatic Inference Systems: Relation-based System as Natural Deduction and Region-based System as Resolution *

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Abstract

Since the 1990s, reasoning with Venn and Euler diagrams has been studied from mathematical and logical viewpoints. The standard approach to a formalization is a "region-based" approach, where a diagram is defined as a set of regions. An alternative is a "relation-based" approach, where a diagram is defined in terms of topological relations (inclusion and exclusion) between circles and points. In this paper, we compare these two approaches from a proof-theoretical point of view. In general, diagrams correspond to formulas in symbolic logic, and diagram manipulations correspond to applications of inference rules in a certain logical system. From this perspective, we demonstrate the following correspondences. On the one hand, a diagram construed as a set of regions corresponds to a disjunctive normal form formula and the inference system based on such diagrams corresponds to a resolution calculus. On the other hand, a diagram construed as a set of topological relations corresponds to an implicational formula and the inference system based on such diagrams corresponds to a natural deduction system. Based on these correspondences, we discuss advantages and disadvantages of each framework.

Contents

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1 Introduction

Proof theory in logic has traditionally been developed based on linguistic (symbolic) representations of logical proofs. Recently, logical reasoning based on diagrammatic or graphical representations has been investigated by many logicians. In particular, Euler diagrams, introduced in the 18th century to illustrate syllogistic reasoning, began to be studied in the 1990s from a mathematical and formal logical viewpoint. However, until now, proof theory of Euler diagrams has not been that well developed.

In literature on diagrammatic reasoning, Euler diagrams have been formalized based on the method developed in the study on Venn diagrams. A Venn diagram is abstractly defined as a set of regions, where some of them may be shaded. In the same way, an Euler diagram is defined by considering shaded regions of a Venn diagram as "missing" regions. (E.g., Howse et al. [8]; for a survey, see Stapleton [21].) Thus, both Venn and Euler diagrams are abstractly defined in terms of regions, and hence we call this framework a "region-based" framework. Moreover, the inference rule of *unification*, which plays a central role in Euler diagrammatic reasoning, is defined by means of superpositions of Venn diagrams. The operation of superposition is uniformly defined for any two Venn diagrams that have the same circles as a simple union operation of shaded regions of the given diagrams. This uniformity of superposition produces an effectiveness that makes it easy to control theorem proving using diagrams, see, e.g. Stapleton et al. [22, 5].

Nevertheless, the superposition rule has some disadvantages. In particular, by making a detour to Venn diagrams, some redundant steps are introduced in formalizing simple processes employed in Euler diagrammatic reasoning. For example, in order to derive $\mathcal{E}$ of Fig. 1 from given diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$, they are first transformed into Venn diagrams $\mathcal{D}_1^v$ and $\mathcal{D}_2^v$ of Fig. 2, respectively; then, by superposing the shaded regions of $\mathcal{D}_1^v$ and $\mathcal{D}_2^v$, and by erasing the circle B, the Venn diagram $\mathcal{E}^v$ is obtained, which is transformed into the region-based Euler diagram $\mathcal{E}$. Thus, within the region-based framework, it is difficult to capture the appropriate notion of Euler diagrammatic proofs, in particular that of "normal diagrammatic proofs." Accordingly, the notion of proof normalization, which plays an essential role in proof theory, has not been developed to date for diagrammatic proofs.

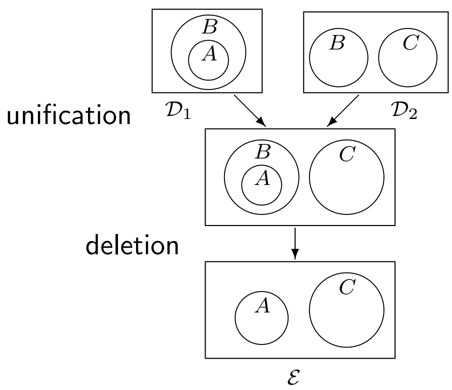


Fig. 1 Syllogism with Euler diagrams

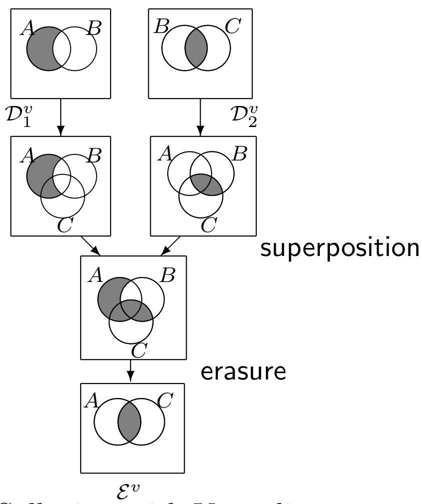


Fig. 2 Syllogism with Venn diagrams

In contrast to studies elaborated using this region-based framework, we introduced in

[12, 10] an Euler diagrammatic reasoning system, where diagrams are defined in terms of topological relations (inclusion and exclusion relations) between circles and points. We describe our approach as "relation-based." We formalized the unification of Euler diagrams in the manner developed in natural deduction, a well-known formalization of logical reasoning in traditional symbolic logic. Inference rules of Gentzen's natural deduction are intended to be as close as possible to actual reasoning ([6]). Along similar lines, our unification rule is designed to be as natural as possible to reflect intuitive manipulations of Euler diagrams as seen in Fig. 1.

In this paper, we discuss from a proof-theoretical viewpoint the following contrast between the two frameworks: At the representation level, diagrams abstractly defined in the relation-based framework correspond to implicational formulas, whereas diagrams defined in the region-based framework correspond to disjunctive normal form formulas. At the inference level, unification rules in the relation-based framework correspond to natural deduction style inference rules associated with the implicational connective, whereas inference rules in the region-based framework, in particular the erasure rule, correspond to the resolution principle. Thus, an inference system in the relation-based framework corresponds to a natural deduction system, and an inference system in the region-based framework corresponds to a resolution calculus. The contrast between the two frameworks is summarized in the following table:

		Relation-based framework (Euler diagrams)	Region-based framework (Venn diagrams)
Representation	Diag.	topological relations	regions with shading
	Ling.	*implicational formulas*	*disjunctive normal formulas*
Inference	Diag.	unification and deletion	superposition and erasure
	Ling.	*natural deduction*	*resolution calculus*

Table 1

These correspondences enable us to apply well-developed techniques used in traditional proof theory within symbolic logic in the field of diagrammatic reasoning. For our Euler diagrammatic inference system in the relation-based framework, we introduce the notion of a *normal diagrammatic proof*, i.e., a proof in which unification and deletion appear alternately. We show the normalization theorem of [12] by using the correspondence theorem between our Euler diagrammatic inference system and a natural deduction system. The normalization theorem is used to show that each chain of traditional Aristotelian categorical syllogisms corresponds to a normal diagrammatic proof in our system (cf. [10]).

The rest of this paper is organized as follows. In Section 2, we summarize the contrast between natural deduction and resolution. In Section 3, we show that our Euler diagrammatic inference system in the relation-based framework corresponds to a natural deduction system (Theorems 3.11 and 3.17). As a corollary, we show a normalization theorem for Euler diagrammatic proofs (Corollary 3.16). In Section 4, we show that a Venn diagrammatic inference system in the region-based framework corresponds to a resolution calculus (Theorems 4.3 and 4.4). In Section 5, we discuss advantages and disadvantages of the relation-based and the region-based frameworks.

2 Natural deduction and resolution

In this section, we briefly review natural deduction and resolution, by contrasting them in terms of the motivational aspects behind their formalizations. We concentrate on propositional fragments of these systems. For more detailed descriptions, see for example [16, 2] for natural deduction and [4, 2, 9] for resolution.

Natural deduction. Natural deduction was introduced by Gentzen [6], and studied extensively by Prawitz [16]. Natural deduction is one of the major logical inference systems for propositional and first-order logic in proof theory. Gentzen wrote of natural deduction "(Engl. Transl.:) ... I intended first to set up a formal system which comes as close as possible to actual reasoning." ([6, p.68].) Natural deduction is applied in various areas and not just in mathematical logic. Indeed, some cognitive psychologists in their studies on Mental Logic or Formal Rule Theory have admitted the naturalness of the logical inference rules, and adopted it as a theoretical basis for their studies (e.g. Rips [17], Braine-O'Brien [1]).

We denote atoms (propositional variables) by $A, B, C, \ldots$. Formulas are defined inductively as usual by using connectives $\land, \lor, \rightarrow, \neg, \bot$, and formulas are denoted by $\varphi, \psi, \theta, \ldots$. Sets of formulas are denoted by $\Gamma, \Delta, \Lambda, \ldots$. Inference rules of natural deduction have a particular property: Each inference rule is associated with a logical connective as follows.

$$\frac{\begin{matrix}\vdots & \vdots\\ \varphi & \psi\end{matrix}}{\varphi \land \psi}\land I \qquad \frac{\begin{matrix}\vdots\\ \varphi_1 \land \varphi_2\end{matrix}}{\varphi_i}\land E_{i\,(i=1,2)} \qquad \frac{\begin{matrix}\vdots\\ \varphi_i\end{matrix}}{\varphi_1 \lor \varphi_2}\lor I_{i\,(i=1,2)} \qquad \frac{\begin{matrix} & [\varphi]^n & [\psi]^n\\ \vdots & \vdots & \vdots\\ \varphi \lor \psi & \theta & \theta\end{matrix}}{\theta}\lor E, n$$

$$\frac{\begin{matrix}[\varphi]^n\\ \vdots\\ \psi\end{matrix}}{\varphi \rightarrow \psi}\rightarrow I, n \qquad \frac{\begin{matrix}\vdots & \vdots\\ \varphi & \varphi \rightarrow \psi\end{matrix}}{\psi}\rightarrow E \qquad \frac{\begin{matrix}[\varphi]^n\\ \vdots\\ \bot\end{matrix}}{\neg\varphi}\neg I, n \qquad \frac{\begin{matrix}\vdots & \vdots\\ \varphi & \neg\varphi\end{matrix}}{\bot}\neg E$$

Introduction (I) rules are set based on the operational meaning of each logical connective in question. Elimination (E) rules, which define the use of the corresponding connective, are then justifiable by the definition. For example, once we define the operational meaning of $\varphi \land \psi$ ("φ and ψ") as "φ is assertible and ψ is assertible," then the $\land I$ rule expresses the definition. Then the corresponding $\land E_i$ rules are justifiable as valid with this meaning of $\land$ (cf. [14]).

