Remarks on Semantic Completeness for Proof-terms with Laird's Dual Affine/Intuitionistic λ-calculus

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Abstract. The purpose of this note is to give a demonstration of the completeness theorem of type assignment system for λ-terms of [Hindley 83] and [Coquand 05] with two directions of slight extensions. Firstly, using the idea of [Okada 96], [Okada-Terui 99] and [Hermant-Okada 07], we extend their completeness theorem to a stronger form which implies a normal form theorem. Secondly, we extend the simple type (the implicational fragment of intuitionistic logic) framework of [Hindley 83] and [Coquand 05] to a linear (affine) types (the $\{\multimap, \&, \rightarrow\}$-fragment of affine logic) framework of [Laird 03, 05].

1 Introduction

Using the traditional Tarskian or Kripke model, Hindley [1983] and Coquand [2005] showed completeness at the level of "proof-terms" (λ-terms), in place of the traditional completeness for "provability," with the implicational fragment of intuitionistic logic. Compared with some recent works on full completeness and full abstraction employing Scott-Plotkin's denotational semantics or game semantics (e.g., [O'Hearn-Riecke 95], [Hyland-Ong 00]), their completeness proofs can be considered natural extensions of traditional completeness proofs for provability. On the other hand, in our previous work [Okada 96, 02] and [Okada-Terui 99], we remarked that a slight change in the phase semantic completeness proof of [Girard 87] leads to the cut-elimination theorem (or, existence theorem of a normal proof) for "provability." In this note we combine these two ideas to show that a slight change in the setting of Hindley-Coquand's semantic completeness proof for "proof-terms" leads to the normal form theorem for "proof-terms." We demonstrate this for Laird's dual affine/intuitionistic λ-calculus.

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Girard proved, in his completeness proof in [Girard 87] at the level of provability, a lemma of the form

$$A^* = [\![A]\!],$$

where A^* (called the inner-value of A in [Okada 02]) is the interpretation of a formula A in the canonical model, and $[\![A]\!] = \{\Gamma \mid \Gamma \vdash A \text{ is provable}\}$ (called the outer-value). We remarked in [Okada 96], [Okada-Terui 99] (cf. also [Hermant-Okada 07] for higher order cases) that the cut-elimination theorem can be obtained by changing this lemma slightly into the following form:

$$A \in A^* \subseteq [\![A]\!],$$

where $[\![A]\!] = \{\Gamma \mid \Gamma \vdash A \ \text{ is provable without the cut-rule }\}$. Indeed, the completeness proofs for "proof-terms" in [Hindley 83] and [Coquand 05] use the same form $A^* = [\![A]\!]$ as in [Girard 87], where A^* and $[\![A]\!]$ are adapted to a semantics for "proof-terms." We shall remark in this note that a modification of the lemma similar to [Okada 96, 02] and [Okada-Terui 99] can show the normal form theorems for "proof-terms," with a suitable change in the definition of $[\![A]\!]$.

As was mentioned above, we also extend completeness of [Hindley 83] and [Coquand 05] for the simple type (the implicational fragment of intuitionistic logic) framework to a linear (affine) types (the $\{\multimap, \&, \rightarrow\}$-fragment of affine logic) framework. In order to do this, we consider dual affine/intuitionistic λ-calculus (λ_{Aff}) of [Laird 05, 03], which is a fragment of Dual Intuitionistic Linear Logic (**DILL**) of Barber and Plotkin (see [Barber 96]). Based on the linear types of the forms $A \multimap B$ and $A \,\&\, B$, λ_{Aff} has a part of the exponential as the intuitionistic function type of the form $A \rightarrow B$, which corresponds to $!A \multimap B$ with the exponential !. (Hasegawa [2002b] also considers a similar system in the framework of classical logic.) In [Laird 05, 03], λ_{Aff} is introduced as a target language of Continuation-Passing-Style (CPS) translation of simple types of the call-by-value λ-calculus. Introducing game semantics for λ_{Aff}, Laird gives a semantic analysis of λ_{Aff} and CPS translation. CPS translation is developed as an evaluation-order independent λ-encoding of λ-terms (see [Plotkin 75], [Fischer 93] etc.). Although the standard call-by-value CPS translation has been shown to be equationally sound and complete, it is not *full*: there are inhabitants of the interpreted types which are not in the image of the transformation. (See [Hasegawa 02a].) Using game semantics, Laird showed the fullness of CPS translation by adopting λ_{Aff} as the target language.

Our completeness proof for "proof-terms" could be understood as a variant of the well-known Tait-Girard's computability/reducibility argument of proof-terms normalization.

The rest of this note is organized as follows. In Section 2, we mainly concern ourselves with the level of "provability" of Laird's dual affine/intuitionistic λ-calculus (λ_{Aff}) ([Laird 05, 03]) and its phase semantics. In Section 3, we first review the type assignment system of Laird's λ_{Aff}, and then we introduce phase semantics for proof-terms of this system. We prove, in Section 4, the soundness and the completeness theorems of λ_{Aff}. Using completeness, we show a normal form theorem of λ_{Aff}.

2 Phase semantics for the provability of λ_{Aff}

In this section, we first briefly review the logic of Laird's dual affine/intuitionistic λ-calculus (λ_{Aff}) ([Laird 05, 03]). Then we introduce phase semantics for the provability of this system for the preparation of phase semantics for proof-terms, which will be developed in Section 3.2.

λ_{Aff} takes dual contexts of linear and non-linear in the style of Barber-Plotkin's Dual Intuitionistic Linear Logic (**DILL**) (see [Barber 96]). A sequent in this system has the form $\Phi \,;\, \Gamma \vdash A$, where Φ is a set of formulas (types) called "intuitionistic context", and Γ is a multiset of formulas (types) called "linear context", with A a formula (type). The symbol " ; " is used to distinguish an intuitionistic context and a linear context. We call $\Phi \,;\, \Gamma$ a dual context. We write $\emptyset$ for the empty sequent.

Syntactical notions such as formulas (types) and contexts of λ_{Aff} are defined as the type assignment system for λ-terms in Section 3.1, and we only indicate the inference rules in Table 1 in this section.

