Proof-Theoretical Investigation of Venn Diagrams: a Logic Translation and Free Rides

Ryo Takemura
Nihon University, Japan.
takemura.ryo@nihon-u.ac.jp

Abstract

In the literature on diagrammatic reasoning, Venn diagrams are abstractly formalized in terms of minimal regions. In view of the cognitive process to recognize Venn diagrams, we modify slightly the formalization by distinguishing conjunctive, negative, and disjunctive regions among possible regions in Venn diagrams. Then we study a logic translation of the Venn diagrammatic system with the aim of investigating how our inference rules are rendered to resolution calculus. We further investigate the free ride property of the Venn diagrammatic system. Free ride is one of the most basic properties of diagrammatic systems and it is mainly discussed in cognitive science literature as an account of the inferential efficacy of diagrams. The soundness of our translation shows that a free ride occurs between the Venn diagrammatic system and resolution calculus. Furthermore, our translation provides a more in-depth analysis of the free ride. In particular, we calculate how many pieces of information are obtained in the manipulation of Venn diagrams.

1 Introduction

Venn and Euler diagrams are two of the most basic diagrams for logical reasoning, originally introduced to illustrate syllogisms. These diagrams have been studied since the 1990s from both a mathematical and formal logic viewpoint, and have been applied, beyond simple syllogisms, to various areas such as knowledge representation and ontologies.

In the literature on diagrammatic reasoning, a Venn diagram is abstractly defined as a set of *minimal regions* (refer to Howse et al. [4] and to [11] for a survey).

For example, the Venn diagram, $\mathcal{V}$, in Fig. 1 is specified as a diagram in which "the region inside A, but outside B and C (denoted by $A\overline{B}\overline{C}$) is empty," and "the region inside A and C, but outside B ($A\overline{B}C$) is empty." Thus, $\mathcal{V}$ is abstractly formalized as the set of shaded minimal regions $\{A\overline{B}\overline{C}, A\overline{B}C\}$.

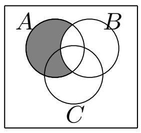


Fig. 1 Venn diagram, $\mathcal{V}$

Although we can read from this $\mathcal{V}$, "the region inside A, but outside B ($A\overline{B}$) is empty," such a (non minimal) region is generally defined as the union of minimal regions, i.e., $A\overline{B}\overline{C} \cup A\overline{B}C$, and hence, only minimal regions are mathematically sufficient for the abstract formalization of Venn diagrams. Based on this formalization of Venn diagrams, Euler diagrams are formalized by considering shaded regions of Venn diagrams as "missing" regions (cf. [4]). Thus, both Venn and Euler diagrams are abstractly formalized in terms of minimal regions, and we call this framework a "region-based" framework.

Based on the above formalization of Venn and Euler diagrams, their influence on human reasoning, particularly syllogistic reasoning, is discussed in the cognitive science literature by Gurr-Lee-Stenning [3] and Shimojima [9], amongst others. From a cognitive viewpoint, it is often observed that Venn diagrams are harder to handle in actual reasoning than Euler diagrams. However, from the mathematical formalization of Venn and Euler diagrams in the region-based framework, it is difficult to derive any differences between them especially in their cognitive complexities [3].

In contrast to studies using the region-based framework, Mineshima-Okada-Takemura [5] introduced another framework to formalize Euler diagrams. The framework is based on the idea that, from a cognitive viewpoint, the most basic and essential component of Euler diagrams is topological (inclusion and exclusion) relations between circles. Thus, in this framework, Euler diagrams and manipulations thereof are formalized in terms of the topological relations. The inference rules are designed to be as natural as possible to reflect intuitive manipulations of Euler diagrams. We describe this approach being "relation-based."

Along similar lines, if we consider processes to recognize Venn diagrams, it seems that the existing mathematical formalization in terms of minimal regions is somewhat inadequate. For example, in the diagram $\mathcal{V}$ in Fig. 1, in order to read "There is nothing that is A but not B," i.e., "All A are B," it is not sufficient to grasp the emptiness of the minimal regions $A\overline{B}\overline{C}$ and $A\overline{B}C$ discretely. Instead it is necessary to grasp the emptiness of the union of these regions as a whole. Thus, it is considered appropriate for the formalization of Venn diagrams to take into account the unions of certain minimal regions. Based on this idea, we modify slightly the region-based formalization of Venn diagrams. Of the possible regions in Venn diagrams, we distinguish conjunctive, negative, and disjunctive regions. In particular, each conjunctive region denotes "Things that are X_1 and ... and X_n, but not Y_1 or ... or Y_m." As the most basic system, we are mainly concerned with a formal Venn diagrammatic system based on conjunctive regions in Section 2, and we define an abstract Venn diagram as a set of shaded conjunctive regions.