We denote natural deduction proofs by $\pi, \pi', \pi_1, \pi_2, \ldots$. In general, a natural deduction proof may contain some redundant inference steps and formulas called **maximal formulas**. For example, the formula $\varphi_1 \land \varphi_2$ and the pair of applications of $\land I$ and $\land E_1$ rules on the left in the following proof are redundant in the sense that without them we already have a proof π_1 of φ_1 as illustrated on the right.

$$\frac{\dfrac{\begin{matrix}\vdots\,\pi_1 & \vdots\,\pi_2\\ \varphi_1 & \varphi_2\end{matrix}}{\varphi_1 \land \varphi_2}\land I}{\varphi_1}\land E_1 \qquad\qquad \begin{matrix}\vdots\,\pi_1\\ \varphi_1\end{matrix}$$

A natural deduction proof is said to be in **normal form** when it does not contain any maximal formula. The following **normalization theorem** holds for the natural deduction system: *If φ is provable from Γ then there is a normal proof of φ from Γ.* [1] Normal proofs have an essential property called the **subformula property**: *A normal proof π of φ from Γ contains only subformulas of φ and Γ.* Normalization and normal proofs play a central role in the development of 20th century proof theory. Gentzen called the normalization theorem *Hauptsatz (Main Theorem)* since most of the proof-theoretical results depend on this theorem. [2] We also use the theorem to show the correspondence between natural deduction and our Euler diagrammatic inference system in Section 3.

The above introduction and elimination rules for the logical connectives, $\wedge$, $\vee$, $\rightarrow$, $\neg$, compose the natural deduction system for **minimal logic**, which is the most basic part of natural deduction. The system is extended to those for *intuitionistic logic* and for usual *classical logic* by introducing the following additional inference rules $\perp E$ (the absurdity rule for intuitionistic logic) and $\neg\neg E$ (the double-negation rule for classical logic), respectively:

$$\frac{\begin{array}{c}\vdots\\ \perp\end{array}}{\varphi}\ \perp E \qquad\qquad \frac{\begin{array}{c}\vdots\\ \neg\neg\varphi\end{array}}{\varphi}\ \neg\neg E$$

To determine whether or not a given formula is provable, we would try to construct a proof of the formula in the natural deduction system. However, such a proof-search procedure in natural deduction is complicated and not very efficient, since the way to handle assumptions is not well-suited to computer implementation (see [2]). In contrast, resolution, which we review in the next section, provides a more efficient procedure to decide provability of formulas.

Resolution. Propositional and first-order resolution were introduced by Robinson "for use as a basic theoretical instrument of the computer theorem-proving program" [18]. Thus, from the outset, it is machine-oriented rather than human-oriented as exemplified by natural deduction. Whereas natural deduction has many inference rules, two rules associated with each logical connective, there is only a single rule in the resolution calculus. The single rule, called the "resolution principle," is very powerful in that it forms by itself a complete system of propositional and first-order logic. The efficiency of the rule makes it easy to implement decision procedures involved in establishing the provability of given formulas, and today's most automated theorem-proving programs adopt essentially this principle. The resolution principle also plays an important role in the design of logic programming such as Prolog. (See, e.g. [2].) Resolution is defined over a particular class of formulas, called "clauses." A clause is essentially a *logic-free* structure in the sense that it consists only of atoms and their negation, with no other logical connectives explicitly occurring (see [9]). Although a clause is usually interpreted as a conjunctive normal form (CNF) formula, it can also be interpreted dually as being of disjunctive normal form (DNF) formula. This highlights a characteristic feature of resolution; the principle can be applied without giving a specific interpretation. We refer to [15, 2] for the following definition.

A **literal** is defined as either an atom A or its negation $\overline{A}$. In the context of resolution, we use here the "overbar" symbol for negation instead of the usual $\neg$. We denote literals by $L, M, N, \ldots$. If L is a literal of the form $\overline{A}$, then $\overline{L}$ denotes the unnegated literal A. A **clause**

[1] This theorem is sometimes called the **normal form theorem** to distinguish it from the following stronger form: *Any proof reduces to a normal proof.*

[2] Notions and properties related to normalization are more neatly formulated in Gentzen's *sequent calculus*, which is a refined system of natural deduction (cf. [6, 2]).

is a finite set of literals, and it is denoted by $x, y, z, \ldots$. When a clause x is $\{L_1, \ldots, L_n\}$, it is sometimes denoted as $L_1 \cdots L_n$. In particular, the empty-clause is denoted by $\Box$. A non-empty set of clauses is called a **clause set**, and it is denoted by $\Gamma, \Delta, \Sigma, \ldots$. Let x_1 and x_2 be clauses such that $L \in x_1$ and $\bar{L} \in x_2$. The **resolution principle** is defined as the following operation to derive the clause $(x_1 \setminus \{L\}) \cup (x_2 \setminus \{\bar{L}\})$, called the **resolvent** of x_1 and x_2:

$$\frac{x_1 \qquad\qquad x_2}{(x_1 \setminus \{L\}) \cup (x_2 \setminus \{\bar{L}\})}\ L$$

A **resolution derivation** is then defined as the process of deriving a clause x from a given clause set Γ by applying the resolution principle.

In the literature on automated theorem proving, many refined strategies to construct resolution derivations have been studied (see Chang-Lee [4]). Among them, we mention a naive strategy, called the Davis-Putnam procedure: For any ordering of atoms $A_1, A_2, \ldots, A_m$, a derivation is required to have a sequence of nested resolutions; i.e. first resolutions with respect to A_1, then resolutions with respect to A_2, etc., concluding with resolutions with respect to A_m. With this stipulation, we slightly extend the notion of derivability in our resolution calculus: Δ is derivable from Γ when, for any $x \in \Delta$, x is derivable from Γ.

When $x = \{L_1, \ldots, L_n\}$, we define $\wedge x = L_1 \wedge \cdots \wedge L_n$ and $\vee x = L_1 \vee \cdots \vee L_n$. In particular, $\wedge\Box = \top$ and $\vee\Box = \bot$. For any non-empty clause set $\Gamma = \{x_1, \ldots, x_m\}$, we define $d(\Gamma) = (\wedge x_1) \vee \cdots \vee (\wedge x_m)$, which is DNF, and $c(\Gamma) = (\vee x_1) \wedge \cdots \wedge (\vee x_m)$, which is CNF. Resolution is usually defined to be a *refutation procedure* that refutes the satisfiability of a clause set rather than a proof construction as found for natural deduction. The following completeness then holds: *For any non-empty clause set Γ, the empty clause $\Box$ is derivable from Γ if and only if $c(\Gamma)$ is unsatisfiable* (see, [4, 2]). There is another presentation of completeness, which is dual that given above: *The empty clause $\Box$ is derivable from Γ if and only if $d(\Gamma)$ is a tautology* (see [15]).

Let us summarize some advantages and disadvantages of natural deduction and resolution. In natural deduction, inference rules are defined over the full syntax of propositional/first-order logic and defined as simple as possible. In particular, the introduction rule of each connective can be regarded as the definition of the operational meaning of the connective, and the elimination rule, which defines the use of the connective, is justifiable by the definition. Thus, their operational meaning and validity can be immediately grasped by that definition. However, such rules cause some complications in their implementation.

In contrast, resolution works in essence only for clauses, which are formulas in conjunctive (dually, disjunctive) normal form. The logic-free structure of clauses enables us to formalize a complete system by using only the resolution principle. The effective single rule has the advantage of being amenable to implementation. However, such a refined rule composed of more primitive rules, and it is usually explained in terms of the inference rules of natural deduction (or sequent calculus), and hence its validity and operational meaning are not immediately grasped.

The contrast between the two systems has been summarized in the following Table 2:

	Natural deduction	Resolution
Motivation	human-oriented proof construction	machine-oriented refutation procedure
Formalization	full syntax of the first order logic rules for each logical connective simplicity of each rule normalization	restricted to clause (logic-free) single rule not primitive but effective rule
Implementation	difficult to implement	amenable to implementation

Table 2 The contrast between natural deduction and resolution

3 Relation-based framework and natural deduction

In this section, we show a correspondence between Euler diagrammatic inference system introduced in [12] and natural deduction.

3.1 Euler diagrammatic representation system EUL

We roughly review the definition of EUL-diagrams of [11, 12].

Definition 3.1 (EUL-diagrams) An EUL-*diagram* is a plane ($\mathbb{R}^2$) with a finite number, at least one, of *named simple closed curves* (denoted by $A, B, C, \dots$) and *named points* (denoted by $a, b, c, \dots$), where each named simple closed curve or named point has a unique and distinct name. EUL-diagrams are denoted by $\mathcal{D}, \mathcal{E}, \mathcal{D}_1, \mathcal{D}_2, \dots$. An EUL-diagram consisting of at most two objects is called a *minimal diagram.* Minimal diagrams are denoted by $\alpha, \beta, \gamma, \dots$.

In what follows, a named simple closed curve is called a *named circle.* Named circles and named points are collectively called *objects*, and denoted by $s, t, u, \dots$.

Definition 3.2 EUL-*relations* are the following binary relations:

$A \sqsubset B$	"the interior of A is *inside of* the interior of B,"
$A \vdash\!\dashv B$	"the interior of A is *outside of* the interior of B,"
$A \bowtie B$	"there is at least one *crossing* point between A and B,"
$b \sqsubset A$	"b is *inside of* the interior of A,"
$b \vdash\!\dashv A$	"b is *outside of* the interior of A,"
$a \vdash\!\dashv b$	"a is *outside of* b (i.e. a is not located at the point of b)."

EUL-relation $\sqsubset$ is reflexive asymmetric relation, and $\vdash\!\dashv$ and $\bowtie$ are irreflexive symmetric relations.

Proposition 3.3 *Let $\mathcal{D}$ be an* EUL*-diagram. For any distinct objects s and t of $\mathcal{D}$, exactly one of the* EUL*-relations $s \sqsubset t, t \sqsubset s, s \vdash\!\dashv t, s \bowtie t$ holds.*

Observe that, by Proposition 3.3, the set of EUL-relations holding on a given EUL-diagram $\mathcal{D}$ is uniquely determined. We denote the set by $\mathsf{rel}(\mathcal{D})$. We also denote by $pt(\mathcal{D})$ the set of named points of $\mathcal{D}$, by $cr(\mathcal{D})$ the set of named circles of $\mathcal{D}$, and by $ob(\mathcal{D})$ the set of objects of $\mathcal{D}$. As an illustration, for the diagram $\mathcal{D}_1$ of Fig. 3, we have $pt(\mathcal{D}_1) = \{a\}$, $cr(\mathcal{D}_1) = \{A, B, C\}$, and $\mathsf{rel}(\mathcal{D}_1) = \{A \bowtie B, A \bowtie C, B \bowtie C, a \vdash\!\dashv A, a \sqsubset B, a \vdash\!\dashv C\}$. In the description of a set of relations, we usually omit the reflexive relation $s \sqsubset s$ for each object s.

Definition 3.4 (Equivalence) When any two objects of the same name appear in different diagrams, we identify them up to isomorphism. Any EUL-diagrams $\mathcal{D}$ and $\mathcal{E}$ such that $ob(\mathcal{D}) = ob(\mathcal{E})$ are *syntactically equivalent* when $\mathsf{rel}(\mathcal{D}) = \mathsf{rel}(\mathcal{E})$.