Table 1. Inference rules of λ_{Aff}

$$\frac{}{\Phi \,;\, \Gamma, A \vdash A}\ ax \qquad \frac{\Phi \,;\, \Gamma, A \vdash B}{\Phi, A \,;\, \Gamma \vdash B}\ der$$

$$\frac{\Phi \,;\, \Gamma, A \vdash B}{\Phi \,;\, \Gamma \vdash A \multimap B}\ {\multimap}I \qquad \frac{\Phi \,;\, \Gamma \vdash A \multimap B \quad \Phi \,;\, \Delta \vdash A}{\Phi \,;\, \Gamma, \Delta \vdash B}\ {\multimap}E$$

$$\frac{\Phi, A \,;\, \Gamma \vdash B}{\Phi \,;\, \Gamma \vdash A \to B}\ {\to}I \qquad \frac{\Phi \,;\, \Gamma \vdash A \to B \quad \Phi \,;\, \emptyset \vdash A}{\Phi \,;\, \Gamma \vdash B}\ {\to}E$$

$$\frac{\Phi \,;\, \Gamma \vdash A \quad \Phi \,;\, \Gamma \vdash B}{\Phi \,;\, \Gamma \vdash A \,\&\, B}\ \&I \qquad \frac{\Phi \,;\, \Gamma \vdash A_1 \,\&\, A_2}{\Phi \,;\, \Gamma \vdash A_i}\ \&E_i \quad i = 1, 2$$

We introduce phase semantics for the provability of λ_{Aff}. First for comparison, we briefly recall operations of usual phase semantics for the provability. See [Okada 02] for details. Phase semantics for the provability of the $\{\multimap, \&\}$-fragment of intuitionistic linear logic is based on commutative monoid M and the following operations: For any $\alpha, \beta \subseteq M$,

- $\alpha \multimap \beta = \{m \mid \text{ for any } n \in \alpha,\ m \cdot n \in \beta\}$,
- $\alpha \,\&\, \beta = \alpha \cap \beta$.

We now introduce phase semantics for the provability of λ_{Aff}, based on intuitionistic phase semantics of [Okada 02]. See also [Okada-Terui 99] on phase semantics for affine logic.

Definition 1 (Phase space for the provability). A *phase space $\mathcal{M}$ for the provability of* λ_{Aff} is defined as follows.

- Start with a structure $(M, \star, \cdot, \varepsilon)$ where M is a set equipped with two binary commutative associative operators $\star$ and $\cdot$ which share the same unit ε, and such that $\star$ is also idempotent. Then the phase space which is used is the set of subsets of $\mathcal{M} = (M, \star, \varepsilon) \times (M, \cdot, \varepsilon)$ with the following operations for any $\alpha, \beta \subseteq \mathcal{M}$. We denote an element of $\mathcal{M}$ as $(n\ ;\ l)$ instead of (n, l).

  - $\alpha \multimap \beta = \{(n\ ;\ l) \mid \text{for any } (n'\ ;\ l') \in \alpha,\ (n \star n'\ ;\ l \cdot l') \in \beta\}$
  - $\alpha \to \beta = \{(n\ ;\ l) \mid \text{for any } (n'\ ;\ \varepsilon) \in \alpha,\ (n \star n'\ ;\ l) \in \beta\}$
  - $\alpha \,\&\, \beta = \alpha \cap \beta$.

- $\alpha \subseteq \mathcal{M}$ is called *closed* if it satisfies the following *monotonicity* condition:

(*Monotonicity*) If $(n\ ;\ l \cdot l') \in \alpha$, then $(n \star l' \star m\ ;\ l \cdot k) \in \alpha$ for any $m, k \in M$.

This condition is a combination of *Weakening* and *Dereliction* (*der*-rule).

Definition 2 (Phase model for the provability). A *phase model* $(\mathcal{M}, *)$ *for the provability of* λ_{Aff} consists of a phase space $\mathcal{M}$ for the provability and an interpretation function $*$ from the set of atoms to the set of closed sets of $\mathcal{M}$.

We write $\mathfrak{l}$ for $l_1 \cdot \dots \cdot l_k$ in a linear context, and $\mathfrak{n}$ for $n_1 \star \dots \star n_l$ and $\mathfrak{m}$ for $m_1 \star \dots \star m_k$ in an intuitionistic context.

Definition 3 (Interpretation of sequents).
A sequent $C_1, \dots, C_l\ ;\ D_1, \dots, D_k \vdash A$ is *true* (denoted as $C_1, \dots, C_l\ ;\ D_1, \dots, D_k \models A$) in a phase model $(\mathcal{M}, *)$ for the provability of λ_{Aff}, if, for any $(n_i\ ;\ \varepsilon) \in C_i^*$ and $(m_j\ ;\ l_j) \in D_j^*$, we have $(\mathfrak{n} \star \mathfrak{m}\ ;\ \mathfrak{l}) \in A^*$.

In the above definition of phase semantics, if we define an order relation in a monoid as $m \leq n$ *iff* $\exists l (m \cdot l = n)$, then we obtain a linear (affine) Kripke semantics naturally (see [Okada 04]). Hence our framework can be also considered as a framework of Kripke semantics as [Coquand 05].

Using the method of [Okada 96, 02] and [Okada-Terui 99], we have completeness of λ_{Aff} with respect to phase semantics for the provability. We shall also show that these theorems are obtained by proofs for corresponding theorems in Section 4.

Theorem 1 (Soundness for the provability).
If $C_1, \dots, C_l\ ;\ D_1, \dots, D_k \vdash A$ *is provable in* λ_{Aff}*, then* $C_1, \dots, C_l\ ;\ D_1, \dots, D_k \models A$ *in any phase model for the provability of* λ_{Aff}*.*

In order to show completeness for the provability of λ_{Aff}, we construct a canonical model $(\mathcal{M}_S, *)$ as follows. See Section 4.2 for the detail.

- $\mathcal{M}_S$ is the set of dual contexts, where binary operators $\star$ and $\cdot$ are " , " in intuitionistic contexts (set-union) and " , " in linear contexts (multiset-union), respectively, and the unit is the empty sequent $\emptyset$.
- On this set, we consider the following outer-value:

- $[\![A]\!] = \{\Phi\ ;\ \Gamma \mid \Phi\ ;\ \Gamma \vdash A$ is provable with a normal proof $\}$

- We define an interpretation function $*$ as $X^* = [\![X]\!]$.

Then we have the following main lemma by the induction on the complexity of A as [Okada 96, 02]. See Section 4.2 for the detail.

Lemma 1 (Main lemma). *In $(\mathcal{M}_S, *)$, for any type A,*

$$(\emptyset\ ;\ A) \in A^* \subseteq [\![A]\!].$$

The following completeness is a direct consequence of the main lemma.

