Logic 言語

A Note on Phase Semantics for Proofs (Preliminary Report)

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1 Introduction

One of the basic paradigms of logical studies is to show soundness and completeness of syntactic inference systems with respect to a certain model theoretic semantics. The usual setting of soundness and completeness is provided with respect to "provability". In this note we show that the same paradigm can be extended to the setting of soundness and completeness with respect to "proofs", in place of "provability", for certain logical systems. For that purpose, as a simple setting, we give phase semantic soundness and completeness, for proofs, of the implicational conjunctive fragment of intuitionistic logic (*IL*). We show that the phase semantic soundness and completeness argument for provability (which was presented in eg. [Okada 02]) can be naturally extended to that for proofs in this setting.

Soundness and completeness of *IL* in phase semantics for provability are expressed as the following statement: *A formula A is provable in IL if and only if A is true in any phase model $\mathcal{M}$ for IL.* In phase semantics for proofs, this statement is replaced by the form referring to not only a formula but also a proof explicitly: *s is a proof of formula A if and only if s belongs to the interpretation A^* of formula A in any phase model $\mathcal{M}$ for proofs.*

Our phase semantic model construction is closely related to the computability/reducibility argument for the normalization theorem for λ-calculus (see [Tait 67, 75], [Girard 72], [Prawitz 71] and [Marin-Löf 75] etc.). One of the most notable differences between these approaches and ours is in the way of model construction. The computability/reducibility argument directly refers to such a syntactic notion as normalizability. (See, for example, [Barendregt 92] or [Gallier 90] etc..) This means that a specific single model (namely the syntactic canonical model) is considered in the argument. On the other hand, we give a more general model construction based on the notion of phase space, which is defined independently of such a notion as normal form or normalizability.

In this note we shall present our paradigm with a simple setting, following [Okada 96]. The domain of our models consists of untyped λ-terms with contexts. Logical connectives (type constructors) are defined by the type assignment rules. There is a "closure condition" of the form: *if $s \in X$ and $s \simeq t$, then $t \in X$*, namely the interpretation of types should be closed under a kind of equality (for example $\beta\eta$ equality). Then we shall show soundness and completeness for this semantics by using almost the same argument as the phase semantic argument for provability.

2 Phase semantics for proofs of $IL_{\to\wedge}$

In this section, we shall introduce a type assignment system for the implicational conjunctive fragment of traditional intuitionistic logic

$IL_{\to\wedge}$ and phase semantics for proofs of this system.

2.1 Review of the type assignment system for $IL_{\to\wedge}$

There are two ways to introduce types into λ-calculus. One is due to [Curry 34], and the system is called type assignment system. The other is due to [Church 40], and the system is called explicitly-typed λ-calculus. We shall introduce a type assignment system for $IL_{\to\wedge}$. A type assignment system is designed on the operation of assigning a type to an untyped λ-term. We review the basic notions following [Barendregt 92].

Definition 2.1.1 (Untyped λ-terms) *Untyped λ-terms (λ-terms, for short) are defined as follows:*

- Variables $x, y, z, \ldots$ are λ-terms.
- If x is a variable and s is a λ-term, then $\lambda x.s$ is a λ-term.
- If s, t are λ-terms, then $E^{\to}(s, t)$, $< s, t >, E_1^{\wedge}(s), E_2^{\wedge}(s)$ are λ-terms.

$E^{\to}(s, t)$ is the application of s and t, and it is usually written as (st). $E_1^{\wedge}, E_2^{\wedge}$ are projections, and they are usually written, for example, as $\pi_1(s), \pi_2(s)$. Our notation comes from [Troelstra 92].

We shall define substitution. With α-conversion, we assume that a variable does not occur free and bound in a λ-term.

Definition 2.1.2 (Substitution) *For any λ-terms s, t and a variable x, the substitution of t for the free occurrences of x in s (written $s[x := t]$) is defined as follows, where $x \not\equiv y$:*

$x[x := t] \equiv t$,
$y[x := t] \equiv y$,
$(\lambda y.u)[x := t] \equiv \lambda y.(u[x := t])$,
$(\lambda x.u)[x := t] \equiv \lambda x.u$,
$E^{\to}(u, v)[x := t] \equiv E^{\to}(u[x := t], v[x := t])$,
$< u, v > [x := t] \equiv < u[x := t], v[x := t] >$,
$E_i^{\wedge}(u)[x := t] \equiv E_i^{\wedge}(u[x := t])$ for $i = 1, 2$.

Definition 2.1.3 (Types) *Types are defined as follows:*

- Atomic types $R, Q, \ldots, R_1, R_2, \ldots$ are types.
- If A and B are types, then $A \to B$ and $A \wedge B$ are also types.

Next we define the property of typability of λ-terms.

Definition 2.1.4 (Typable)

- A **statement** is of the form $s : A$ with a λ-term s and a type A.
- A **context** is a finite set of statements such that $x_1 : A_1, \ldots, x_n : A_n$ where all $x_1, \ldots, x_n$ are distinct variables. We write Γ, Δ, Π for any context.
- A λ-term s is **typable** as type A if a sequent $\Gamma \vdash s : A$ is derivable for some context Γ by the following type assignment rules.