Example 3.5 (Equivalence of diagrams) For example, diagrams $\mathcal{D}_1$, $\mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ of Fig. 3 are equivalent since $\mathsf{rel}(\mathcal{D}_1) = \mathsf{rel}(\mathcal{D}_2) = \mathsf{rel}(\mathcal{D}_3) = \mathsf{rel}(\mathcal{D}_4)$.

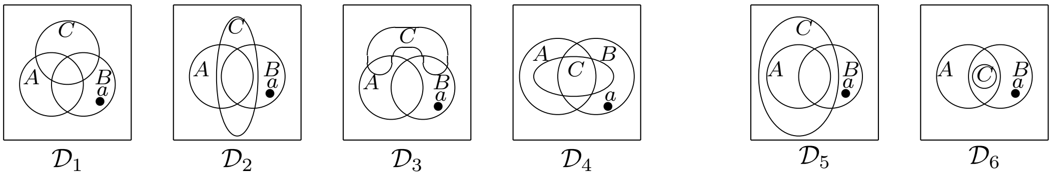


Fig. 3 Equivalence of EUL-diagrams.

On the other hand, although $\mathcal{D}_1$ and $\mathcal{D}_5$ (resp. $\mathcal{D}_1$ and $\mathcal{D}_6$) consists of the same objects, they are not equivalent since different EUL-relations hold on them: $A \sqsubset C$ holds on $\mathcal{D}_5$ in place of $A \bowtie C$ of $\mathcal{D}_1$ (resp. $C \sqsubset A$ and $C \sqsubset B$ hold on $\mathcal{D}_6$ in place of $A \bowtie C$ and $C \bowtie B$ of $\mathcal{D}_1$).

See [11], in which EUL is extended by introducing intersection, union, and complement regions, respectively, as diagrammatic objects, and $\mathcal{D}_1, \mathcal{D}_2$, $\mathcal{D}_3$, and $\mathcal{D}_4$ are distinguished.

In what follows, the diagrams which are syntactically equivalent are identified, and they are referred by a single name. Note that, given the equivalence of EUL-diagrams, there is a one-to-one correspondence between minimal diagrams and EUL-relations. Thus we also denote EUL-relations by $\alpha, \beta, \gamma, \ldots$.

Our semantics is distinct from usual ones, e.g., [7, 8] in that diagrams are interpreted in terms of binary relations. In order to interpret the EUL-relations $\sqsubset$ and $\vdash\!\dashv$ uniformly as the subset relation and the disjointness relation, respectively, we regard each point of EUL as a special circle which does not contain, nor cross, any other objects.

Definition 3.6 (Model) A *model* M is a pair (U, I), where U is a non-empty set (the domain of M), and I is an interpretation function which assigns to each named circle or point a non-empty subset of U such that $I(a)$ is a singleton for any named point a, and $I(a) \neq I(b)$ for any points a, b of distinct names.

Definition 3.7 (Truth conditions) Let $\mathcal{D}$ be an EUL-diagram. $M = (U, I)$ is a *model of* $\mathcal{D}$, written as $M \models \mathcal{D}$, if the following *truth-conditions* (1) and (2) hold: For all objects s, t of $\mathcal{D}$,
(1) $I(s) \subseteq I(t)$ if $s \sqsubset t$ holds on $\mathcal{D}$, and (2) $I(s) \cap I(t) = \emptyset$ if $s \vdash\!\dashv t$ holds on $\mathcal{D}$.

Remark 3.8 (Semantic interpretation of $\bowtie$-relation) By Definition 3.7, the EUL-relation $\bowtie$ does not contribute to the truth-condition of EUL-diagrams. Informally speaking, $s \bowtie t$ may be understood as $I(s) \cap I(t) = \emptyset$ *or* $I(s) \cap I(t) \neq \emptyset$, which is true in any model.

The semantic consequence relation, $\models$ between EUL-diagrams is defined as usual. (See [12] for a detailed description.)

3.2 Euler diagrammatic inference system GDS as natural deduction

In this section, we show that the diagrammatic inference system for EUL-diagrams, called the Generalized Diagrammatic Syllogistic inference system GDS of [12, 11], corresponds to the natural deduction system for disjunction-free minimal logic. We first give a translation of each EUL-diagram to a propositional implicational formula. In what follows, we consider the $\wedge$ connective as an n-ary connective for an appropriate n. Furthermore, we denote simply by a sequence (set) $\varphi_1, \ldots, \varphi_n$ a conjunction $\varphi_1 \wedge \cdots \wedge \varphi_n$, where we assume all conjuncts are distinct. We also generalize $\wedge I$ and $\wedge E$ rules of natural deduction to those for the n-ary $\wedge$ connective.

Definition 3.9 (Translation of EUL-diagrams) Each named circle or named point is translated into an atom. Then each EUL-relation α is translated into an implicational formula α° as follows:

$$(s \sqsubset t)^\circ := s \to t \qquad (s \vdash\!\dashv t)^\circ := s \to \neg t \qquad (s \bowtie t)^\circ := s \to s, t \to t$$

Let $\mathcal{D}$ be an EUL-diagram whose set of relations $\mathsf{rel}(\mathcal{D})$ is $\{\alpha_1, \ldots, \alpha_n\}$. The diagram $\mathcal{D}$ is translated into the conjunction $\mathcal{D}^\circ := \alpha_1^\circ, \ldots, \alpha_n^\circ$.

We next give a translation of inference rules of GDS [12, 11]. GDS consists of two inference rules: *unification* and *deletion*. Two kinds of constraint are imposed on unification. One is the *constraint for determinacy*, which blocks the disjunctive ambiguity with respect to locations of named points. For example, two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ in Fig. 4 are not permitted to be unified into one diagram since the location of the point c is not determined.

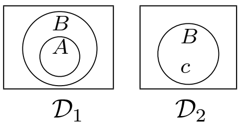


Fig. 4 Indeterminacy

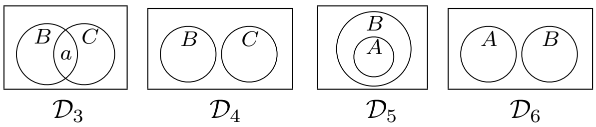


Fig. 5 Inconsistency

The other is the *constraint for consistency*, which blocks representing inconsistent information in a single diagram. For example, the diagrams $\mathcal{D}_3$ and $\mathcal{D}_4$ (resp. $\mathcal{D}_5$ and $\mathcal{D}_6$) in Fig. 5 are not permitted to be unified since they contradict each other based on the semantics of EUL. Unification rules of two diagrams are formalized by restricting one of the two diagrams to being a *minimal diagram*, except for one rule called the Point Insertion-rule. The restriction of unification makes the operational meaning of it clear. See also Appendix A for a more detailed description of inference rules of GDS. Our completeness ([12]) ensures that any diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ may be unified, under the constraints for determinacy and consistency, into one diagram whose semantic information is equivalent to the conjunction of those of $\mathcal{D}_1, \ldots, \mathcal{D}_n$. Inference rules are described in terms of EUL-relations: Given a diagram $\mathcal{D}$ and a minimal diagram α, the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ for the unified diagram $\mathcal{D} + \alpha$ is defined.

We give the definition of each inference rule of GDS, and in parallel with it, we give a translation of each rule into a combination of inference rules of natural deduction. Unification between $\mathcal{D}$ and α such that $\mathsf{rel}(\mathcal{D} + \alpha) = \mathsf{rel}(\mathcal{D}) \cup \{\gamma_1, \ldots, \gamma_n\}$ is translated schematically as Fig. 6:

In the following natural deduction proofs, by $(\varphi_n)_n$ we mean the set of formulas $\varphi_1, \ldots, \varphi_n$. Furthermore, for each formula φ_n, the repetition of the same inference steps is expressed as the skeleton of a proof as in Fig. 7:

$$\dfrac{\mathcal{D}^\circ \qquad \begin{array}{c}\mathcal{D}^\circ\ \alpha^\circ\\ \vdots\\ \gamma_1^\circ\end{array} \qquad \cdots \qquad \begin{array}{c}\mathcal{D}^\circ\ \alpha^\circ\\ \vdots\\ \gamma_n^\circ\end{array}}{\mathcal{D}^\circ, \gamma_1^\circ, \dots, \gamma_n^\circ}\ \land I$$

Fig. 6

$$\left(\begin{array}{c}\vdots\\ \varphi_n\end{array}\right)_n$$

Fig. 7

Definition 3.10 (Translation of GDS) Inference rules of GDS are translated as follows. We show here only U7 and U9 rules. See Appendix A for the remaining cases.

U7: If $A \vdash\!\dashv B$ holds on α and $A \in cr(\mathcal{D})$, and if $c \sqsubset A$ holds for all $c \in pt(\mathcal{D})$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \{C \vdash\!\dashv B \mid C \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{c \vdash\!\dashv B \mid c \in pt(\mathcal{D})\}$$
$$\cup \{C \bowtie B \mid A \sqsubset C \text{ or } A \vdash\!\dashv C \text{ or } A \bowtie C \in \mathsf{rel}(\mathcal{D})\} \cup \{B \sqsubset B\}$$

U7 is translated as follows:

$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{[C_n]^1 \quad \dfrac{\mathcal{D}^\circ}{C_n \to A}}{A} \quad A \to \neg B}{\neg B}}{C_n \to \neg B}\ 1\right)_n \quad \left(\dfrac{\dfrac{\dfrac{[c_m]^1 \quad \dfrac{\mathcal{D}^\circ}{c_m \to A}}{A} \quad A \to \neg B}{\neg B}}{c_m \to \neg B}\ 1\right)_m \quad \dfrac{[B]^1}{B \to B}\ 1}{\mathcal{D}^\circ, (C_n \to \neg B)_n, (c_m \to \neg B)_m, B \to B}$$

U9: If $A \sqsubset B$ holds on α and $A \bowtie B$ holds on $\mathcal{D}$, and if there is no object s such that $s \sqsubset A$ and $s \vdash\!\dashv B$ hold on $\mathcal{D}$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\big(\mathsf{rel}(\mathcal{D}) \setminus \{D \bowtie C \mid D \sqsubset A \text{ and } B \sqsubset C \in \mathsf{rel}(\mathcal{D})\} \setminus \{C \bowtie D \mid C \sqsubset A \text{ and } D \vdash\!\dashv B \in \mathsf{rel}(\mathcal{D})\}\big)$$
$$\cup \{D \sqsubset C \mid D \sqsubset A \text{ and } B \sqsubset C \in \mathsf{rel}(\mathcal{D})\} \cup \{C \vdash\!\dashv D \mid C \sqsubset A \text{ and } D \vdash\!\dashv B \in \mathsf{rel}(\mathcal{D})\}$$

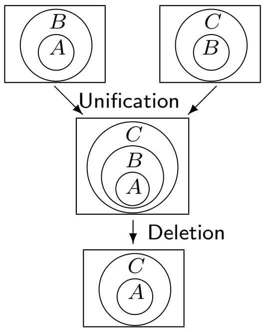
U9 is translated as follows:

$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{\dfrac{[D_n]^1 \quad \dfrac{\mathcal{D}^\circ}{D_n \to A}}{A} \quad A \to B}{B} \quad \dfrac{\mathcal{D}^\circ}{B \to C_m}}{C_m}}{D_n \to C_m}\ 1\right)_{n,m} \quad \left(\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[C_k]^2 \quad \dfrac{\mathcal{D}^\circ}{C_k \to A}}{A} \quad A \to B}{B} \quad \dfrac{[D_l]^1 \quad \dfrac{\mathcal{D}^\circ}{D_l \to \neg B}}{\neg B}}{\bot}}{\neg D_l}\ 1}{C_k \to \neg D_l}\ 2\right)_{k,l}}{\mathcal{D}^\circ, (D_n \to C_m)_{n,m}, (C_k \to \neg D_l)_{k,l}}$$

Definition 3.10 gives, by induction, a translation of any *diagrammatic proof* π of GDS into a natural deduction proof π°. Hence the following theorem is immediate:

Theorem 3.11 (Translation of GDS) *Let $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}$ be* EUL*-diagrams. If π is a diagrammatic proof of $\mathcal{E}$ from $\mathcal{D}_1, \dots, \mathcal{D}_n$ in* GDS*, then π° is a natural deduction proof of $\mathcal{E}^\circ$ from $\mathcal{D}_1^\circ, \dots, \mathcal{D}_n^\circ$.*

Example 3.12 (Barbara in GDS) The following diagrammatic proof on the left, which expresses the famous valid syllogism called *Barbara*, is translated by Definition 3.10 into the natural deduction proof on the right. For the sake of simplicity, we omit tautologies of the form $A \to A$ in the proof:



$$\dfrac{\dfrac{A \to B \quad B \to C \qquad \dfrac{\dfrac{\dfrac{[A]^1 \quad A \to B}{B}\to E \qquad B \to C}{C}\to E}{A \to C}\to I,1}{A \to B, B \to C, A \to C}\wedge I}{A \to C}\wedge E$$

Observe that for the simulation of GDS, we use only a particular class of inference rules: $\wedge I, \wedge E, \to I, \to E, \neg I, \neg E$, which form a subsystem of classical logic, i.e., minimal logic without disjunction. We denote the system as NM. Further note that the natural deduction proof in Example 3.12 above is not in normal form since it contains redundant steps: without applying $\wedge I$ and $\wedge E$ rules, we already have a proof of $A \to C$. By reducing the above proof, we obtain the following normal proof in Fig. 8:

$$\dfrac{\dfrac{\dfrac{[A]^1 \quad A \to B}{B}\to E \qquad B \to C}{C}\to E}{A \to C}\to I,1$$

Fig. 8 Normal proof for Barbara

$$\dfrac{A \to B \quad B \to C}{A \to C}\sqsubset$$

$$\dfrac{A \to B \quad B \to \neg C}{A \to \neg C}\vdash\!\!\dashv$$

Fig. 9 Derived rules

The above normal proof of Fig. 8 provides a derived rule called $\sqsubset$-rule in Fig. 9, and this rule makes explicit the correspondence between the natural deduction proof with $\sqsubset$-rule and the diagrammatic proof for Barbara.

Lemma 3.13 ($\sqsubset$ and $\vdash\!\!\dashv$ rules) *$\sqsubset$-rule and $\vdash\!\!\dashv$-rule of Fig. 9 are derived rules in disjunction-free minimal logic.*

Proposition 3.14 (Translation of NM) *Let $\vec{\mathcal{D}}$ be a set of EUL-diagrams which has a model. Let α be a minimal diagram. Any proof of α° from $\vec{\mathcal{D}}^\circ$ in disjunction-free minimal logic is transformed into a diagrammatic proof of α from $\vec{\mathcal{D}}$ in GDS.*

Proof. By the normalization theorem for disjunction-free minimal logic, any proof of α° from $\vec{\mathcal{D}}^\circ$ is transformed into a normal proof. Let π be such a normal proof. Note that, by the translation of EUL-diagrams, $\wedge$ and $\to$ connectives are completely separated without nesting of them, and hence, $\vec{\mathcal{D}}$ is decomposed into minimal diagrams by $\wedge E$-rule in any normal natural deduction proof of α°. Hence, we assume, without loss of generality, that $\vec{\mathcal{D}}$ is a set of minimal diagrams $\beta_1, \ldots, \beta_n$, which is denoted by $\vec{\beta}^\circ$.

Note that the assertion is immediate, when there exists $1 \le i \le n$ such that $\beta_i^\circ \equiv \alpha^\circ$, or when $\alpha^\circ \equiv s \to \neg t$ and there exists $1 \le i \le n$ such that $\beta_i^\circ \equiv t \to \neg s$. Otherwise, we divide the following two cases according to the form of α°.

(1) When $\alpha^\circ \equiv s \to t$, it is obtained by $\to I$-rule since π is in normal form. Note that by our translation of diagrams, an atomic formula, other than $\bot$, is obtained by only $\to E$-rule. Thus t is obtained by successive applications of $\to E$-rule, and then π has the following form:

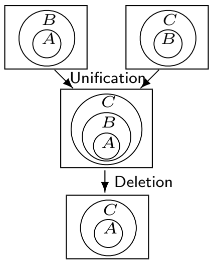
$$\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[s]^1 \quad s\to s_1}{s_1}{\scriptstyle \to E} \quad s_1\to s_2}{s_2}{\scriptstyle \to E}\\ \vdots \\ s_n\end{array} \quad s_n \to t}{t}{\scriptstyle \to E}}{s\to t}{\scriptstyle \to I,1}$$

where $\vdots$ means successive applications of $\to E$-rule. Note that formulas $s \to s_1$, $s_i \to s_{i+1}$, and $s_n \to t$ are open assumptions, since π is in normal form, in which they cannot be obtained by $\to I$-rule. Then π is transformed into the following π' by using $\sqsubset$-rule:

$$\dfrac{\begin{array}{c}\dfrac{s\to s_1 \quad s_1\to s_2}{s\to s_2}\sqsubset\\ \vdots\\ s\to s_n\end{array}\quad s_n\to t}{s\to t}\sqsubset$$

Observe that each application of $\sqsubset$-rule of the following form on the left is translated into the diagrammatic proof on the right by using a pair of Unification and Deletion rules:

$$\dfrac{A\to B \quad B\to C}{A\to C}\sqsubset$$



Thus π' is translated into a diagrammatic proof of α from $\vec{\beta}$.

(2) When $\alpha^\circ \equiv s \to \neg t$, it is obtained by $\to I$-rule since π is in normal form. Note that $\neg t$ may be obtained by $\to E$ or $\neg I$ rule. In case it is obtained by $\to E$-rule, π has a similar form to the case (1). On the other hand, if $\neg t$ is obtained by $\neg I$-rule, it is shown that π has the following form:

$$\dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\dfrac{[s]^2 \quad s\to s_1}{s_1}{\scriptstyle\to E}\quad s_1\to s_2}{s_2}{\scriptstyle\to E}\\ \vdots\\ u\end{array} \quad \begin{array}{c}\dfrac{\dfrac{[t]^1 \quad t\to\varphi_1}{\varphi_1}{\scriptstyle\to E}\quad \varphi_1\to\varphi_2}{\varphi_2}{\scriptstyle\to E}\\ \vdots\\ \neg u\end{array}}{\bot}{\scriptstyle\neg E}}{\neg t}{\scriptstyle\neg I,1}}{s\to\neg t}{\scriptstyle\to I,2}$$

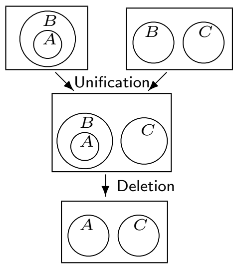
where $\vdots$ means successive applications of $\to E$-rule, and each φ_j is an atom or its negation. Note that, by the presupposition of the semantic consistency of $\vec{\beta}$, i.e., it has a model, there is no case where $s \equiv t$ and they are closed at the same time by the application of either $\neg I$ or $\to I$ (in such a case, by $\to I$ or $\neg I$, respectively, the empty assumption is closed).

Furtherer note that, since π is in normal form, $\neg u$ cannot be obtained by $\neg I$-rule, and hence, $\neg u$ is obtained by $\rightarrow E$-rule. Thus π has the above form, and then it is transformed into the following π' by using $\vdash\!\dashv$-rule and $\sqsubset$-rule:

$$\dfrac{\dfrac{\dfrac{s \rightarrow s_1 \quad s_1 \rightarrow s_2}{s \rightarrow s_2}\,\sqsubset}{\vdots \atop s \rightarrow u} \qquad \dfrac{\dfrac{t \rightarrow \varphi_1 \quad \varphi_1 \rightarrow \varphi_2}{t \rightarrow \varphi_2}\,\sqsubset / \vdash\!\dashv}{\vdots \atop t \rightarrow \neg u}}{s \rightarrow \neg t}\,\vdash\!\dashv$$

Observe that each application of $\vdash\!\dashv$-rule of the following form on the left is translated into the diagrammatic proof on the right by using a pair of Unification and Deletion rules:

$$\dfrac{A \rightarrow B \quad B \rightarrow \neg C}{A \rightarrow \neg C}\,\vdash\!\dashv$$



Thus π' is translated into a diagrammatic proof of α from $\vec{\beta}$. ■

Note that when premises and the conclusion are restricted to being minimal diagrams in Proposition 3.14, a diagrammatic proof obtained from the above translation has the following particular form, cf. Example 3.12:

Definition 3.15 ($\pm$-normal form) A diagrammatic proof π is in $\pm$-*normal form* if a unification (+) and a deletion ($-$) appear alternately in π.

Thus Proposition 3.14 gives a normalization theorem of GDS:

Corollary 3.16 ($\pm$-normal form) *Let $\vec{\beta}$ be a set of minimal diagrams which has a model. Let α be a minimal diagram. If α is provable from $\vec{\beta}$ in* GDS*, then there is a $\pm$-normal diagrammatic proof of α from $\vec{\beta}$.*

See [12] for a semantic proof of the theorem. This $\pm$-normal form is important because it serves to show the correspondence between diagrammatic proofs and chains of Aristotelian categorical syllogisms. (Cf. [10].)

By applying the construction given in [12] of a general diagram $\mathcal{E}$ from minimal diagrams obtained from $\mathcal{E}$, Proposition 3.14 is naturally extended to the general case, where the conclusion is not restricted to be minimal:

Theorem 3.17 (Translation of NM) *Let $\vec{\mathcal{D}}$ be a set of* EUL*-diagrams which has a model. Let $\mathcal{E}$ be an* EUL*-diagram. Any proof of $\mathcal{E}^\circ$ from $\vec{\mathcal{D}}^\circ$ in disjunction-free minimal logic is transformed into a diagrammatic proof of $\mathcal{E}$ from $\vec{\mathcal{D}}$ in* GDS*.*

Proof. By the normalization theorem for minimal logic, any proof of $\mathcal{E}^\circ$ is transformed into a normal proof of $\mathcal{E}^\circ$. Let π be such a normal proof of $\mathcal{E}^\circ$ from $\vec{\mathcal{D}}^\circ$. Let $\mathcal{E}^\circ$ be $\alpha_1^\circ, \ldots, \alpha_m^\circ$. Then $\mathcal{E}^\circ$ is obtained by $\wedge E$ or $\wedge I$ rule.