Theorem 2 (Completeness for the provability).
If $C_1, \ldots, C_l\ ;\ D_1, \ldots, D_k \models A$ in any phase model for the provability of λ_{Aff}, then $C_1, \ldots, C_l\ ;\ D_1, \ldots, D_k \vdash A$ is provable with a normal proof in λ_{Aff}.

Then with the soundness theorem, the following normal form theorem is obtained.

Theorem 3 (Normal form theorem for λ_{Aff}).
If $C_1, \ldots, C_l\ ;\ D_1, \ldots, D_k \vdash A$ is provable in λ_{Aff}, then it is provable with a normal proof.

3 Phase semantics for proof-terms of $\boldsymbol{\lambda_{\mathrm{Aff}}}$

In this section, we review the type assignment system of Laird's dual affine/intuitionistic λ-calculus (λ_{Aff}) ([Laird 05, 03]), and we introduce phase semantics for proof-terms of this system.

3.1 Type assignment system for $\boldsymbol{\lambda_{\mathrm{Aff}}}$

Although Barber [1996] and others, who study **DILL** or linear CPS-translation, introduce the linear λ-terms, we take, following [Laird 05, 03], the usual (non linear) untyped λ-terms with pairs for λ_{Aff}: Variables $x, y, z, \ldots$; abstraction $\lambda x.s$; application (st); pair $\langle s, t\rangle$; and projection $\pi_1(s), \pi_2(s)$.

We write $s[x := t]$ for the substitution of a λ-term t for the free occurrences of x in a λ-term s.

Definition 4 (Types). The linear types of λ_{Aff} is defined as follows:

- Atomic types $X, Y, \ldots, X_1, X_2, \ldots$ are types.
- If A and B are types, then $A \multimap B, A \rightarrow B$ and $A \,\&\, B$ are also types.

We now define the type assignment rules of Laird's λ_{Aff}.

Definition 5 (Type asignment rules).

- A *declaration* (*assumption*) is of the form $x\colon A$ with a variable x and a type A.
- A *dual context* is of the form $y_1\colon C_1, \ldots, y_l\colon C_l \ ; \ x_1\colon D_1, \ldots, x_k\colon D_k$ where the left of " ; " ($y_1\colon C_1, \ldots, y_l\colon C_l$) is a finite *set* of declarations called an "intuitionistic context" and the right of " ; " ($x_1\colon D_1, \ldots, x_k\colon D_k$) is a finite *multiset* of declarations called a "linear context". We write $\Phi, \Psi, \ldots$ for any intuitionistic contexts and $\Gamma, \Delta, \ldots$ for any linear contexts. We write $\emptyset$ for the empty context.
- A λ-term s is *typable* as type A if a sequent $\Phi \ ; \ \Gamma \vdash s\colon A$ is derivable for some dual context $\Phi \ ; \ \Gamma$ by the type assignment rules of Table 2.

Table 2. Type assignment rules of λ_{Aff}

$$\frac{}{\Phi \ ; \ \Gamma, x\colon A \vdash x\colon A}\ ax \qquad \frac{\Phi \ ; \ \Gamma, x\colon A \vdash s\colon B}{\Phi, x\colon A \ ; \ \Gamma \vdash s\colon B}\ der$$

$$\frac{\Phi \ ; \ \Gamma, x\colon A \vdash s\colon B}{\Phi \ ; \ \Gamma \vdash \lambda x.s\colon A \multimap B}\ {\multimap}I \qquad \frac{\Phi \ ; \ \Gamma \vdash s\colon A \multimap B \quad \Phi \ ; \ \Delta \vdash t\colon A}{\Phi \ ; \ \Gamma, \Delta \vdash (st)\colon B}\ {\multimap}E$$

where s contains at most one free occurrence of x.

$$\frac{\Phi, x\colon A \ ; \ \Gamma \vdash s\colon B}{\Phi \ ; \ \Gamma \vdash \lambda x.s\colon A \to B} \to I \qquad \frac{\Phi \ ; \ \Gamma \vdash s\colon A \to B \quad \Phi \ ; \ \emptyset \vdash t\colon A}{\Phi \ ; \ \Gamma \vdash (st)\colon B} \to E$$

$$\frac{\Phi \ ; \ \Gamma \vdash s\colon A \quad \Phi \ ; \ \Gamma \vdash t\colon B}{\Phi \ ; \ \Gamma \vdash \langle s, t\rangle\colon A \,\&\, B}\ \&I \qquad \frac{\Phi \ ; \ \Gamma \vdash s\colon A_1 \,\&\, A_2}{\Phi \ ; \ \Gamma \vdash \pi_i(s)\colon A_i}\ \&E_i \quad i = 1, 2$$

We introduce the $\beta\eta$-equality relation $\simeq_{\beta\eta}$ on λ-terms as the usual conversion relation, see [Hindley-Seldin 86], [Barendregt 92]. A λ-term is in *normal form*, if it contains no redex of the form $(\lambda x.st)$, $\pi_i(\langle s, t\rangle)$ for $i = 1$ and 2, $\lambda x.(sx)$ for $x \notin FV(s)$, or $\langle \pi_1(s), \pi_2(s)\rangle$, where $FV(s)$ means the set of free variables of the λ-term s.

Note that there is a slight difference between our λ_{Aff} and Barber-Plotkin's **DILL**-style λ_{Aff}. In **DILL**-style λ_{Aff}, it is assumed that Φ and Γ are disjoint in a dual context $\Phi \ ; \ \Gamma$, and further that Γ and Δ are disjoint in a linear context Γ, Δ. (See [Barber 96].) In our λ_{Aff}, in place of this assumption, we assume that ${\multimap}I$-rule of Table 2 is applicable when s contains at most one free occurrence of x. This change does not affect the type assignment system of λ_{Aff}, which can be stated in Proposition 1 bellow.

Note first that, if a sequent $\Phi \ ; \ \Gamma \vdash s\colon A$ is derivable in **DILL**-style λ_{Aff}, then the variables of the linear context Γ occur at most once in s due to the assumption of disjointness of linear contexts (see [Barber 96]). Hence ${\multimap}I$-rule of **DILL**-style system satisfies the restriction of ${\multimap}I$-rule of our λ_{Aff}.

On the other hand, the following proposition means that any derivation of our λ_{Aff} can be simulated in **DILL**-style λ_{Aff} by considering duplicated formulas in a linear context of our λ_{Aff} as an intuitionistic assumption of **DILL**-style λ_{Aff}.