Definition 2.1.5 (Type assignment rules)

- Axiom
$$\Gamma, x : A \vdash x : A$$
- $\to I$
$$\frac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.s : A \to B} \to I$$

- $\to E$

$$\frac{\Gamma \vdash s : A \to B \quad \Delta \vdash t : A}{\Gamma \cup \Delta \vdash E^{\to}(s,t) : B} \to E$$

$\cup$ is the set theoretical union operator.
We sometimes write Γ, Δ in place of $\Gamma \cup \Delta$.

- $\wedge I$

$$\frac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash < s,t >: A \wedge B} \wedge I$$

- $\wedge E_1$
- $\wedge E_2$

$$\frac{\Gamma \vdash s : A \wedge B}{\Gamma \vdash E_1^{\wedge}(s) : A} \wedge E_1 \quad \frac{\Gamma \vdash s : A \wedge B}{\Gamma \vdash E_2^{\wedge}(s) : B} \wedge E_2$$

We shall introduce $\beta\eta$ equality relation $\simeq$.

Definition 2.1.6 ($\beta\eta$ equality relation)

- β-rule

$$E^{\to}(\lambda x.s, t) \simeq s[x := t]$$
$$E_1^{\wedge}(< s,t >) \simeq s$$
$$E_2^{\wedge}(< s,t >) \simeq t$$

- η-rule

$$\lambda x.E^{\to}(s,x) \simeq s \text{ if } x \notin FV(s)$$
$$< E_1^{\wedge}(s), E_2^{\wedge}(s) >\simeq s$$

- Congruence rules

$$\frac{s \simeq t}{E^{\to}(u,s) \simeq E^{\to}(u,t)} \quad \frac{s \simeq t}{E^{\to}(s,u) \simeq E^{\to}(t,u)}$$
$$\frac{s \simeq t}{< u,s >\simeq< u,t >} \quad \frac{s \simeq t}{< s,u >\simeq< t,u >}$$
$$\frac{s \simeq t}{E_1^{\wedge}(s) \simeq E_1^{\wedge}(t)} \quad \frac{s \simeq t}{E_2^{\wedge}(s) \simeq E_2^{\wedge}(t)}$$

- ξ-rule

$$\frac{s \simeq t}{\lambda x.s \simeq \lambda x.t}$$

- Reflection rule

$$s \simeq s$$

- Symmetry rule

$$\frac{s \simeq t}{t \simeq s}$$

- Transitivity rule

$$\frac{s \simeq t \quad t \simeq u}{s \simeq u}$$

– A **redex** is a λ-term of the form $E^{\to}(\lambda x.s, t)$ or $E_i^{\wedge}(< s,t >)$ for $i = 1, 2$.

– A λ-term s is in **normal form**, if s contains no redex.

2.2 Phase semantics for proofs of $IL_{\to\wedge}$

In this subsection, we shall introduce phase semantics for proofs of the implicational conjunctive fragment. Operations $\to$ and $\wedge$ are introduced in a manner similar to those operations in phase semantics for provability, namely based on the $\to$ elimination rule and the set theoretical intersection operator.

The domain of our models consists of untyped λ-terms with contexts. The reason why we need not only λ-terms but also contexts shall be stated as Remark 3.3.1 in the end of the proof of the completeness theorem.

Definition 2.2.1 (Phase model) *A phase model is* $(\mathcal{M}, \mathcal{P}, *)$ *such that:*

- $\mathcal{M}$ is a set.
- $\mathcal{P} = \{(M \rhd s) | M \subseteq \mathcal{M}$ and s is an untyped λ-term $\}$. We sometimes call this set the domain of labeled λ-terms.
- There are some operations for any $\alpha, \beta \subseteq \mathcal{P}$ defined as follows:
  - $\alpha \to \beta = \{(M \rhd s) | (M \cup N \rhd E^{\to}(s,t)) \in \beta$ for any $(N \rhd t) \in \alpha\}$,
  - $E_i^{\wedge}(\alpha) = \{(M \rhd s) | (M \rhd E_i^{\wedge}(s)) \in \alpha\}$ for $i = 1, 2$,
  - $\alpha \wedge \beta = E_1^{\wedge}(\alpha) \cap E_2^{\wedge}(\beta)$.
- c is a closure operator from the power set of $\mathcal{P}$ to the power set of $\mathcal{P}$ such that:
  - if $s \in \alpha^c$ and $s \simeq t$, then $t \in \alpha^c$.
- Further we assume that α^c satisfies the *monotonicity* property, namely if $(M \rhd s) \in \alpha^c$ and $M \subseteq N$ then $(N \rhd s) \in \alpha^c$.
- $\alpha \subseteq \mathcal{P}$ is called closed if $\alpha = \beta^c$ for some $\beta \subseteq \mathcal{P}$.
- We call $(\mathcal{M}, \mathcal{P})$ phase space.
- $*$ is an interpretation function such that:

- $R^* = \alpha$ for some closed set α,
- $(A \to B)^* = A^* \to B^*$,
- $(A \wedge B)^* = A^* \wedge B^*$.