In Section 3, we investigate a proof theory of the Venn diagrammatic system. In particular, we study a logic translation of the system with the aim of investigating how our Venn diagrammatic inference rules are rendered to resolution calculus. The logic translation enables us various further investigation of the Venn diagrammatic system. We investigate, among others, the free ride property of our system in Section 4. Free ride occurs when by adding a certain piece of information to a diagram, the resulting diagram somehow comes to represent pieces of information not contained in the given diagram or in the original piece of information. Shimojima [9] analyzed its semantic conditions within the framework of Barwise-Seligman's channel theory [1].

In contrast to the semantic investigation, Takemura [12] introduced a proof-theoretical framework to analyze the free ride through logic translations. It was shown that the soundness of a translation between two logical systems implies the occurrence of free ride (in the sense of [9, 1]) between the systems. Based on the result, we show that free ride occurs between our Venn diagrammatic system and resolution calculus in Section 4. Our logic translation gives us a more in-depth analysis of free ride. We are able to investigate what kind of and how many inference steps are required in terms of the translated logical system, to derive freely obtained pieces of information in the manipulation of diagrams. In particular, we investigate the number of pieces of information, that we call "free rides," occurring in an application of a Venn diagrammatic inference rule, which serves to estimate recognition steps for the free rides.

Since our formalization and analysis of the free ride is based on the general method of logic translation in the proof theory, they are also applicable to other appropriately defined diagrammatic systems.

In Section 5, based on the analysis given in Section 4 and that given in [12], we discuss a difference between the Venn and Euler diagrammatic systems with respect to free rides.

2 Venn diagrammatic system

In this section, we recall the syntax, semantics, and inference rules for Venn diagrams. See, e.g., Howse et al. [4], Shin [10] for detailed and formal descriptions thereof. In particular, in Section 2.1 we introduce a classification among regions in Venn diagrams, namely conjunctive, negative, and disjunctive regions. Inference rules, described in Section 2.2, are those defined in [6].

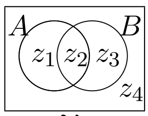
2.1 Venn diagrams

A (concrete) **Venn diagram** (denoted by $\mathcal{V}, \mathcal{V}_1, \mathcal{V}_2, \dots$) consists of a finite number of named circles (i.e., simple closed curves) on a plane $\mathbb{R}^2$ enclosed by a boundary rectangle that satisfies the *partial-overlapping* condition, i.e., all possible intersections of circles must occur.

A **minimal region** (denoted by $z, z_1, z_2, \dots$) is a part of the plane that lies inside some of the circles and outside the remaining circles of the diagram. (Although our "minimal region" may be more appropriately called a **zone** as in [4], we prefer to retain our terminology in this paper.) Thus, if $\mathcal{L}$ is the set of names of circles in a Venn diagram $\mathcal{V}$, each minimal region is specified using the names in $\mathcal{L}$ as $X_1 \dots X_n \overline{Y_1} \dots \overline{Y_m}$, where $\{X_1, \dots, X_n, Y_1, \dots, Y_m\} = \mathcal{L}$, and $X_1, \dots, X_n$ are the names of the circles enclosing the region, $Y_1, \dots, Y_m$ are the names of the circles outside of which the region lies.

A (general) **region** is the union of certain minimal regions.

Example 2.1 The following Venn diagram $\mathcal{V}$ consists of two circles A and B, which have four minimal regions: z_1 is inside A, but outside B (and is denoted by $A\overline{B}$); z_2 is inside both A and B (AB); z_3 is outside A, but inside B ($\overline{A}B$); and z_4 is outside both A and B ($\overline{A}\,\overline{B}$).