Proposition 1 (Simulation). *Let $x_i\vec{:}A_i$ be a sequence $x_i\colon A_i, \ldots, x_i\colon A_i$ of the same assumptions. Assume that no variable appears twice in a sequence of assumptions $\Phi, \Gamma, x_1\colon A_1, \ldots, x_n\colon A_n$. For $0 \leq k \leq n$,*
if $\Phi, x_1\colon A_1, \ldots, x_k\colon A_k \ ; \ \Gamma, x_1\vec{:}A_1, \ldots, x_n\vec{:}A_n \vdash s\colon B$ is derivable in our λ_{Aff}, then $\Phi, x_1\colon A_1, \ldots, x_n\colon A_n \ ; \ \Gamma \vdash s\colon B$ is derivable in **DILL**-*style λ_{Aff}.*

This proposition is obtained by the following lemma of **DILL**-style λ_{Aff}.

Lemma 2 (Environment strengthening (Cf. [Barber 96])).
If $\Phi, x\colon A \ ; \ \Gamma \vdash s\colon B$ is derivable in **DILL**-*style λ_{Aff} and s contains at most one free occurrence of x, then $\Phi \ ; \ \Gamma, x\colon A \vdash s\colon B$ is derivable in* **DILL**-*style λ_{Aff}.*

Let us consider the following example of an application of $\multimap I$

$$\frac{\Phi \ ; \ \Gamma, x\colon A, x\colon A \vdash s\colon B}{\Phi \ ; \ \Gamma, x\colon A \vdash \lambda x.s\colon A \multimap B} \multimap I \quad \text{in our } \lambda_{\mathrm{Aff}}.$$

For simplicity, we assume that no variable other than x occurs twice in the dual context.

This form of $\multimap I$-rule is simulated by **DILL**-style λ_{Aff} as follows: First we consider the duplicated assumptions $(x\colon A, x\colon A)$ in the linear context of the upper sequent as an intuitionistic assumption as

$$\Phi, x\colon A \ ; \ \Gamma \vdash s\colon B$$

in **DILL**-style λ_{Aff}. Note that by the definition of $\multimap I$-rule of our λ_{Aff}, the λ-term s contains at most one free occurrence of x. Hence by Lemma 2, we have $\Phi \ ; \ \Gamma, x\colon A \vdash s\colon B$. Thus by applying $\multimap I$-rule of **DILL**-style λ_{Aff}, we have the following derivation:

$$\frac{\Phi \ ; \ \Gamma, x\colon A \vdash s\colon B}{\Phi \ ; \ \Gamma \vdash \lambda x.s\colon A \multimap B} \multimap I \quad \text{in } \mathbf{DILL}\text{-style } \lambda_{\mathrm{Aff}}.$$

Then by Weakening (cf. [Barber 96]) we have $\Phi, x\colon A \ ; \ \Gamma \vdash \lambda x.s\colon A \multimap B$ in **DILL**-style λ_{Aff}.

3.2 Phase semantics for proof-terms of λ_{Aff}

In this subsection, we introduce phase semantics for proof-terms of λ_{Aff}. The domain of our model consists of untyped λ-terms with dual contexts, which corresponds to [Coquand 05]'s model. The operations $\multimap, \rightarrow$ and & are based on the $\multimap, \rightarrow$ elimination rules and the set theoretical intersection operation, which are natural extensions of those operations of phase semantics for the provability of λ_{Aff} (Definition 1). The following phase space for proof-terms is obtained by augmenting proof-terms (λ-terms) to a phase space for the provability of λ_{Aff}.

Definition 6 (Phase space for proof-terms). A *phase space for proof-terms* of λ_{Aff} is $(\mathcal{M}, \mathcal{P})$ where

- $\mathcal{M}$ is a phase space for the provability of λ_{Aff}.
- $\mathcal{P} = \{((n\ ;\ l), s) \mid (n\ ;\ l) \in \mathcal{M}$ and s is a λ-term$\}$.
  We write $(n\ ;\ l \rhd s)$ for $((n\ ;\ l), s)$.

  There are the following operations for any $\alpha, \beta \subseteq \mathcal{P}$:

  - $\alpha \multimap \beta = \{(n\ ;\ l \rhd s) \mid$ for any $(n'\ ;\ l' \rhd t) \in \alpha,\ (n \star n'\ ;\ l \cdot l' \rhd (st)) \in \beta\}$
  - $\alpha \to \beta = \{(n\ ;\ l \rhd s) \mid$ for any $(n'\ ;\ \varepsilon \rhd t) \in \alpha,\ (n \star n'\ ;\ l \rhd (st)) \in \beta\}$
  - $\alpha \,\&\, \beta = \pi_1(\alpha) \cap \pi_2(\beta)$,
    where $\pi_i(\alpha) = \{(n\ ;\ l \rhd s) \mid (n\ ;\ l \rhd \pi_i(s)) \in \alpha\}$ for $i = 1, 2$.

- We call $\alpha \subseteq \mathcal{P}$ *closed* if it satisfies the following *equality-closed* and *monotonicity* conditions:

(*Equality-closed*) If $t \simeq_{\beta\eta} s$ and $(n\ ;\ l \rhd s) \in \alpha$, then $(n\ ;\ l \rhd t) \in \alpha$;
(*Monotonicity*) If $(n\ ;\ l \cdot l' \rhd s) \in \alpha$, then $(n \star m \star l'\ ;\ l \cdot k \rhd s) \in \alpha$ for any $m, k \in M$.

These two conditions are collectively called the *closure condition.*

Note that, in the above definition of phase space for proof-terms of λ_{Aff}, if we forget λ-terms (proof-terms), then (1) $\mathcal{P}$ coincides with $\mathcal{M}$, (2) operations $\{\multimap, \to, \&\}$ exactly correspond to those of Definition 1 of phase space for the provability of λ_{Aff}, and (3) only monotonicity remains as the closure condition. Hence a phase space for the provability is obtained by forgetting λ-terms of a phase space for proof-terms.

It is easily shown that the closure condition is preserved under $\{\multimap, \to, \&\}$-operations.

Lemma 3. *For any $\alpha, \beta \subseteq \mathcal{P}$, if β is closed then $\alpha \multimap \beta$ and $\alpha \to \beta$ are closed; if α and β are closed then $\alpha \,\&\, \beta$ is closed.*

Definition 7 (Phase model for proof-terms). A *phase model* $(\mathcal{M}, \mathcal{P}, *)$ *for proof-terms* of λ_{Aff} consists of

- a phase space $(\mathcal{M}, \mathcal{P})$ for proof-terms of λ_{Aff};
- an interpretation function $*$ form the set of atomic types to the set of closed sets of $(\mathcal{M}, \mathcal{P})$ such that

$(A \multimap B)^* = A^* \multimap B^*, \quad (A \to B)^* = A^* \to B^*, \quad (A \,\&\, B)^* = A^* \,\&\, B^*.$

Note that any interpretation A^* of a type is closed from Lemma 3.