Lemma 2.2.1 *For any α and β, $\alpha \to \beta^c$ and $\alpha^c \wedge \beta^c$ are closed, namely: if $(M \rhd s) \in \alpha \to \beta^c$ and $s \simeq t$, then $(M \rhd t) \in \alpha \to \beta^c$; and if $(M \rhd s) \in \alpha^c \wedge \beta^c$ and $s \simeq t$, then $(M \rhd t) \in \alpha^c \wedge \beta^c$.*

Proof: Assume $(M \rhd s) \in \alpha \to \beta^c$ and $s \simeq t$, then we have $(M \cup N \rhd E^{\to}(s, u)) \in \beta^c$ for any $(N \rhd u) \in \alpha$. Since β^c is closed and we have $E^{\to}(s, u) \simeq E^{\to}(t, u)$ from the hypothesis $s \simeq t$, we have $(M \cup N \rhd E^{\to}(t, u)) \in \beta^c$ for any $(N \rhd u) \in \alpha$. Hence $(M \rhd t) \in \alpha \to \beta^c$.

Let $(M \rhd s) \in \alpha^c \wedge \beta^c$, namely $(M \rhd E_1^{\wedge}(s)) \in \alpha^c$ and $(M \rhd E_2^{\wedge}(s)) \in \beta^c$. From the hypothesis $s \simeq t$, we have $E_1^{\wedge}(s) \simeq E_1^{\wedge}(t)$ and $E_2^{\wedge}(s) \simeq E_2^{\wedge}(t)$. Since α^c and β^c are closed, we have $(M \rhd E_1^{\wedge}(t)) \in \alpha^c$ and $(M \rhd E_2^{\wedge}(t)) \in \beta^c$, namely $(M \rhd t) \in \alpha^c \wedge \beta^c$.

■

From this lemma and the definition of the interpretation function, any interpretation A^* of type A is closed.

3 Soundness and completeness of $IL_{\to\wedge}$

In this section, we shall prove the soundness theorem and the completeness theorem. Our argument below is essentially an extension of the completeness argument of phase semantics for provability in [Okada 96,99,02].

We shall also prove the normal form theorem for $IL_{\to\wedge}$ with the following form: *if a λ-term s is typable, then there is a λ-term t in normal form such that $s \simeq_{\beta\eta} t$* . This theorem is obtained from the soundness theorem and the completeness theorem.

3.1 Soundness theorem

We shall show the soundness theorem.

Theorem 3.1.1 (Soundness theorem) *If $x_1 : A_1, \ldots, x_k : A_k \vdash s : B$, then, for any phase model $(\mathcal{M}, \mathcal{P}, *)$, $(\bigcup_{1 \le i \le k} M_i \rhd s[x_1 := s_1, \ldots, x_k := s_k]) \in B^*$ for any $(M_i \rhd s_i) \in A_i^*$.*

Proof: We shall prove this theorem by induction on the construction of derivation for $x_1 : A_1, \ldots, x_k : A_k \vdash s : B$. We shall abbreviate $x_1 : A_1, \ldots, x_k : A_k$ as Γ and $y_1 : B_1, \ldots, y_l : B_l$ as Δ. We shall also abbreviate $\bigcup_{1 \le i \le k} M_i$ as $\bigcup M_k$ if $1 \le i \le k$ is clear from the context.

1. When $\Gamma, x : A \vdash x : A$.

   We should show $(\bigcup M_k \cup M \rhd x[x_1 := s_1, \ldots, x_k := s_k, x := t]) \in A^*$ for any $(M \rhd t) \in A^*$ and for any $(M_i \rhd s_i) \in A_i^*$, but this is obvious from the monotonicity condition.

2. When $\dfrac{\Gamma, x : A \vdash s : B}{\Gamma \vdash \lambda x.s : A \to B} \to I$.

   From the induction hypothesis, we have $(\bigcup M_k \cup N \rhd s[x_1 := s_1, \ldots, x_k := s_k, x := t]) \in B^*$ for any $(M_i \rhd s_i) \in A_i^*, (N \rhd t) \in A^*$. Because B^* is closed and $s[x_1 := s_1, \ldots, x_k := s_k, x := t] \simeq E^{\to}(\lambda x.s, t)[x_1 := s_1, \ldots, x_k := s_k]$, we have $(\bigcup M_k \cup N \rhd E^{\to}(\lambda x.s, t)[x_1 := s_1, \ldots, x_k := s_k]) \in B^*$, for any $(M_i \rhd s_i) \in A_i^*, (N \rhd t) \in A^*$, this means $(\bigcup M_k \rhd \lambda x.s[x_1 := s_1, \ldots, x_k := s_k]) \in A^* \to B^*$, for any $(M_i \rhd s_i) \in A_i^*$ from the definition of $\to$.

3. When $\dfrac{\Gamma \vdash s : A \to B \quad \Delta \vdash t : A}{\Gamma \cup \Delta \vdash E^{\to}(s, t) : B} \to E$.

   From the induction hypothesis, we have $(\bigcup M_k \rhd s[x_1 := s_1, \ldots, x_k := s_k]) \in (A \to B)^*$ for any $(M_i \rhd s_i) \in A_i^*$, and $(\bigcup N_l \rhd t[y_1 := t_1, \ldots, y_l := t_l]) \in A^*$ for any $(N_j \rhd$

$t_j) \in B_j^*$. Hence it is immediate, from the definition of $\rightarrow$, that $(\bigcup M_k \cup \bigcup N_l \rhd E^{\rightarrow}(s,t)[x_1 := s_1, \ldots, x_k := s_k, y_1 := t_1, \ldots, y_l := t_l]) \in B^*$.