(1) When $\alpha_1^\circ, \ldots, \alpha_m^\circ$ is obtained by applying $\wedge E$ to $\alpha_1^\circ, \ldots, \alpha_m^\circ, \Gamma$, since π is in normal form, it is again obtained by $\wedge E$ rule. In this way, π consists of successive applications of $\wedge E$ to some assumption $\mathcal{D}_i^\circ \in \vec{\mathcal{D}}^\circ$. Thus π is transformed into a diagrammatic proof of $\mathcal{E}$ from $\vec{\mathcal{D}}$ in GDS, where $\mathcal{E}$ is derived from $\mathcal{D}_i$ by successive applications of Deletion rule.

(2) When $\mathcal{E}^\circ$ is obtained by an application of $\wedge I$-rule, without loss of generality, we assume that it is obtained by a single application of $\wedge I$-rule, and that π is in the following form:

$$\dfrac{\begin{array}{c}\vec{\mathcal{D}}^\circ\\ \vdots\ \pi_1\\ \alpha_1^\circ\end{array} \quad \cdots \quad \begin{array}{c}\vec{\mathcal{D}}^\circ\\ \vdots\ \pi_m\\ \alpha_m^\circ\end{array}}{\alpha_1^\circ, \ldots, \alpha_m^\circ} \wedge I$$

Then, by Proposition 3.14, each proof π_i of α_i° for $1 \leq i \leq m$ is transformed into a diagrammatic proof of α_i° in GDS. Note that, by the presupposition that $\mathcal{E}^\circ$ is the translation of an EUL-diagram, the last application of $\wedge I$-rule is translated into some applications of Unification of GDS. A concrete procedure to transform the $\wedge I$-rule is given in the completeness proof of GDS in [12].

Thus, we have a d-proof of $\mathcal{E}$ from $\vec{\mathcal{D}}$ in GDS. ■

4 Region-based framework and resolution calculus

In this section, we show a correspondence between a Venn diagrammatic inference system and the resolution calculus.

4.1 Venn diagrammatic representation system

We briefly recall the syntax of Venn diagrams. See, e.g., [8, 20] for detailed descriptions. We define Venn diagrams in terms of shaded regions. Euler diagrams in the region-based framework, called *Euler diagrams with shading*, are obtained by considering some shaded regions of Venn diagrams as "missing" regions.

A concrete Venn diagram consists of finite numbers of named circles (simple closed curves) on a plane enclosed by a boundary rectangle which satisfies the *partial-overlapping* condition, i.e., all possible intersections of circles must occur. A zone or minimal region is a connected component of the complement of the contour set, which may be shaded. Independently of a concrete plane diagram, an abstract Venn diagram is defined in terms of names of circles as follows: Let $\mathcal{L}$ be the finite set of the names of circles for a diagram $\mathcal{V}$. An **abstract zone** (or **minimal region**) z is defined as $z = (in(z), out(z))$, where $in(z)$ and $out(z)$ are finite subsets of $\mathcal{L}$ such that $in(z) \cap out(z) = \emptyset$ and $in(z) \cup out(z) = \mathcal{L}$. Let $zone(\mathcal{V})$ be the set of zones such that $\{in(z) \mid z \in zone(\mathcal{V})\}$ is the power set $\mathcal{P}(\mathcal{L})$ of $\mathcal{L}$. Then an **abstract Venn diagram** is defined as the set $shade(\mathcal{V})$ of shaded zones which is a subset of $zone(\mathcal{V})$. When $shade(\mathcal{V}) = \emptyset$, $\mathcal{V}$ is called a *primary diagram*. We denote Venn diagrams by $\mathcal{V}, \mathcal{W}, \ldots$.

A **model** of Venn diagrams is a pair $M = (U, I)$ where U is a non-empty set called the universe, and I is an interpretation function which assigns to each circle a subset of U. The interpretation function I is naturally extended to interpret zones as follows: For any zone $z = (in(z), out(z))$ the interpretation $I(z)$ of a zone is defined by $I(z) = \bigcap_{X \in in(z)} I(X) \cap \bigcap_{Y \in out(z)} \overline{I(Y)}$, where $\overline{I(Y)}$ is the complement of $I(Y)$. $M = (U, I)$ is a model of a Venn diagram $\mathcal{V}$, denoted as $M \models \mathcal{V}$, if each shaded zone is interpreted as the empty set, i.e., $\bigcup_{z \in shade(\mathcal{V})} I(z) = \emptyset$. Cf. [8, 20].

In a similar way as EUL-diagrams, points (with linking) in Venn diagrams can be considered as special circles. For example, the Venn diagram on the left in Fig. 10, which contains two occurrences of a point c and a linking of them, can be replaced by the diagram on the right:

Fig. 10 Points as circles in Venn diagrams

Note that each point expresses the existence of an object, and hence, when a point as a circle is completely shaded, it expresses the contradiction.

4.2 Venn diagrammatic inference system as resolution calculus

In this section, we show that a Venn diagrammatic inference system corresponds to the resolution calculus over clauses. We first describe a translation of Venn diagrams into sets of clauses.

Definition 4.1 (Translation of Venn diagrams) Let $\mathcal{V}$ be a Venn diagram whose set of shaded zones $shade(\mathcal{V})$ is $\{z_1, \dots, z_n\}$.

- Each shaded zone $z_i = (\{A_1, \dots, A_j\}, \{B_1, \dots, B_k\})$ is interpreted as a clause $z_i^* = \{A_1, \dots, A_j, \overline{B_1}, \cdots, \overline{B_k}\}$, which is abbreviated as $A_1 \cdots A_j \overline{B_1} \cdots \overline{B_k}$.
- The Venn diagram $\mathcal{V}$ is translated into the set of clauses $\mathcal{V}^* = \{z_1^*, \dots, z_n^*\}$.

  In particular, the completely shaded diagram without any circle is translated as $\{\Box\}$, and each primary diagram is translated as $\emptyset$.

In what follows, we denote by $\vec{L}$ a set of literals, i.e., a clause, $L_1 \cdots L_n$ for an appropriate n. For a clause z, we denote by $|z|$ the set of atoms which appear in z. For a given clause set Γ, let $|\Gamma| = \bigcup\{|z| \mid z \in \Gamma\}$.

For example, the Venn diagram on the right in Fig. 10 is translated into a set of clauses $\{ABc, \overline{A}\overline{B}c\}$. For the interpretation of Venn diagrams, a clause, i.e., a zone of a Venn diagram, say $z = \overline{A}\overline{B}C$, is interpreted as the conjunctive formula $\wedge z = \overline{A} \wedge \overline{B} \wedge C$; then a clause set, i.e., a Venn diagram, say $\mathcal{V} = \{ABC, \overline{A}\overline{B}C\}$, is interpreted as the disjunctive normal form (DNF) formula $d(\mathcal{V}) = (A \wedge B \wedge C) \vee (\overline{A} \wedge \overline{B} \wedge C)$. Cf. the semantics of Venn diagrams [8, 20].

Based on the above translation of Venn diagrams into sets of clauses, inference rules for Venn diagrams are described as the following inference rules over clauses. We call the system VR. In particular, any primary diagram is an axiom. See, e.g. [8, 20] for formal descriptions of Venn diagrammatic inference rules.

Definition 4.2 (Translation of VR) Inference rules for Venn diagrams are translated as follows:

Introduction of a circle: A new circle A may be added to a diagram observing the partial-overlapping rule, i.e., each zone splits into two zones with the introduction of A. If the zone is shaded, then both corresponding new zones are shaded.

Let $\mathcal{V}$ be a Venn diagram such that $\mathcal{V}^* = \{\vec{L}_1, \ldots, \vec{L}_n\}$ and $A \notin |\vec{L}_i|$:

$$\frac{\{\vec{L}_1, \ldots, \vec{L}_n\}}{\{A\vec{L}_1, \ldots, A\vec{L}_n,\ \overline{A}\vec{L}_1, \ldots, \overline{A}\vec{L}_n\}}\ Intro, A$$

Superposition of diagrams (Combining diagrams): Two diagrams that have the same circles may be combined into a diagram whose semantic information is equivalent to the conjunction of those of the original diagrams. Shaded zones in the combined diagram are shaded in one (or both) of the original diagrams.

Let $\mathcal{V}_1^*$ be $\{\vec{L}_1, \ldots, \vec{L}_n\}$ and $\mathcal{V}_2^*$ be $\{\vec{M}_1, \ldots, \vec{M}_m\}$ such that $|\mathcal{V}_1^*| = |\mathcal{V}_2^*|$:

$$\frac{\{\vec{L}_1, \ldots, \vec{L}_n\} \quad \{\vec{M}_1, \ldots, \vec{M}_m\}}{\{\vec{L}_1, \ldots, \vec{L}_n\} \cup \{\vec{M}_1, \ldots, \vec{M}_m\}}\ Sup$$

Erasure of shaded zones: Any shaded zones may be erased from a diagram.

Let $\mathcal{V}$ be a Venn diagram such that $\mathcal{V}^* = \{\vec{L}_1, \ldots, \vec{L}_n, \vec{L}_{n+1}, \ldots, \vec{L}_m\}$:

$$\frac{\{\vec{L}_1, \ldots, \vec{L}_n, \vec{L}_{n+1}, \ldots, \vec{L}_m\}}{\{\vec{L}_1, \ldots, \vec{L}_n\}}\ ErS$$

Erasure of a circle: A circle A may be erased from a diagram so that any shading remaining in only a part of a zone should also be erased.

Let $\mathcal{V}$ be a Venn diagram such that $\mathcal{V}^* = \{A\vec{L}_1, \ldots, A\vec{L}_n,\ \overline{A}\vec{M}_1, \ldots, \overline{A}\vec{M}_m\}$:

$$\frac{\{A\vec{L}_1, \ldots, A\vec{L}_n,\ \overline{A}\vec{M}_1, \ldots, \overline{A}\vec{M}_m\}}{\{\vec{L}_i\vec{M}_j \mid \neg\exists L(L \in \vec{L}_i \ and\ \overline{L} \in \vec{M}_j)\}}\ Er, A$$

We show that VR is simulated by the resolution calculus. In the literature on automated theorem proving, many refined strategies to construct resolution derivations have been studied (see Chang-Lee [4]). Among them, we mention a naive strategy, called the Davis-Putnam procedure: For any ordering of atoms $A_1, A_2, \ldots, A_m$, a derivation is required to have a sequence of nested resolutions; i.e. first resolutions with respect to A_1, then resolutions with respect to A_2, etc., concluding with resolutions with respect to A_m. With this stipulation, we slightly extend the notion of derivability in our resolution calculus: Δ is derivable from Γ when, for any $x \in \Delta$, x is derivable from Γ.