Notation: We write $s[\mathfrak{t}, \mathfrak{s}]$ for $s[y_1 := t_1, \ldots, y_l := t_l, x_1 := s_1, \ldots, x_k := s_k]$. We also write $\mathfrak{l}$ for $l_1 \cdot \cdots \cdot l_k$ in a linear context, and $\mathfrak{n}$ for $n_1 \star \cdots \star n_l$ and $\mathfrak{m}$ for $m_1 \star \cdots \star m_k$ in an intuitionistic context.

Definition 8 (Interpretation of sequents).
A sequent $y_1\!:C_1, \ldots, y_l\!:C_l\ ;\ x_1\!:D_1, \ldots, x_k\!:D_k \vdash s\!:A$ is *true* (denoted as $y_1\!:C_1, \ldots, y_l\!:C_l\ ;\ x_1\!:D_1, \ldots, x_k\!:D_k \models s\!:A$) in a phase model $(\mathcal{M}, \mathcal{P}, *)$ for proof-terms of λ_{Aff}, if, for any $(n_i\ ;\ \varepsilon \rhd t_i) \in C_i^*$ and $(m_j\ ;\ l_j \rhd s_j) \in D_j^*$, we have $(\mathfrak{n} \star \mathfrak{m}\ ;\ \mathfrak{l} \rhd s[\mathfrak{t}, \mathfrak{s}]) \in A^*$.

4 Soundness and completeness of $\boldsymbol{\lambda}_{\mathbf{Aff}}$

In this section, we prove the soundness and the completeness theorems. Our proof below is an extension of the completeness proof in [Okada 96, 02] for phase semantics for the provability of intuitionistic linear logic. Using completeness, we show a normal form theorem of λ_{Aff}: *if a λ-term s is typable in λ_{Aff}, then there is a λ-term t in normal form such that $s \simeq_{\beta\eta} t$* .

By the following completeness proof for proof-terms (λ-terms), we obtain completeness for the provability if we ignore the part related to λ-terms.

4.1 Soundness theorem

We first show the soundness theorem.

Theorem 4 (Soundness for proof-terms).
If $y_1{:}\,C_1, \ldots, y_l{:}\,C_l \ ; \ x_1{:}\,D_1, \ldots, x_k{:}\,D_k \vdash s{:}\,A$ is derivable in λ_{Aff}, then $y_1{:}\,C_1, \ldots, y_l{:}\,C_l \ ; \ x_1{:}\,D_1, \ldots, x_k{:}\,D_k \models s{:}\,A$ in any phase model for proof-terms of λ_{Aff}.

Proof. By induction on the construction of derivation. As for elimination rules, the assertion is more or less immediate by the induction hypothesis since the corresponding operations in our phase model are defined based on the elimination rules. Further, since $\to E$-rule is a particular case of $\multimap I$-rule, it is treated by the same way as $\multimap I$ case. Thus we show only for $\multimap I$ and $\& I$ rules.

In the following, Φ denotes $y_1{:}\,C_1, \ldots, y_l{:}\,C_l$, and Γ denotes $x_1{:}\,D_1, \ldots, x_k{:}\,D_k$.

Case $\dfrac{\Phi \ ; \ \Gamma, x{:}\,A \vdash s{:}\,B}{\Phi \ ; \ \Gamma \vdash \lambda x.s{:}\,A \multimap B} \multimap I.$

Let $(n_i \ ; \ \varepsilon \rhd t_i) \in C_i^*$ and $(m_j \ ; \ l_j \rhd s_j) \in D_j^*$ for each $1 \le i \le l$ and $1 \le j \le k$. Then by the induction hypothesis, for any $(n \ ; \ l \rhd u) \in A^*$, we have $(\mathfrak{n} \star \mathfrak{m} \star n \ ; \ \mathfrak{l} \cdot l \rhd s[\mathfrak{t}, \mathfrak{s}, x := u]) \in B^*$. Since B^* is closed, we have $(\mathfrak{n} \star \mathfrak{m} \star n \ ; \ \mathfrak{l} \cdot l \rhd (\lambda x.su)[\mathfrak{t}, \mathfrak{s}]) \in B^*$. That is $(\mathfrak{n} \star \mathfrak{m} \ ; \ \mathfrak{l} \rhd \lambda x.s[\mathfrak{t}, \mathfrak{s}]) \in A^* \multimap B^*$ by the definition of $\multimap$.

Case $\dfrac{\Phi \ ; \ \Gamma \vdash s{:}\,A \quad \Phi \ ; \ \Gamma \vdash t{:}\,B}{\Phi \ ; \ \Gamma \vdash \langle s, t \rangle{:}\,A \,\&\, B} \& I.$

Let $(n_i \ ; \ \varepsilon \rhd t_i) \in C_i^*$ and $(m_j \ ; \ l_j \rhd s_j) \in D_j^*$. Then by the induction hypothesis, we have $(\mathfrak{n} \star \mathfrak{m} \ ; \ \mathfrak{l} \rhd s[\mathfrak{t}, \mathfrak{s}]) \in A^*$ and $(\mathfrak{n} \star \mathfrak{m} \ ; \ \mathfrak{l} \rhd t[\mathfrak{t}, \mathfrak{s}]) \in B^*$. Since both A^* and B^* are closed, we have $(\mathfrak{n} \star \mathfrak{m} \ ; \ \mathfrak{l} \rhd \pi_1 \langle s, t \rangle [\mathfrak{t}, \mathfrak{s}]) \in A^*$ and $(\mathfrak{n} \star \mathfrak{m} \ ; \ \mathfrak{l} \rhd \pi_2 \langle s, t \rangle [\mathfrak{t}, \mathfrak{s}]) \in B^*$. Thus, from the definition of $\&$, we have $(\mathfrak{n} \star \mathfrak{m} \ ; \ \mathfrak{l} \rhd \langle s, t \rangle [\mathfrak{t}, \mathfrak{s}]) \in A^* \,\&\, B^*$. ■

Note that the soundness theorem (Theorem 1) for the provability is a direct corollary of this soundness theorem for proof-terms.