4. When $\dfrac{\Gamma \vdash s : A \quad \Gamma \vdash t : B}{\Gamma \vdash < s,t >: A \wedge B} \wedge I$.

   From the induction hypothesis, we have $(\bigcup M_k \rhd s[x_1 := s_1, \ldots, x_k := s_k]) \in A^*$ and $(\bigcup M_k \rhd t[x_1 := s_1, \ldots, x_k := s_k]) \in B^*$ for any $(M_i \rhd s_i) \in A_i^*$. Since $s[x_1 := s_1, \ldots, x_k := s_k] \simeq E_1^{\wedge}(< s,t >)[x_1 := s_1, \ldots, x_k := s_k]$ and $t[x_1 := s_1, \ldots, x_k := s_k] \simeq E_2^{\wedge}(< s,t >)[x_1 := s_1, \ldots, x_k := s_k]$, and since A^*, B^* are closed, we have $E_1^{\wedge}(< s,t >)[x_1 := s_1, \ldots, x_k := s_k] \in A^*$ and $E_2^{\wedge}(< s,t >)[x_1 := s_1, \ldots, x_k := s_k] \in B^*$. Hence, from the definition, we have $< s,t > [x_1 := s_1, \ldots, x_k := s_k] \in A^* \wedge B^*$.

5. When $\dfrac{\Gamma \vdash s : A \wedge B}{\Gamma \vdash E_1^{\wedge}(s) : A} \wedge E_1$.

   From the induction hypothesis, we have $(\bigcup M_k \rhd s[x_1 := s_1, \ldots, x_k := s_k]) \in A^* \wedge B^*$ for any $(M_i \rhd s_i) \in A_i^*$. From the definition, we have $(\bigcup M_k \rhd E_1^{\wedge}(s)[x_1 := s_1, \ldots, x_k := s_k]) \in A^*$.

6. When $\dfrac{\Gamma \vdash s : A \wedge B}{\Gamma \vdash E_2^{\wedge}(s) : B} \wedge E_2$.

   From the induction hypothesis, we have $(\bigcup M_k \rhd s[x_1 := s_1, \ldots, x_k := s_k]) \in A^* \wedge B^*$ for any $(M_i \rhd s_i) \in A_i^*$. From the definition, we have $(\bigcup M_k \rhd E_2^{\wedge}(s)[x_1 := s_1, \ldots, x_k := s_k]) \in B^*$.

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3.2 Completeness theorem

In this subsection, we shall prove the completeness theorem. Here we consider "completeness" of the form $(\Gamma, x : A \rhd x) \in A^* \subseteq [\![A]\!]$ for any Γ with $x : A \notin \Gamma$ and for any A, where A^* is the interpretation of A in the canonical model called the inner-value of A and $[\![A]\!]$ is a certain set in the canonical model called the outer-value of A in [Okada 96]. Note that Girard's original version of the completeness theorem is of the form $A^* = [\![A]\!]$ in [Girard 87], where $[\![A]\!] = \{\Gamma | \Gamma \vdash A \text{ is provable}\}$. [1] However, this original form of the completeness theorem does not imply the proof normalization theorem. Okada introduced, with respect to phase semantics for provability, the completeness theorem of the form $A \in A^* \subseteq [\![A]\!]$, where $[\![A]\!] = \{\Gamma | \Gamma \vdash A \text{ is provable without cut-rule }\}$, then he proved the cut-elimination theorem via soundness and completeness. See [Okada 96,99,02]. In order to prove the normal form theorem, we shall apply this technique to our phase semantics for proofs.

We construct a canonical model $(\mathcal{M}, \mathcal{P}, *)$ as follows:

- $\mathcal{M}$ is the set of statements of the form $x : A$ for any variable term x and any type A.
- $\mathcal{P} = \{(\Gamma \rhd s) | \Gamma \subseteq \mathcal{M} \text{ and } s \text{ is an untyped } \lambda\text{-term }\}$.
- On this set, we shall consider the following outer-values:
  - $[\![A]\!]_T = \{(\Gamma \rhd s) | s \simeq t$ for some t such that $\Gamma \vdash t : A\}$,
  - $[\![A]\!]_W = \{(\Gamma \rhd s) | s \simeq t$ for some t in normal form such that $\Gamma \vdash t : A\}$.
- We shall write simply $[\![A]\!]$ without the subscription, when the definition or the property to be shown holds for both cases.
- Then we shall define an interpretation function as $R^* = [\![R]\!]$.

According to the definition of outer-values, there are two canonical models, namely $(\mathcal{M}, \mathcal{P}, *)$ with the interpretation $R^* = [\![R]\!]_T$ and with the interpretation $R^* = [\![R]\!]_W$. We

[1] Girard considered in one-sided sequent calculus.

call the later the canonical model for the normal form theorem.

The above construction indeed produce phase models.

Lemma 3.2.1 *These models satisfy the definition of phase model.*

In order to show this lemma, it is only necessary to prove the following two facts: *A^* is closed for any A*; and *if $(\Gamma \rhd s) \in A^*$ and $\Gamma \subseteq \Delta$, then $(\Delta \rhd s) \in A^*$.*

- A^* is closed for any A.