For any clause sets Γ and Δ, we denote $\Gamma \simeq \Delta$ when they are semantically equivalent, that is, $I(d(\Gamma)) = I(d(\Delta))$ for any model $M = (U, I)$.

Theorem 4.3 (Translation of VR) *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n, \mathcal{V}$ be Venn diagrams. If $\mathcal{V}$ is derivable from $\mathcal{V}_1, \ldots, \mathcal{V}_n$ in* VR*, then there is a Venn diagram $\mathcal{W}$ such that $\mathcal{V}_1^* \cup \cdots \cup \mathcal{V}_n^* \simeq \mathcal{W}^*$ and $\mathcal{V}^*$ is derivable from $\mathcal{W}^*$ in the resolution calculus.*

Proof. Note that there is a derivation of $\mathcal{V}$ such that applications of $Er(S)$ rule are delayed, in the sense that successive applications of *Intro* and *Sup* rules are followed by successive applications of $Er(S)$ rule. Thus, there is a Venn diagram $\mathcal{W}$ which consists of all circles contained in $\mathcal{V}_1, \ldots, \mathcal{V}_n$ (cf. Shin's maximal diagram [20]). Observe that $Intro, A$ rule is described by the following semantic equations:

$$\vec{L}_1 \vee \cdots \vee \vec{L}_n = (\vec{L}_1 \vee \cdots \vee \vec{L}_n) \wedge (A \vee \overline{A}) = A\vec{L}_1 \vee \cdots \vee A\vec{L}_n \vee \overline{A}\vec{L}_1 \vee \cdots \vee \overline{A}\vec{L}_n$$

This means that when $\mathcal{W}_2$ is obtained from $\mathcal{W}_1$ by *Intro* rule, $\mathcal{W}_1 \simeq \mathcal{W}_2$. Thus, since *Sup* rule is simply the union operation, we have $\mathcal{V}_1^* \cup \cdots \cup \mathcal{V}_n^* \simeq \mathcal{W}^*$.

Furthermore, note that applications of ErS rule are further delayed, i.e., each ErS rule is applied in the last part of the derivation. We assume without loss of generality that no primary diagram is derived by $Er(S)$ rule, since it is an axiom. Then Er, A rule corresponds to the following applications of resolution with respect to A:

$$\frac{A\vec{L}_i \quad \overline{A}\vec{M}_j}{\vec{L}_i\vec{M}_j}\ A$$

Thus, for any $z^* \in \mathcal{V}^*$, z^* is derivable from $\mathcal{W}^*$ in the resolution calculus. ■

The above proof shows that VR is a resolution calculus over *full* DNF, i.e., a DNF such that each of its atoms appears exactly once in all conjuncts.

The converse of Theorem 4.4 also holds.

Theorem 4.4 (Translation of resolution) *Let $\mathcal{V}_1, \ldots, \mathcal{V}_n, \mathcal{V}$ be Venn diagrams. If $\mathcal{V}^*$ is derivable from $\mathcal{V}_1^* \cup \cdots \cup \mathcal{V}_n^*$ in the resolution calculus, then $\mathcal{V}$ is derivable from $\mathcal{V}_1, \ldots, \mathcal{V}_n$ in the Venn diagrammatic inference system* VR.

Proof. Let us consider the following resolution with respect to A on the left:

$$\frac{AL \quad \overline{A}M}{LM}\ A \qquad\qquad \cfrac{\cfrac{\cfrac{\{AL\}}{\{ALM, AL\overline{M}\}}\ Intro, M \qquad \cfrac{\{\overline{A}M\}}{\{\overline{A}ML, \overline{A}M\overline{L}\}}\ Intro, L}{\{ALM, AL\overline{M}, \overline{A}ML, \overline{A}M\overline{L}\}}\ Sup}{\{LM\}}\ Er, A$$

It is shown by induction (the base case is obtained by using ErS rule) that there are Venn diagrams which correspond to AL and $\overline{A}M$, respectively. Then the resolution is simulated by the above derivation on the right in VR. The same applies to the general case in which L and M are extended to sequences of literals $\vec{L}$ and $\vec{M}$. Thus there is a derivation of each zones $z \in \mathcal{V}$ in VR. It is obvious that they are superposed into the Venn diagram $\mathcal{V}$. ■

5 Discussion and future work

We showed that the Euler diagrammatic inference system GDS, which is formalized in the relation-based framework, corresponds to the natural deduction system for disjunction-free minimal logic (Theorems 3.11 and 3.17), and that the Venn diagrammatic inference system VR, which is formalized in the region-based framework, corresponds to the resolution calculus

(Theorems 4.3 and 4.4). These correspondences highlight both advantages and disadvantages of the two frameworks. On the one hand, inference rules in the relation-based framework provide a natural notion of normal diagrammatic proofs, which neatly characterizes the notion of linguistic chains of syllogisms; but the drawback is that they may cause some complications in their implementation. On the other hand, inference rules in the region-based framework are easy to control so that they are utilized to automated theorem proving; but such resolution-style rules do not provide an appropriate notion of diagrammatic proofs. (See also [19] for our cognitive psychological experiments, which compare Euler and Venn diagrams in actual human reasoning assignments.)

Furthermore, with these correspondences, there arises the possibility of applying well-developed techniques in traditional proof theory within symbolic logic in the field of diagrammatic reasoning. Firstly, it is well-known that the Curry-Howard correspondence between natural deduction and typed λ-calculus provides correspondences between formulas and types, between proofs and λ-terms, and between reduction steps of proofs and λ-terms (see [2]). Under the correspondence between natural deduction and GDS, then, we can explore computational aspects of Euler diagrammatic proofs by defining their syntactic rewriting procedure in GDS. Secondly, in studies on resolution for theorem proving within symbolic logic, many effective strategies are developed (see, e.g. [2, 4]). We may apply such strategies for the study of theorem proving using Venn diagrams. Thirdly, complexity of proofs in resolution calculus has been very well studied, e.g. [3]. Such results may be applied to the complexity analysis on Venn diagrammatic proofs.

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A Diagrammatic inference system GDS

In this appendix, we review Generalized Diagrammatic Syllogistic inference system GDS of [12, 11], which consists of two inference rules: *unification* and *deletion*. We also give the full description of Definition 3.10, i.e., the translation of each inference rule of GDS into a combination of inference rules of natural deduction.

In order to motivate our definition of *unification*, let us consider the following question: Given the following diagrams $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{D}_3$ of Fig. 11, what diagrammatic information on A, B and c can be obtained? (In what follows, in order to avoid notational complexity in a diagram, we express each named point, say $\overset{c}{\bullet}$, simply by its name c.) Fig. 11 represents a way of solving the question.

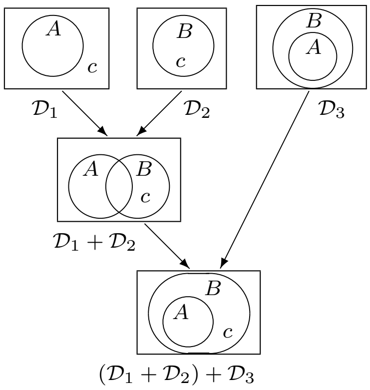


Fig. 11 Unification

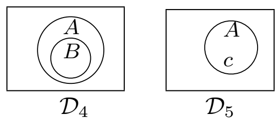


Fig. 12 Indeterminacy

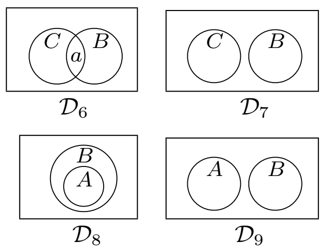


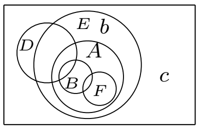
Fig. 13 Inconsistency

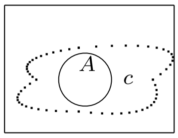
In Fig. 11, at the first step, two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ are unified to obtain $\mathcal{D}_1 + \mathcal{D}_2$, where point c in $\mathcal{D}_1$ and $\mathcal{D}_2$ are identified, and B is added to $\mathcal{D}_1$ so that c is inside of B and B overlaps with A without any implication of a relationship between A and B. We formalize such cases, where two given diagrams share one object, by U1–U8 rules of group (I) of Definition A.1. At the second step, $\mathcal{D}_1 + \mathcal{D}_2$ is combined with another diagram $\mathcal{D}_3$ to obtain $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. Note that the diagrams $\mathcal{D}_1 + \mathcal{D}_2$ and $\mathcal{D}_3$ share two circles A and B: $A \bowtie B$ holds on $\mathcal{D}_1 + \mathcal{D}_2$ and $A \sqsubset B$ holds on $\mathcal{D}_3$. Since the semantic information of $A \sqsubset B$ on $\mathcal{D}_3$ is more accurate than that of $A \bowtie B$ on $\mathcal{D}_1 + \mathcal{D}_2$, according to our semantics of EUL (recall that $A \bowtie B$ means just "true" in our semantics), one keeps the relation $A \sqsubset B$ in the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$. We formalize such cases, where two given diagrams share two objects, by U9–U10 rules of group (II) of Definition A.1. Observe that the unified diagram $(\mathcal{D}_1 + \mathcal{D}_2) + \mathcal{D}_3$ of Fig. 11 represents the information of these diagrams $\mathcal{D}_1, \mathcal{D}_2$, and $\mathcal{D}_3$, that is, their conjunction.

We impose two kinds of constraints on unification. One is the *constraint for determinacy*, which blocks the disjunctive ambiguity with respect to locations of named points. For example, two diagrams $\mathcal{D}_4$ and $\mathcal{D}_5$ in Fig. 12 are not permitted to be unified into one diagram since the location of the point c is not determined (it can be inside B or outside B). The other is the *constraint for consistency*, which blocks representing inconsistent information in a single diagram. For example, the diagrams $\mathcal{D}_6$ and $\mathcal{D}_7$ (resp. $\mathcal{D}_8$ and $\mathcal{D}_9$) in Fig. 13 are not permitted to be unified since they contradict each other. Recall that each circle is interpreted by non-empty set in our semantics of Definition 3.6, and hence $\mathcal{D}_8$ and $\mathcal{D}_9$ are also incompatible. [3]

We formalize our unification [4] of two diagrams by restricting one of them to be a *minimal diagram*, except for one rule called the Point Insertion-rule. Our completeness (Theorem A.2) ensures that any diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$ may be unified, under the constraints for determinacy and consistency, into one diagram whose semantic information is equivalent to the conjunction of that of $\mathcal{D}_1, \ldots, \mathcal{D}_n$. We give a formal description of inference rules in terms of EUL-relations: Given a diagram $\mathcal{D}$ and a minimal diagram α, the set of relations $\mathsf{rel}(\mathcal{D} + \alpha)$ for the unified diagram $\mathcal{D}+\alpha$ is defined. It is easily checked that according to our constraints for determinacy, locations of points are determined in a unified diagram. (See also [11], where we give a graph-theoretical representation of unification.)