4.2 Completeness theorem

In this subsection, we present the completeness theorem. For the sake of a normal form theorem, we consider the following form of main lemma for completeness:

$$(\emptyset\ ;\ x\colon A \rhd x) \in A^* \subseteq [\![A]\!]$$

for any variable x and any type A, where A^* is the interpretation of A in the canonical model, and $[\![A]\!] = \{(\Phi\ ;\ \Gamma \rhd s) \mid \Phi\ ;\ \Gamma \vdash s\colon A$ and there is t in normal form such that $s \simeq_{\beta\eta} t\}$ (called the outer-value of A). This is a modification of the lemma of [Coquand 05] following the method of [Okada 96, 02].

In order to show completeness of λ_{Aff}, we add the following *equality*-rule:

$$\frac{\Phi\ ;\ \Gamma \vdash s : A \quad s \simeq_{\beta\eta} t}{\Phi\ ;\ \Gamma \vdash t\colon A}\ eq$$

This type of equality-rule is used in [Hindley 83] and [Coquand 05] to show completeness.

Now we construct a canonical model $(\mathcal{M}_S, \mathcal{P}_S, *)$ as follows.

- $\mathcal{M}_S$ is the set of dual contexts, where binary operators $\star$ and $\cdot$ are " , " in intuitionistic contexts (set-union) and " , " in linear contexts (multiset-union), respectively, and the unit is the empty sequent $\emptyset$.
- $\mathcal{P}_S = \{(\Phi\ ;\ \Gamma \rhd s) \mid (\Phi\ ;\ \Gamma) \in \mathcal{M}_S$ and s is a λ-term $\}$.
- On this set, we consider the following outer-value:
  - $[\![A]\!] = \{(\Phi\ ;\ \Gamma \rhd s) \mid \Phi\ ;\ \Gamma \vdash s\colon A$ and there is t in normal form such that $s \simeq_{\beta\eta} t\}$.
- Then we define an interpretation function $*$ as $X^* = [\![X]\!]$.

Note that we can define another outer-value as $[\![A]\!]_T = \{(\Phi\ ;\ \Gamma \rhd s) \mid \Phi\ ;\ \Gamma \vdash s\colon A\}$ without referring any normal form nor equality relation on λ-terms. The completeness theorem of [Coquand 05] is obtained by this outer-value.

The above construction indeed produces a phase model for proof-terms.

Lemma 4. *$(\mathcal{M}_S, \mathcal{P}_S, *)$ is a phase model for proof-terms of λ_{Aff}.*

Now we prove the following main lemma for our completeness theorem.

Lemma 5 (Main lemma). *In $\mathcal{M}_S$, for any variable x and any type A,*

$$(\emptyset\ ;\ x\colon A \rhd x) \in A^* \subseteq [\![A]\!].$$

Proof. In order to prove this lemma, we first introduce the following meta expression for a type and a λ-term.

For any type $B_1, \ldots, B_n$, we first define, by the induction on n, a meta expression $E(B_1, \ldots, B_n)$ of a type as follows:

$(n = 1)$ $E(B_1)$ means B_1;
$(n > 1)$ $E(B_1, \ldots, B_n)$ means one of the followings;
- $E(B_1, \ldots, B_{n-2}, (B_{n-1} \multimap B_n))$ or;
- $E(B_1, \ldots, B_{n-2}, (B_{n-1} \to B_n))$ or;
- $E(B_1, \ldots, B_{n-2}, (B_n \,\&\, B_{n-1}))$ or;
- $E(B_1, \ldots, B_{n-2}, (B_{n-1} \,\&\, B_n))$.

Next, for any term $s_1, \ldots, s_n$ which have types $B_1, \ldots, B_n$ respectively, we introduce a meta expression $E(s_n, s_1, \ldots, s_{n-1})$ of a λ-term, which depends on the form of a type $E(B_1, \ldots, B_n)$ as follows:
$E(s_n)$ is s_n;
$E(s_n, s_1, \ldots, s_{n-1})$ is one of the following:

- an application $(E(s_n, s_1, \ldots, s_{n-2})s_{n-1})$
  if $E(B_1, \ldots, B_{n-1}, B_n)$ is $E(B_1, \ldots, (B_{n-1} \multimap B_n))$ or $E(B_1, \ldots, (B_{n-1} \to B_n))$;
- a projection $\pi_1(E(s_n, s_1, \ldots, s_{n-2}))$
  if $E(B_1, \ldots, B_{n-1}, B_n)$ is $E(B_1, \ldots, (B_n \,\&\, B_{n-1}))$;
- a projection $\pi_2(E(s_n, s_1, \ldots, s_{n-2}))$
  if $E(B_1, \ldots, B_{n-1}, B_n)$ is $E(B_1, \ldots, (B_{n-1} \,\&\, B_n))$.

If we consider only $\{\multimap, \to\}$-fragment, then a λ-term $E(x, s_1, \ldots, s_n)$ is just an application $(\cdots(xs_1)\cdots s_n)$, and from a proof-theoretical point of view, it correspond to a proof structure whose main branch consists only of $\{\multimap, \to\}$-elimination rules. (Cf. [Martin-Löf 71].)

Then we prove, by induction on the complexity of A, the conjunction of the following two statements (1) and (2), which is a generalized form of the main lemma.

(1) $A^* \subseteq [\![A]\!]$;
(2) *For any type* $E(B_1, \ldots, B_n, A)$,
  if $(\Phi \,;\, \Gamma_i \rhd s_i) \in [\![B_i]\!]$ *for any* $i \leq n$ *such that* $E(B_1, \ldots, B_{i-1}, (B_i \multimap D))$; *and*
  if $(\Phi \,;\, \emptyset \rhd s_j) \in [\![B_j]\!]$ *for any* $j \leq n$ *such that* $E(B_1, \ldots, B_{j-1}, (B_j \to D))$,
  then we have $(\Phi \,;\, \vec{\Gamma_i}, x\colon E(B_1, \ldots, B_n, A) \rhd E(x, s_1, \ldots, s_n)) \in A^*$,
  where $\vec{\Gamma_i}$ *is a sequence of contexts appearing in the premise of the statement.*

In particular, we have $(\emptyset \,;\, x\colon A \rhd x) \in A^*$ from the case $n = 0$ of (2).

We write $E(\vec{B_n}, A)$ for the type $E(B_1, \ldots, B_n, A)$, and write $E(x, \vec{s_n})$ for the λ-term $E(x, s_1, \ldots, s_n)$.

(Case $A \equiv X$)
(1) $X^* \subseteq [\![X]\!]$ is obvious from the definition.