Proof: We shall prove this fact by induction on the complexity of A.

1. When $A \equiv R$.
   Since $R^* = [\![R]\!]$, we shall show that $(\Gamma \rhd s) \in [\![R]\!]$ and $s \simeq t$, then $(\Gamma \rhd t) \in [\![R]\!]$.
   Let $(\Gamma \rhd s) \in [\![R]\!]$ and $s \simeq t$, then we have $s \simeq u$ for some u (in normal form) such that $\Gamma \vdash u : R$. Hence we have $t \simeq s \simeq u$ for some u (in normal form) such that $\Gamma \vdash u : R$. Hence $(\Gamma \rhd t) \in [\![R]\!] = R^*$.

2. When $A \equiv B \to C$.
   Since C^* is closed from the induction hypothesis, $B^* \to C^*$ is closed from the lemma 2.2.1.

3. When $A \equiv B \wedge C$.
   Since B^* and C^* are closed from the induction hypothesis, $B^* \wedge C^*$ is closed from the lemma 2.2.1.

- If $(\Gamma \rhd s) \in A^*$ and $\Gamma \subseteq \Delta$, then $(\Delta \rhd s) \in A^*$.

We shall show this fact by induction on the complexity of A.

1. When $A \equiv R$.
   Let $(\Gamma \rhd s) \in R^* = [\![R]\!]$. Then we have $s \simeq v$ for some v (in normal form) such that $\Gamma \vdash v : R$. From $\Gamma \subseteq \Delta$, we also have $s \simeq v$ for some v (in normal form) such that $\Delta \vdash v : R$, namely $(\Delta \rhd s) \in [\![R]\!] = R^*$.

2. When $A \equiv B \to C$.
   Let $(\Gamma \rhd s) \in B^* \to C^*$, namely $(\Gamma \cup \Pi \rhd E^{\to}(s,t)) \in C^*$ for any $(\Pi \rhd t) \in B^*$. From the induction hypothesis for C^*, we have $(\Gamma \cup \Pi \cup \Delta \rhd E^{\to}(s,t)) \in C^*$ for any $(\Pi \rhd t) \in B^*$ and for any Δ such that $\Gamma \subseteq \Delta$. Hence we have $(\Gamma \cup \Delta \rhd s) \in B^* \to C^*$. Since $\Gamma \subseteq \Delta$, we have $(\Delta \rhd s) \in B^* \to C^*$.

3. When $A \equiv B \wedge C$.
   Let $(\Gamma \rhd s) \in B^* \wedge C^*$, namely $(\Gamma \rhd E_1^{\wedge}(s)) \in B^*$ and $(\Gamma \rhd E_2^{\wedge}(s)) \in C^*$. From the induction hypothesis for B^* and C^*, we have $(\Delta \rhd E_1^{\wedge}(s)) \in B^*$ and $(\Delta \rhd E_2^{\wedge}(s)) \in C^*$ for any Δ such that $\Gamma \subseteq \Delta$. Hence, from the definition, we have $(\Delta \rhd s) \in B^* \wedge C^*$.

Therefore each $(\mathcal{P}, \mathcal{M}, *)$ is a phase model.

■

In order to express the completeness theorem, we shall introduce some notations.

We introduce a meta expression $E(s_0 \dots s_n s_{n+1})$, which roughly means a λ-term obtained from s_0 by repeated elimination rules.

Definition 3.2.1

$E(s_0)$ means a λ-term s_0;
$E(s_0 \dots s_{n+1})$ means $E^{\to}(E(s_0 \dots s_n), s_{n+1})$ or $E_i^{\wedge}(E(s_0 \dots s_k))$ for some $k \leq n$, for $i = 1, 2$.

We also introduce a meta expression for the corresponding type $E(A_0 \dots A_n)$ of $E(s_0 \dots s_n)$, if s_i has type A_i for each i.

$$
\begin{array}{lcl}
E(A_0) & \equiv & A_0 \\
E(A_1 \dots A_{n+1} A_0) & \equiv & E(A_1 \dots A_n (A_{n+1} \to A_0)) \\
 & & \textit{if } E(s_0 \dots s_{n+1}) \textit{ is } \\
 & & E^{\to}(E(s_0 \dots s_n), s_{n+1}) \\
 & \equiv & E(A_1 \dots A_n (A_0 \wedge A_{n+1})) \\
 & & \textit{if } E(s_0 \dots s_{n+1}) \textit{ is } \\
 & & E_1^{\wedge}(E(s_0 \dots s_n)) \\
 & \equiv & E(A_1 \dots A_n (A_{n+1} \wedge A_0)) \\
 & & \textit{if } E(s_0 \dots s_{n+1}) \textit{ is } \\
 & & E_2^{\wedge}(E(s_0 \dots s_n))
\end{array}
$$

Note that if $E(s_0 \dots s_n)$ is given as a concrete λ-term, then we can find the corresponding type $E(A_1 \dots A_n A_0)$ as a concrete type. [2]

Theorem 3.2.1 (Completeness theorem) *$(\Gamma, x : A \rhd x) \in A^* \subseteq [\![A]\!]$ for any Γ with $x : A \notin \Gamma$ and for any A.*

Proof: We show the conjunction of the following two facts by induction on the complexity of A.