For a better understanding of our unification rule, we also give a schematic diagrammatic representation and a concrete example of each rule. In the schematic representation of diagrams, to indicate the occurrence of some objects in a context on a diagram, we write the indicated objects explicitly and indicate the context by "dots" as in the diagram to the right below. For example, when we need to indicate only A and c on the left hand diagram, we could write it as shown on the right.





Definition A.1 Axiom, **unification**, and **deletion** of GDS are defined as follows.

Axiom:

A1: For any circles A and B, any minimal diagram where $A \bowtie B$ holds is an axiom.

A2: Any EUL-diagram which consists only of points is an axiom.

Unification: We denote by $\mathcal{D}+\alpha$ the unified diagram of $\mathcal{D}$ with a minimal diagram α. $\mathcal{D}+\alpha$ is defined when $\mathcal{D}$ and α share one or two objects. We distinguish the following two cases: (I) When $\mathcal{D}$ and α share one object, they may be unified to $\mathcal{D}+\alpha$ by rules U1–U8 according to the shared object and the relation holding on α. Each rule of (I) has a constraint for determinacy. (II) When $\mathcal{D}$ and α share two circles, if the relation which holds on α also holds on $\mathcal{D}$, $\mathcal{D}+\alpha$ is $\mathcal{D}$ itself; otherwise, they may be unified to $\mathcal{D}+\alpha$ by rules U9 or U10 according to the relation holding on α. Each rule of (II) has a constraint for consistency. Moreover, there is another unification rule called the Point Insertion-rule (III).

(I) The case $\mathcal{D}$ and α share one object:

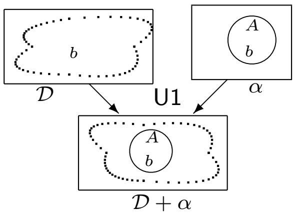
[3] In place of our syntactic constraint, it is possible to allow unification of inconsistent diagrams such as $\mathcal{D}_6$ and $\mathcal{D}_7$ (resp. $\mathcal{D}_8$ and $\mathcal{D}_9$) by extending GDS with an inference rule corresponding to the absurdity rule of Gentzen's natural deduction system: We can infer any diagram from a pair of inconsistent diagrams. (For natural deduction systems, see, for example, [6, 16].) However, such a rule requires linguistic symbol, say $\perp$, or some arbitrary convention to represent inconsistency, and hence we prefer our syntactic constraint in our framework of a diagrammatic inference system.

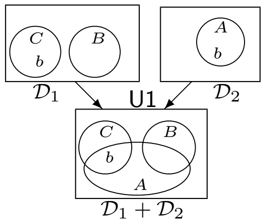
[4] The following definition of inference rules of GDS is slightly different from that of [12, 10] since we regard $\sqsubset$-relation as reflexive relation in this paper.

U1: If $b \sqsubset A$ holds on α and $pt(\mathcal{D}) = \{b\}$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \sqsubset A\} \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\} \cup \{A \sqsubset A\}$$

U1 is applied as follows:





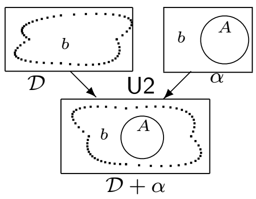
U1 is translated as follows:

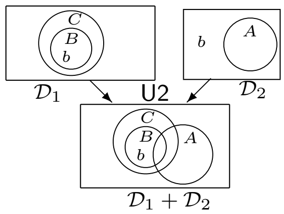
$$\dfrac{\mathcal{D}^\circ \quad b \to A \quad \dfrac{[A]^1}{A \to A}\,1}{\mathcal{D}^\circ, b \to A, A \to A}\,\wedge I$$

U2: If $b \vdash\!\dashv A$ holds on α and $pt(\mathcal{D}) = \{b\}$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \vdash\!\dashv A\} \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\} \cup \{A \sqsubset A\}$$

U2 is applied as follows:





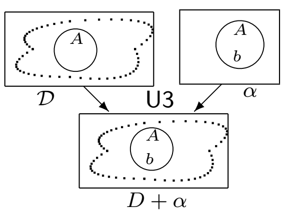
U2 is translated as follows:

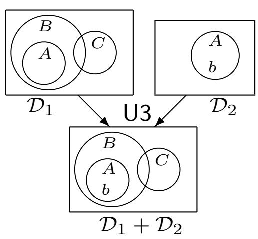
$$\dfrac{\mathcal{D}^\circ \quad b \to \neg A \quad \dfrac{[A]^1}{A \to A}\,1}{\mathcal{D}^\circ, b \to \neg A, A \to A}\,\wedge I$$

U3: If $b \sqsubset A$ holds on α and $A \in cr(\mathcal{D})$, and if $A \sqsubset X$ or $A \vdash\!\dashv X$ holds for all circle X in $\mathcal{D}$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \sqsubset X \mid A \sqsubset X \in \mathsf{rel}(\mathcal{D})\} \cup \{b \vdash\!\dashv X \mid A \vdash\!\dashv X \in \mathsf{rel}(\mathcal{D})\}$$
$$\cup \{b \vdash\!\dashv x \mid x \in pt(\mathcal{D})\} \cup \{b \sqsubset b\}$$

U3 is applied as follows:





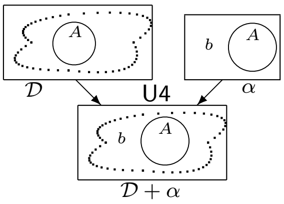
U3 is translated as follows:

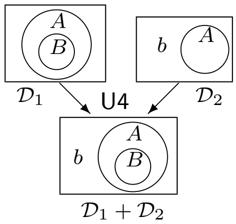
$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{[b]^1 \quad b \to A}{A} \quad \dfrac{\mathcal{D}^\circ}{A \to X_n}}{X_n}}{b \to X_n}\,1\right)_n \quad \left(\dfrac{\dfrac{\dfrac{[b]^1 \quad b \to A}{A} \quad \dfrac{\mathcal{D}^\circ}{A \to \neg Y_m}}{\neg Y_m}}{b \to \neg Y_m}\,1\right)_m \quad \dfrac{[b]^1}{b \to b}\,1}{\mathcal{D}^\circ, (b \to X_n)_n, (b \to \neg Y_m)_m, b \to b}$$

U4: If $b \vDash A$ holds on α and $A \in cr(\mathcal{D})$, and if $X \sqsubset A$ holds for all circle X in $\mathcal{D}$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \{b \vDash X \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{b \vDash x \mid x \in pt(\mathcal{D})\} \cup \{b \sqsubset b\}$$

U4 is applied as follows:





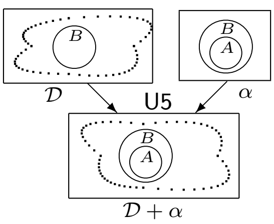
U4 is translated as follows:

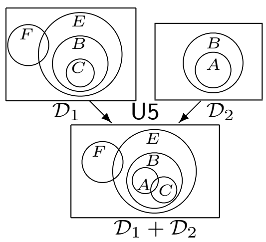
$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{\dfrac{[X_n]^1 \quad \dfrac{\mathcal{D}^\circ}{X_n \to A}}{A} \quad \dfrac{[b]^2 \quad b \to \neg A}{\neg A}}{\bot}}{\neg X_n}\,1}{b \to \neg X_n}\,2\right)_n \quad \dfrac{[b]^1}{b \to b}\,1}{\mathcal{D}^\circ, (b \to \neg X_n)_n, b \to b}$$

U5: If $A \sqsubset B$ holds on α and $B \in cr(\mathcal{D})$, and if $c \vDash B$ holds for all $c \in pt(\mathcal{D})$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \{A \sqsubset C \mid B \sqsubset C \in \mathsf{rel}(\mathcal{D})\} \cup \{A \bowtie C \mid C \sqsubset B \text{ or } C \bowtie B \in \mathsf{rel}(\mathcal{D})\}$$
$$\cup \{A \vDash C \mid C \vDash B \in \mathsf{rel}(\mathcal{D})\} \cup \{c \vDash A \mid c \in pt(\mathcal{D})\} \cup \{A \sqsubset A\}$$

U5 is applied as follows:





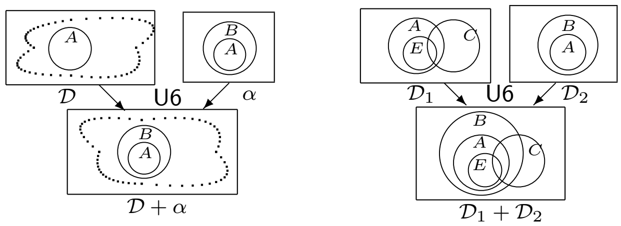
U5 is translated as follows:

$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{[A]^1 \quad A \to B}{B} \quad \dfrac{\mathcal{D}^\circ}{B \to C_n}}{C_n}}{A \to C_n}\,1\right)_n \quad \left(\dfrac{\dfrac{\dfrac{\dfrac{[A]^2 \quad A \to B}{B} \quad \dfrac{[D_m]^1 \quad \dfrac{\mathcal{D}^\circ}{D_m \to \neg B}}{\neg B}}{\bot}}{\neg D_m}\,1}{A \to \neg D_m}\,2\right)_m \quad \left(\dfrac{\dfrac{\dfrac{\dfrac{[A]^1 \quad A \to B}{B} \quad \dfrac{[c_l]^2 \quad \dfrac{\mathcal{D}^\circ}{c_l \to \neg B}}{\neg B}}{\bot}}{\neg A}\,1}{c_l \to \neg A}\right)_l \quad \dfrac{[A]^1}{A \to A}\,1}{\mathcal{D}^\circ, (A \to C_n)_n, (A \to \neg D_m)_m, (c_l \to \neg A)_l, A \to A}$$

U6: If $A \sqsubset B$ holds on α and $A \in cr(\mathcal{D})$, and if $x \sqsubset A$ holds for all $x \in pt(\mathcal{D})$, then $\mathsf{rel}(\mathcal{D}+\alpha)$ is defined as follows:

$$\begin{aligned}&\mathsf{rel}(\mathcal{D}) \cup \{X \sqsubset B \mid X \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{x \sqsubset B \mid x \in pt(\mathcal{D})\}\\ &\quad \cup \{X \bowtie B \mid A \sqsubset X \text{ or } A \vDash X \text{ or } A \bowtie X \in \mathsf{rel}(\mathcal{D})\} \cup \{B \sqsubset B\}\end{aligned}$$

U6 is applied as follows:



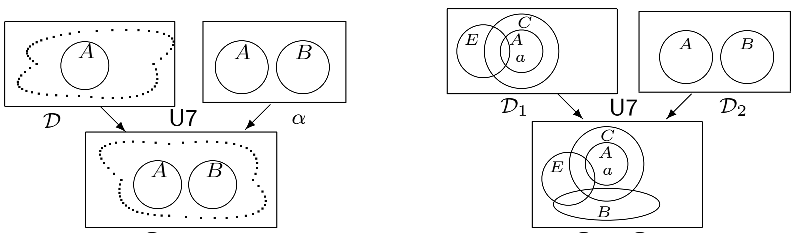
U6 is translated as follows:

$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{[X_n]^1 \quad \dfrac{\mathcal{D}^\circ}{X_n \to A}}{A} \quad A \to B}{B}}{X_n \to B}\,1\right)_n \quad \left(\dfrac{\dfrac{\dfrac{[x_m]^1 \quad \dfrac{\mathcal{D}^\circ}{x_m \to A}}{A} \quad A \to B}{B}}{x_m \to B}\,1\right)_m \quad \dfrac{[B]^1}{B \to B}\,1}{\mathcal{D}^\circ, (X_n \to B)_n, (x_m \to B)_m, B \to B}$$

U7: If $A \vDash B$ holds on α and $A \in cr(\mathcal{D})$, and if $c \sqsubset A$ holds for all $c \in pt(\mathcal{D})$, then $\mathsf{rel}(\mathcal{D}+\alpha)$ is defined as follows:

$$\begin{aligned}&\mathsf{rel}(\mathcal{D}) \cup \{C \vDash B \mid C \sqsubset A \in \mathsf{rel}(\mathcal{D})\} \cup \{c \vDash B \mid c \in pt(\mathcal{D})\}\\ &\quad \cup \{C \bowtie B \mid A \sqsubset C \text{ or } A \vDash C \text{ or } A \bowtie C \in \mathsf{rel}(\mathcal{D})\} \cup \{B \sqsubset B\}\end{aligned}$$

U7 is applied as follows:



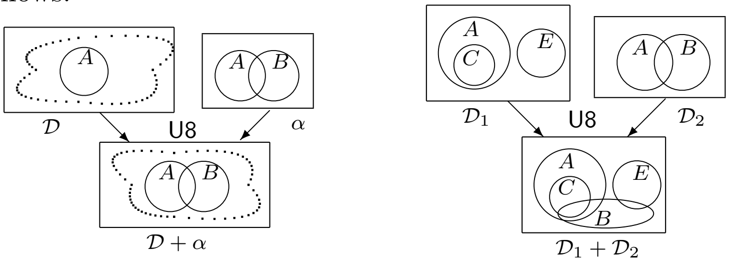
U7 is translated as follows:

$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{[C_n]^1 \quad \dfrac{\mathcal{D}^\circ}{C_n \to A}}{A} \quad A \to \neg B}{\neg B}}{C_n \to \neg B}\ 1\right)_n \quad \left(\dfrac{\dfrac{\dfrac{[c_m]^1 \quad \dfrac{\mathcal{D}^\circ}{c_m \to A}}{A} \quad A \to \neg B}{\neg B}}{c_m \to \neg B}\ 1\right)_m \quad \dfrac{[B]^1}{B \to B}\ 1}{\mathcal{D}^\circ, (C_n \to \neg B)_n, (c_m \to \neg B)_m, B \to B}$$

U8: If $A \bowtie B$ holds on α and $A \in cr(\mathcal{D})$, and if $pt(\mathcal{D}) = \emptyset$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}) \cup \{X \bowtie B \mid X \in cr(\mathcal{D})\} \cup \{B \sqsubset B\}$$

U8 is applied as follows:



U8 is translated as follows:

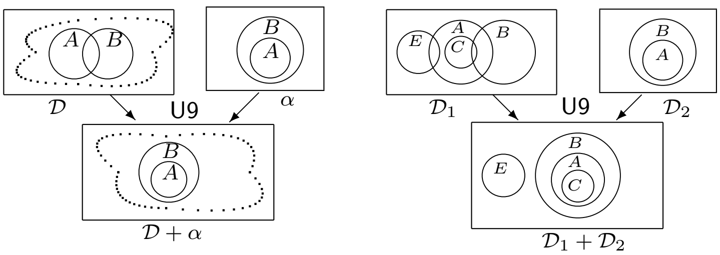
$$\dfrac{\mathcal{D}^\circ \quad \dfrac{[B]^1}{B \to B}\ 1}{\mathcal{D}^\circ, B \to B}$$

(II) When $\mathcal{D}$ and α share two circles, they may be unified to $\mathcal{D} + \alpha$ by the following U9 and U10 rules.

U9: If $A \sqsubset B$ holds on α and $A \bowtie B$ holds on $\mathcal{D}$, and if there is no object s such that $s \sqsubset A$ and $s \vDash B$ hold on $\mathcal{D}$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\begin{aligned}&\big(\mathsf{rel}(\mathcal{D}) \setminus \{D \bowtie C \mid D \sqsubset A \text{ and } B \sqsubset C \in \mathsf{rel}(\mathcal{D})\} \setminus \{C \bowtie D \mid C \sqsubset A \text{ and } D \vDash B \in \mathsf{rel}(\mathcal{D})\}\big)\\ &\quad \cup \{D \sqsubset C \mid D \sqsubset A \text{ and } B \sqsubset C \in \mathsf{rel}(\mathcal{D})\} \cup \{C \vDash D \mid C \sqsubset A \text{ and } D \vDash B \in \mathsf{rel}(\mathcal{D})\}\end{aligned}$$

U9 is applied as follows:



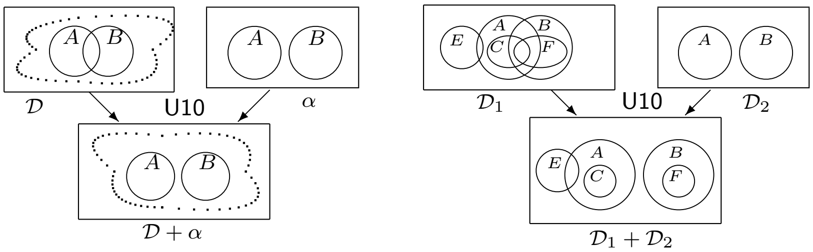
U9 is translated as follows:

$$\dfrac{\mathcal{D}^\circ \quad \left(\dfrac{\dfrac{\dfrac{\dfrac{[D_n]^1 \quad \dfrac{\mathcal{D}^\circ}{D_n \to A}}{A} \quad A \to B}{B} \quad \dfrac{\mathcal{D}^\circ}{B \to C_m}}{C_m}}{D_n \to C_m}\ 1\right)_{n,m} \quad \left(\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[C_k]^2 \quad \dfrac{\mathcal{D}^\circ}{C_k \to A}}{A} \quad A \to B}{B} \quad \dfrac{[D_l]^1 \quad \dfrac{\mathcal{D}^\circ}{D_l \to \neg B}}{\neg B}}{\bot}}{\neg D_l}\ 1}{C_k \to \neg D_l}\ 2\right)_{k,l}}{\mathcal{D}^\circ, (D_n \to C_m)_{n,m}, (C_k \to \neg D_l)_{k,l}}$$

U10: If $A \vdash\!\dashv B$ holds on α and $A \bowtie B$ holds on $\mathcal{D}$, and if there is no object s such that $s \sqsubset A$ and $s \sqsubset B$ hold on $\mathcal{D}$, then $\mathsf{rel}(\mathcal{D} + \alpha)$ is defined as follows:

$$\big(\mathsf{rel}(\mathcal{D}) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}\big) \cup \{X \vdash\!\dashv Y \mid X \sqsubset A \text{ and } Y \sqsubset B \in \mathsf{rel}(\mathcal{D})\}$$

U10 is applied as follows:



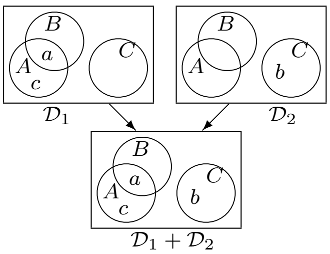
U10 is translated as follows:

$$\dfrac{\mathcal{D}^\circ \qquad \left(\dfrac{\dfrac{\dfrac{\dfrac{[Y_m]^1 \quad \dfrac{\mathcal{D}^\circ}{Y_m \to B}}{B} \qquad \dfrac{\dfrac{[X_n]^2 \quad \dfrac{\mathcal{D}^\circ}{X_n \to A}}{A} \qquad A \to \neg B}{\neg B}}{\bot}}{\neg Y_m}\,1}{X_n \to \neg Y_m}\,2 \right)_{n,m}}{\mathcal{D}^\circ, (X_n \to \neg Y_m)_{n,m}}$$

(III) Point Insertion: If, for any circles X, Y and for any $\Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}$, $X \Box Y \in \mathsf{rel}(\mathcal{D}_1)$ *iff* $X \Box Y \in \mathsf{rel}(\mathcal{D}_2)$ holds, and if $pt(\mathcal{D}_2)$ is a singleton $\{b\}$ such that $b \notin pt(\mathcal{D}_1)$, then $\mathsf{rel}(\mathcal{D}_1 + \mathcal{D}_2)$ is defined as follows:

$$\mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{b \vdash\!\dashv x \mid x \in pt(\mathcal{D}_1)\}$$

Point Insertion is applied as follows:



Point Insertion is translated as follows:

$$\frac{\mathcal{D}_1^\circ \quad \mathcal{D}_2^\circ}{\mathcal{D}_1^\circ, \mathcal{D}_2^\circ}\ \wedge I$$

Deletion: When t is an object of $\mathcal{D}$, t may be deleted from $\mathcal{D}$ to obtain a diagram $\mathcal{D} - t$ under the constraint that $\mathcal{D} - t$ has at least one objects.

$$\mathsf{rel}(\mathcal{D}) \setminus \{t \Box s \mid s \in ob(\mathcal{D}), \Box \in \{\sqsubset, \sqsupset, \vdash\!\dashv, \bowtie\}\}$$

Deletion rule is translated as follows:

$$\frac{\mathcal{D}^{\circ}}{\mathcal{D}^{\circ} \setminus \{\varphi \mid \varphi \text{ is an implicational formula which contains } t\}} \wedge E$$

The notion of *diagrammatic proofs (or, d-proofs)* is defined inductively as tree structures consisting of unification and deletion steps. The provability relation between EUL-diagrams is defined as usual. We denote by $\boldsymbol{\mathcal{D}}$ a sequence of diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$.

Theorem A.2 (Soundness and completeness of GDS **[12])** *Let $\boldsymbol{\mathcal{D}}, \mathcal{E}$ be* EUL*-diagrams, and let $\boldsymbol{\mathcal{D}}$ have a model. $\mathcal{E}$ is a semantically valid consequence of $\boldsymbol{\mathcal{D}}$ ($\boldsymbol{\mathcal{D}} \models \mathcal{E}$), if, and only if, there is a d-proof of $\mathcal{E}$ from $\boldsymbol{\mathcal{D}}$ ($\boldsymbol{\mathcal{D}} \vdash \mathcal{E}$) in* GDS.