(2) By the assumption we have $\Phi \ ; \ \Gamma_i \vdash s_i{:}\, B_i$ for each i and $\Phi \ ; \ \emptyset \vdash s_j{:}\, B_j$ for each j, hence from the axiom of the form

$$\frac{}{\Phi \ ; \ x{:}\, E(\vec{B_n}, X) \vdash x{:}\, E(\vec{B_n}, X)}\ ax \ ,$$

by applying n-times of the following form of elimination rules:

$$\frac{\Phi \ ; \ \vec{\Gamma_l}, x{:}\, E(\vec{B_n}, X) \vdash E(x, \vec{s_i}) : B_{i+1} \Rightarrow E(B_{i+2}, \ldots, B_n, X) \quad \Phi \ ; \ \Gamma_{i+1} \vdash s_{i+1}{:}\, B_{i+1}}{\Phi \ ; \ \vec{\Gamma_l}, \Gamma_{i+1}, x{:}\, E(\vec{B_n}, X) \vdash (E(x, \vec{s_i}) s_{i+1}){:}\, E(B_{i+2}, \ldots, B_n, X)} \Rightarrow E$$

where $\Rightarrow$ is $\multimap$ or $\rightarrow$, and Γ_{i+1} is $\emptyset$ if $\Rightarrow$ is $\rightarrow$,

or

$$\frac{\Phi \ ; \ \vec{\Gamma_l}, x{:}\, E(\vec{B_n}, X) \vdash E(x, \vec{s_i}){:}\, E(B_{i+2}, \ldots, B_n, X) \ \& \ B_{i+1}}{\Phi \ ; \ \vec{\Gamma_l}, x{:}\, E(\vec{B_n}, X) \vdash \pi_1(E(x, \vec{s_i})){:}\, E(B_{i+2}, \ldots, B_n, X)} \ \&E_1$$

similar for $\&E_2$,

we obtain $\Phi \ ; \ \vec{\Gamma_i}, x{:}\, E(\vec{B_n}, X) \vdash E(x, \vec{s_n}){:}\, X$.

On the other hand there are λ-terms $u_1, \ldots, u_n$ in normal forms such that $s_k \simeq u_k$ for each $1 \le k \le n$ by the assumption. Hence we have $E(x, s_1, \ldots, s_n) \simeq_{\beta\eta} E(x, u_1, \ldots, u_n)$ where $E(x, u_1, \ldots, u_n)$ is in normal form.

Thus we obtain $(\Phi \ ; \ \vec{\Gamma_i}, x{:}\, E(\vec{B_n}, X) \rhd E(x, s_1, \ldots, s_n)) \in [\![X]\!] = X^*$.

(Case $A \equiv B \multimap C$**)**

(1) We show $B^* \multimap C^* \subseteq [\![B \multimap C]\!]$. Let $(\Phi \ ; \ \Gamma \rhd s) \in B^* \multimap C^*$. Then for any $(\Psi \ ; \ \Delta \rhd t) \in B^*$, we have $(\Phi, \Psi \ ; \ \Gamma, \Delta \rhd (st)) \in C^*$. Since we have $(\emptyset \ ; \ x{:}\, B \rhd x) \in B^*$ for any x by the induction hypothesis on B of the case $n = 0$, we have $(\Phi \ ; \ \Gamma, x{:}\, B \rhd (sx)) \in C^*$ for $x \notin FV(s)$. Then by the induction hypothesis on C, we have $\Phi \ ; \ \Gamma, x{:}\, B \vdash (sx){:}\, C$ and there is u in normal form such that $(sx) \simeq_{\beta\eta} u$. Then by applying the following $\multimap I$-rule

$$\frac{\Phi \ ; \ \Gamma, x{:}\, B \vdash (sx){:}\, C}{\Phi \ ; \ \Gamma \vdash \lambda x.(sx){:}\, B \multimap C} \multimap I \ ,$$

we have $\Phi \ ; \ \Gamma \vdash \lambda x.(sx){:}\, B \multimap C$ and $\lambda x.(sx) \simeq_{\beta\eta} \lambda x.u$.

Note that $\lambda x.u$ is not necessarily in normal form even if u is in normal form. Thus we divide the following two cases depending on u.

(i) If u is of the form (vx), then we have $s \simeq_\eta \lambda x.(sx) \simeq \lambda x.u \equiv \lambda x.(vx) \simeq_\eta v$. Note that v is in normal form since u is in normal form, and that we have $\Phi \ ; \ \Gamma \vdash s{:}\, B$ by the equality-rule.

(ii) If otherwise, $\lambda x.u$ is in normal form since u is in normal form. Hence from $(sx) \simeq u$, we have $s \simeq_\eta \lambda x.(sx) \simeq \lambda x.u$, and we have $\Phi \ ; \ \Gamma \vdash s{:}\, B$ by the equality-rule.

Hence in either case, we have $(\Phi \ ; \ \Gamma \rhd s) \in [\![B \multimap C]\!]$.

(2) We show that for any $(\Phi \ ; \ \Gamma \rhd t) \in B^*$, we have $(\Phi \ ; \ \vec{\Gamma_i}, \Gamma, x{:}\, E(B_1, \ldots, B_n, (B \multimap C)) \rhd (E(x, s_1, \ldots, s_n)t)) \in C^*$. From the assumption $(\Phi \ ; \ \Gamma \rhd t) \in B^*$, we have

$(\Phi\ ;\ \Gamma \rhd t) \in [\![B]\!]$ by the induction hypothesis on B. Since $E(B_1, \ldots, B_n, (B \multimap C))$ is expressed as $E(B_1, \ldots, B_n, B, C)$ and $(E(x, s_1, \ldots, s_n)t)$ is $E(x, s_1, \ldots, s_n, t)$, we have the assertion by the induction hypothesis on C.

(Case $A \equiv B \to C$**)**
We omit a proof for this case since it is treated by the similar way as $\multimap$.

(Case $A \equiv B \,\&\, C$**)**
(1) We show $B^* \,\&\, C^* \subseteq [\![B \,\&\, C]\!]$. Let $(\Phi\ ;\ \Gamma \rhd s) \in B^* \,\&\, C^*$. Then we have $(\Phi\ ;\ \Gamma \rhd \pi_1(s)) \in B^*$ and $(\Phi\ ;\ \Gamma \rhd \pi_2(s)) \in C^*$. Then by the induction hypothesis on B and C respectively, we have $\Phi\ ;\ \Gamma \vdash \pi_1(s){:}\, B$ and there is u in normal form such that $\pi_1(s) \simeq_{\beta\eta} u$, and we have $\Phi\ ;\ \Gamma \vdash \pi_2(s){:}\, C$ and there is v in normal form such that $\pi_2(s) \simeq_{\beta\eta} v$. Thus by the following $\&I$-rule

$$\frac{\Phi\ ;\ \Gamma \vdash \pi_1(s){:}\, B \quad \Phi\ ;\ \Gamma \vdash \pi_2(s){:}\, C}{\Phi\ ;\ \Gamma \vdash \langle \pi_1(s), \pi_2(s)\rangle {:}\, B \,\&\, C}\ \&I\ ,$$

we have $\Phi\ ;\ \Gamma \vdash \langle \pi_1(s), \pi_2(s)\rangle {:}\, B \,\&\, C$ and $\langle \pi_1(s), \pi_2(s)\rangle \simeq_{\beta\eta} \langle u, v\rangle$.