- If for any $(\Gamma_i \rhd s_i) \in [\![B_i]\!]$ for each $0 \le i \le n$, then $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n A) \rhd E(x s_1 \dots s_n)) \in A^*$ for any A, B_i and for any Γ with $x : E(B_1 \dots B_n A) \notin \Gamma \cup \bigcup \Gamma_n$.
- $A^* \subseteq [\![A]\!]$.

In particular, we have $(\Gamma, x : A \rhd x) \in A^*$ from the above fact when $n = 0$.

1. When $A \equiv R$.
   Since $R^* = [\![R]\!]$, we show $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n R) \rhd E(x s_1 \dots s_n)) \in [\![R]\!]$.

   From the hypothesis $(\Gamma_i \rhd s_i) \in [\![B_i]\!]$, there is a λ-term v_i (in normal form) such that $v_i \simeq s_i$ and $\Gamma_i \vdash v_i : B_i$ for each i. From congruence rules, we have $E(x s_1 \dots s_n) \simeq E(x v_1 \dots v_n)$, and we can find the corresponding type $E(B_1 \dots B_n R)$. Then from the axiom $\Gamma, x : E(B_1 \dots B_n R) \vdash x : E(B_1 \dots B_n R)$, we obtain $\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n R) \vdash E(x v_1 \dots v_n) : R$ by repeated applications of elimination rules of the following forms:

$$\frac{\bigcup \Gamma_i, \Gamma, x : E(\overrightarrow{B_n} R) \vdash E(x\overrightarrow{v_i}) : B_{i+1} \to E(B_{i+2} \dots B_n R) \qquad \Gamma_{i+1} \vdash v_{i+1} : B_{i+1}}{\bigcup \Gamma_{i+1}, \Gamma, x : E(\overrightarrow{B_n} R) \vdash E^{\to}(E(x\overrightarrow{v_i}), v_{i+1}) : E(B_{i+2} \dots B_n R)} \to E$$

   or

$$\frac{\bigcup \Gamma_i, \Gamma, x : E(\overrightarrow{B_n} R) \vdash E(x\overrightarrow{v_i}) : E(B_{i+2} \dots B_n R) \wedge B_{i+1}}{\bigcup \Gamma_{i+1}, \Gamma, x : E(\overrightarrow{B_n} R) \vdash E_1^{\wedge}(E(x\overrightarrow{v_i})) : E(B_{i+2} \dots B_n R)} \wedge E1$$

   or

$$\frac{\bigcup \Gamma_i, \Gamma, x : E(\overrightarrow{B_n} R) \vdash E(x\overrightarrow{v_i}) : B_{i+1} \wedge E(B_{i+2} \dots B_n R)}{\bigcup \Gamma_{i+1}, \Gamma, x : E(\overrightarrow{B_n} R) \vdash E_2^{\wedge}(E(x\overrightarrow{v_i})) : E(B_{i+2} \dots B_n R)} \wedge E2$$

   [3]

   For $[\![\ ]\!]_T$:
   since
   we have $E(x s_1 \dots s_n) \simeq E(x v_1 \dots v_n)$ and $\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n R) \vdash E(x v_1 \dots v_n) : R$, we have $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n R) \rhd E(x s_1 \dots s_n)) \in [\![R]\!]_T$.

   For $[\![\ ]\!]_W$:
   since v_i is in normal form for each i, $E(x v_1 \dots v_n)$ is also in normal form. Hence we have $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n R) \rhd E(x s_1 \dots s_n)) \in [\![R]\!]_W$.

   $R^* \subseteq [\![R]\!]$ is obvious from the definition.

2. When $A \equiv B \to C$.
   We show $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n (B \to C)) \rhd E(x s_1 \dots s_n)) \in B^* \to C^*$.

   Since $x : E(B_1 \dots B_n (B \to C))$ is $x : E(B_1 \dots B_n B C)$, we show $(\bigcup \Gamma_n, \Delta, \Gamma, x : E(B_1 \dots B_n B C) \rhd E^{\to}(E(x s_1 \dots s_n), t) \in C^*$ for any $(\Delta \rhd t) \in B^*$. This is true, since $E^{\to}(E(x s_1 \dots s_n), t)$ is $E(x s_1 \dots s_n t)$ and we can apply the induction hypothesis for

[2] If $E(s_0 \dots s_n)$ is given as a concrete λ-term, for example, $E_1^{\wedge}(E_2^{\wedge}(E^{\to}(E^{\to}(x, s_1), s_2)))$,
then
we can find the corresponding type $E(B_1 B_2 B_3 B_4 R)$ as a concrete type.
From $E_1^{\wedge}(E_2^{\wedge}(E^{\to}(E^{\to}(x, s_1), s_2)))$, $E(B_1 B_2 B_3 B_4 R)$ is $E(B_1 B_2 B_3 (R \wedge B_4))$.
Then from $E_2^{\wedge}(E^{\to}(E^{\to}(x, s_1), s_2))$, $E(B_1 B_2 B_3 (R \wedge B_4))$ is $E(B_1 B_2 (B_3 \wedge (R \wedge B_4)))$.
Then from $E^{\to}(E^{\to}(x, s_1), s_2)$, $E(B_1 B_2 (B_3 \wedge (R \wedge B_4)))$ is $E(B_1 (B_2 \to (B_3 \wedge (R \wedge B_4))))$.
Then from $E^{\to}(x, s_1)$, $E(B_1 (B_2 \to (B_3 \wedge (R \wedge B_4))))$ is $B_1 \to (B_2 \to (B_3 \wedge (R \wedge B_4)))$.

[3] For example,

$$\cfrac{\cfrac{\cfrac{\cfrac{\Gamma, x : E(B_1 B_2 B_3 B_4 R) \vdash x : B_1 \to (B_2 \to (B_3 \wedge (R \wedge B_4))) \qquad \Gamma_1 \vdash v_1 : B_1}{\Gamma_1, \Gamma, x : E(B_1 B_2 B_3 B_4 R) \vdash E^{\to}(x, v_1) : B_2 \to (B_3 \wedge (R \wedge B_4))} \qquad \Gamma_2 \vdash v_2 : B_2}{\bigcup \Gamma_2, \Gamma, x : E(B_1 B_2 B_3 B_4 R) \vdash E^{\to}(E^{\to}(x, v_1), v_2) : B_3 \wedge (R \wedge B_4)}}{\bigcup \Gamma_3, \Gamma, x : E(B_1 B_2 B_3 B_4 R) \vdash E_2^{\wedge}(E^{\to}(E^{\to}(x, v_1), v_2)) : R \wedge B_4}}{\bigcup \Gamma_4, \Gamma, x : E(B_1 B_2 B_3 B_4 R) \vdash E_1^{\wedge}(E_2^{\wedge}(E^{\to}(E^{\to}(x, v_1), v_2))) : R}$$

B and C.

Next, we shall prove $B^* \to C^* \subseteq [\![B \to C]\!]$.
Let $(\Gamma \rhd s) \in B^* \to C^*$, namely $(\Gamma \cup \Delta \rhd E^{\to}(s,t)) \in C^*$ for any $(\Delta \rhd t) \in B^*$. In particular, using the induction hypothesis and the above fact, we can take $(x : B \rhd x) \in B^*$ and then we have $(\Gamma \cup \{x : B\} \rhd E^{\to}(s,t)) \in C^* \subseteq [\![C]\!]$ for $x : B \notin \Gamma$, namely $x \notin FV(s)$. [4]
Hence we have $E^{\to}(s,x) \simeq v$ for some v (in normal form) such that $\Gamma, x : B \vdash v : C$. Then from ξ rule, we have $\lambda x.E^{\to}(s,x) \simeq \lambda x.v$. Since $x \notin FV(s)$, we have $\lambda x.E^{\to}(s,x) \simeq s$ from the η rule. Hence we have $s \simeq \lambda x.v$.

Furthermore, we have the following derivation:

$$\frac{\Gamma, x : B \vdash v : C}{\Gamma \vdash \lambda x.v : B \to C} \to I.$$

For $[\![\ ]\!]_T$:
therefore $(\Gamma \rhd s) \in [\![B \to C]\!]_T$.

For $[\![\ ]\!]_W$:
since v is in normal form, $\lambda x.v$ is also in normal form. Hence $(\Gamma \rhd s) \in [\![B \to C]\!]_W$.

3. When $A \equiv B \wedge C$.
We show $((\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n(B \wedge C)) \rhd E(xs_1 \dots s_n)) \in B^* \wedge C^*$.

There are two cases:
• If $E(xs_1 \dots s_n)$ is $E_1^{\wedge}(E(xs_1 \dots s_{n-1}))$, then $E(B_1 \dots B_n(B \wedge C))$ is $E(B_1 \dots B_n CB)$. Hence from the induction hypothesis for B, we have $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n(B \wedge C)) \rhd E(xs_1 \dots s_n)) \in B^*$.
• If $E(xs_1 \dots s_n)$ is $E_2^{\wedge}(E(xs_1 \dots s_{n-1}))$, then $E(B_1 \dots B_n(B \wedge C))$ is $E(B_1 \dots B_n BC)$. Hence from the induction hypothesis for C, we have $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n(B \wedge C)) \rhd E(xs_1 \dots s_n)) \in C^*$.

[4] Here note that we need the condition $x : B \notin FV(s)$ for the η equality.

Therefore, we have $(\bigcup \Gamma_n, \Gamma, x : E(B_1 \dots B_n(B \wedge C)) \rhd E(xs_1 \dots s_n)) \in B^* \wedge C^*$.

We show $B^* \wedge C^* \subseteq [\![B \wedge C]\!]$.
Let $(\Gamma \rhd s) \in B^* \wedge C^*$, namely $(\Gamma \rhd E_1^{\wedge}(s)) \in B^*$ and $(\Gamma \rhd E_2^{\wedge}(s)) \in C^*$. We have $(\Gamma \rhd E_1^{\wedge}(s)) \in [\![B]\!]$ and $(\Gamma \rhd E_2^{\wedge}(s)) \in [\![C]\!]$ from the induction hypothesis $B^* \subseteq [\![B]\!]$ and $C^* \subseteq [\![C]\!]$.