Thus we divide the following two cases depending on u and v.
(i) If u is of the form $\pi_1(t)$ and v is of the form $\pi_2(t)$, then we have $s \simeq_\eta \langle \pi_1(s), \pi_2(s)\rangle \simeq \langle u, v\rangle \equiv \langle \pi_1(t), \pi_2(t)\rangle \simeq_\eta t$. Note that t is in normal form since u and v are in normal forms, and that we have $\Phi\ ;\ \Gamma \vdash s{:}\, B \,\&\, C$ by the equality-rule.
(ii) If otherwise, $\langle u, v\rangle$ is in normal form since u and v are in normal forms. Hence from $\pi_1(s) \simeq u$ and $\pi_2(s) \simeq v$, we have $s \simeq_\eta \langle \pi_1(s), \pi_2(s)\rangle \simeq \langle u, v\rangle$, and $\Phi\ ;\ \Gamma \vdash s{:}\, B \,\&\, C$ by the equality-rule.
Hence in either case, we have $(\Phi\ ;\ \Gamma \rhd s) \in [\![B \,\&\, C]\!]$.

(2) We show that $(\Phi\ ;\ \vec{\Gamma}_i, x{:}\, E(B_1, \ldots, B_n, (B \& C)) \rhd \pi_1(E(x, s_1, \ldots, s_n))) \in B^*$ and $(\Phi\ ;\ \vec{\Gamma}_i, x{:}\, E(B_1, \ldots, B_n, (B \,\&\, C)) \rhd \pi_2(E(x, s_1, \ldots, s_n))) \in C^*$.

Since $E(B_1, \ldots, B_n, (B \,\&\, C))$ is expressed as $E(B_1, \ldots, B_n, B, C)$ and $\pi_i(E(x, s_1, \ldots, s_n))$ for $i = 1, 2$ are $E(x, s_1, \ldots, s_n, s_{n+1})$ respectively, we have the assertion by the induction hypothesis on B and C respectively. ∎

Assume that, for any $(n_i\ ;\ \varepsilon \rhd t_i) \in C_i^*$ and $(m_j\ ;\ l_j \rhd s_j) \in D_j^*$, we have $(\mathfrak{n} \star \mathfrak{m}\ ;\ \mathfrak{l} \rhd s[\mathfrak{t}, \mathfrak{s}]) \in A^*$ in any phase mode for proof-terms. Then we have $(y_1{:}\, C_1, \ldots, y_l{:}\, C_l\ ;\ x_1{:}\, D_1, \ldots, x_k{:}\, D_k \rhd s) \in A^*$ in the canonical model, since $(y_i{:}\, C_i\ ;\ \emptyset) \in C_i^*$ and $(\emptyset\ ;\ x_j{:}\, D_j) \in D_j^*$ for any C_i and D_j by the main lemma and the monotonicity. Hence by the main lemma $A^* \subseteq [\![A]\!]$, the sequent $y_1{:}\, C_1, \ldots, y_l{:}\, C_l\ ;\ x_1{:}\, D_1, \ldots, x_k{:}\, D_k \vdash s{:}\, A$ is derivable in λ_{Aff} and there is t in normal form such that $s \simeq_{\beta\eta} t$. Thus we obtain the following completeness theorem for proof-terms of λ_{Aff}.

Theorem 5 (Completeness for proof-terms).
If $y_1{:}\, C_1, \ldots, y_l{:}\, C_l\ ;\ x_1{:}\, D_1, \ldots, x_k{:}\, D_k \models s{:}\, A$ *in any phase model for proof-terms of* λ_{Aff}*, then* $y_1{:}\, C_1, \ldots, y_l{:}\, C_l\ ;\ x_1{:}\, D_1, \ldots, x_k{:}\, D_k \vdash s{:}\, A$ *is derivable in* λ_{Aff}*, and there is* t *in normal form such that* $s \simeq_{\beta\eta} t$*.*

Note that the completeness theorem (Theorem 2) for provability is a direct corollary of this completeness theorem for proof-terms.

If a λ-term s is typable in λ_{Aff}, then by the soundness, the premise of the completeness is satisfied. Hence by the completeness, we have the following normal form theorem of λ_{Aff}.

Corollary 1 (Normal form theorem). *If a λ-term s is typable in λ_{Aff}, then there is a λ-term t in normal form such that $s \simeq_{\beta\eta} t$.*

5 Concluding remarks

We remarked that the completeness theorem of type assignment system of [Coquand 05] in the simple type framework can be extended to a stronger form which implies a normal form theorem in a linear (affine) types framework with Laird's dual affine/intuitionistic λ-calculus.

In order to prove completeness, we introduced, following [Hindley 83] and [Coquand 05], an equality-rule. This equality-rule makes some usually untypable λ-terms typable. It seems that this equality-rule is essential to prove completeness in our framework of λ_{Aff} which has η-rule for &, since the subject reduction property fails for this reduction. We remark that, in the framework without & nor η-rule, the following weaker form of completeness holds without having the additional equality-rule: *If $(\Gamma \triangleright s) \in A^*$ for any model, then there is a λ-term t such that $s \simeq t$ and $\Gamma \vdash t{:}\, A$ is derivable*, which naturally implies completeness at the level of the provability.

From the phase semantic point of view, the $\{\multimap, \&, \rightarrow\}$-fragment of **ILL** is complete with respect to intuitionistic phase semantics without any closure condition, which is a generalization of the double negation operation $(\ )^{\perp\perp}$. Cf. [Abrusci 90] and [Okada 02] for the phase semantic closure. Hence our *closure condition* (the equality-closed and the monotonicity conditions) is not related to the closure condition of intuitionistic phase semantics for the provability. As future work, we investigate whether or not our closure condition in phase semantics for proof-terms can be considered as an extension of the closure condition of intuitionistic phase semantics for the provability.

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