Hence we have $E_1^{\wedge}(s) \simeq u$ for some u (in normal form) such that $\Gamma \vdash u : B$, and we have $E_2^{\wedge}(s) \simeq v$ for some v (in normal form) such that $\Gamma \vdash v : C$. From the congruence rule and the η-rule, we have $s \simeq < E_1^{\wedge}(s), E_2^{\wedge}(s) > \simeq < u, v >$, and $s \simeq < u, v >$.

Furthermore, we have the following derivation:

$$\frac{\Gamma \vdash u : B \quad \Gamma \vdash v : C}{\Gamma \vdash < u, v >: B \wedge C} \wedge I$$

For $[\![\ ]\!]_T$:
there fore $(\Gamma \rhd s) \in [\![B \wedge C]\!]_T$.

For $[\![\ ]\!]_W$:
since u and v are in normal form, $< u, v >$ is also in normal form. Hence $(\Gamma \rhd s) \in [\![B \wedge C]\!]_W$.

■

3.3 Normal form theorem with respect to $\simeq_{\beta\eta}$ for $IL_{\to\wedge}$

We show the normal form theorem of the following form:

Theorem 3.3.1 (Normal form theorem) *If a λ-term s is typable, then there is a λ-term t in normal form such that $s \simeq_{\beta\eta} t$.*

Proof: Let $x_1 : A_1, \dots, x_k : A_k \vdash s : B$ be derivable, then from the soundness theorem, for any interpretation $*$, $(\bigcup_{1 \le i \le k} M_i \rhd s[x_1 := s_1, \dots, x_k := s_k]) \in B^*$ for any $(M_i \rhd s_i) \in A_i^*$. In particular, $(\bigcup_{1 \le i \le k} \Gamma_i \rhd s[x_1 :=$

$s_1, \ldots, x_k := s_k]) \in B^*$ for any $(\Gamma_i \rhd s_i) \in A_i^*$ holds in the canonical model for the normal form theorem. Then from the completeness theorem $(x_i : A_i \rhd x_i) \in A_i^*$ and $B^* \subseteq [\![B]\!]_W$, we have $(x_1 : A_1, \ldots, x_k : A_k \rhd s) \in [\![B]\!]_W$. Thus there is a term t in normal form such that $s \simeq t$. ■

Remark 3.3.1 We need contexts even if we consider only the implicational fragment. If we do not consider contexts, in order to show $B^* \to C^* \subseteq [\![B \to C]\!]$, we should show that for any $t \in B^*$, if $E^{\to}(s,t) \in C^*$ then $s \in [\![B \to C]\!]$.
Let $B \equiv R, C \equiv Q, s \equiv \lambda z.E^{\to}(z,w), t \equiv E^{\to}(x,y)$ for atomic types R, Q and variable terms x, y, z, w. Then we have $E^{\to}(s,t) \equiv E^{\to}(\lambda z.E^{\to}(z,w), E^{\to}(x,y)) \in [\![Q]\!]$ and $t \equiv E^{\to}(x,y) \in [\![R]\!]$, however $s \equiv \lambda z.E^{\to}(z,w) \notin [\![R \to Q]\!]$ since $s \equiv \lambda z.E^{\to}(z,w)$ should have type of the form $(X \to Y) \to Y$.
Further we have $s \equiv \lambda z.E^{\to}(z,w) \in R^* \to Q^*$. In order to show this, we should show that for any $u \in [\![R]\!]$, $E^{\to}(\lambda z.E^{\to}(z,w),u) \in [\![Q]\!]$. Let $u \in [\![R]\!]$, then u has type R, hence u has any type. On the other hand, the variable term w also has any type. Hence $E^{\to}(u,w) \in [\![Q]\!]$. Since $[\![Q]\!]$ is closed under $\simeq$, we have $E^{\to}(\lambda z.E^{\to}(z,w),u) \in [\![Q]\!]$. Hence $s \equiv \lambda z.E^{\to}(z,w) \in R^* \to Q^*$.

4 An Extension to a stronger form of the normalization theorem

By modifying the above argument for soundness and completeness with respect to phase semantics for proofs, we can also show the weak normalization theorem of the form: *if a λ-term s is typable, then s is reducible to a normal form.* This form of the weak normalization theorem is proved by making the following small changes for the definition of semantics and for the completeness argument of Section 3:

- For the definition of semantics, the closure condition of the form
  if $s \in \alpha^c$ and $s \simeq t$, then $t \in \alpha^c$
  is replaced with
  if $s \in \alpha^c$ and $s \leftarrow t$, then $t \in \alpha^c$, here $s \leftarrow t$ means t can be obtained from s by one step expansion.

- For the completeness argument, the definition of outer-values is replaced by
  $[\![A]\!]_T = \{(\Gamma \rhd s) | s$ is reducible to some t such that $\Gamma \vdash t : A\}$,
  $[\![A]\!]_W = \{(\Gamma \rhd s) | s$ is reducible to some t in normal form such that $\Gamma \vdash t : A\}$.

Then we can show the above form of the weak normalization theorem by almost the same argument as that of Section 3.

As for the type assignment system, we need the η-rule in order to prove the completeness theorem in Section 3. However, in the above setting, the η-rule is not needed.

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