$\mathcal{V}$

In general, we denote by r a sequence, say $X_1 \dots X_n \overline{Y_1} \dots \overline{Y_m}$, of names and overlined names of circles, in which we ignore the order and repetition of elements, i.e., we regard r as a set. Then, by $|r|$ we denote the set of names $\{X_1, \dots, X_n, Y_1, \dots, Y_m\}$.

For any sequences r and r', the concatenation thereof is denoted by rr'.

Of the regions in a given Venn diagram, we distinguish conjunctive, negative, and disjunctive regions.

Definition 2.2 (Conjunctive region) A **conjunctive region** r of a Venn diagram $\mathcal{V}$ is a region of $\mathcal{V}$ that is specified by the names of circles in $\mathcal{V}$ as follows. For $n \geq 1, m \geq 0$,

$$X_1 \dots X_n \overline{Y_1} \dots \overline{Y_m} \ .$$

A region specified by using only overlined names of the form $\overline{Y_1} \dots \overline{Y_m}$ is called a **negative region**. A region that is neither conjunctive nor negative is called a **disjunctive region**.

Observe that if $\mathcal{L}$ is the set of names of circles in $\mathcal{V}$, a conjunctive or negative region r such that $|r| = \mathcal{L}$ is a usual minimal region. Note also that conjunctive, negative, and disjunctive regions are disjoint.

Example 2.3 In the diagram $\mathcal{V}$ in Example 2.1, the regions z_1 $(= A\overline{B}), z_2$ $(= AB), z_3$ $(= \overline{A}B), z_1 \cup z_2$ $(= A)$, and $z_2 \cup z_3$ $(= B)$ are all conjunctive (the first three are also minimal); regions such as z_4 $(= \overline{A}\overline{B})$, and $z_3 \cup z_4$ $(= \overline{A})$ are negative; while the other regions e.g., $z_1 \cup z_3$, and $z_1 \cup z_2 \cup z_3$, are disjunctive.

Each conjunctive region can naturally be read as "Things that are X_1 and ... and X_n, but not Y_1 or ... or Y_m." We are mainly concerned with conjunctive regions in our formalization of Venn diagrammatic system, which can be extended by including other regions.

Each conjunctive region may be **shaded**. In particular, a Venn diagram in which no regions are shaded is called a **primary diagram**. For a Venn diagram $\mathcal{V}$, we denote by $m(\mathcal{V})$, $shm(\mathcal{V})$, and $shc(\mathcal{V})$ the set of names of minimal regions, shaded minimal regions, and shaded conjunctive regions, respectively of $\mathcal{V}$.

An **abstract Venn diagram** is defined as the set of shaded conjunctive and/or negative, disjunctive regions. Instead of giving a formal definition of the abstract syntax here, we explain it through our translation into a resolution calculus in Section 3.2.

Although we do not deal explicitly with points and the linking thereof in this paper, we may technically regard points (with linking) in Venn diagrams as special circles that do not contain or overlap any other circles. See [6].

We define the semantics of Venn diagrams in the same way as Howse et al. [4]. A **model** is a pair $M = (U, I)$, where U is a non-empty set called the universe, and I is an interpretation function that assigns to each circle a subset of U. The interpretation function I is naturally extended to interpret conjunctive regions as follows. For any conjunctive region r, specified as $X_1 \dots X_n \overline{Y_1} \dots \overline{Y_m}$, the interpretation $I(r)$ is defined by $I(r) = I(X_1) \cap \dots \cap I(X_n) \cap \overline{I(Y_1)} \cap \dots \cap \overline{I(Y_m)}$, where $\overline{I(Y_j)}$ is the complement of the set $I(Y_j)$. $M = (U, I)$ is a model of the Venn diagram $\mathcal{V}$, denoted as $M \models \mathcal{V}$, if each shaded conjunctive region is interpreted as the empty set, i.e., $\bigcup_{r \in shc(\mathcal{V})} I(r) = \emptyset$.

2.2 Venn diagrammatic inference system

We review the most basic Venn diagrammatic inference system V from Mineshima-Okada-Takemura [6]. See [4, 10], for example, for a formal description of rules.

Definition 2.4 (Inference rules for V)

Axiom: Any primary diagram (i.e., a diagram in which no regions are shaded) is an axiom.

Introduction of a circle: Let $\mathcal{V}$ be a Venn diagram that does not contain a circle A. Then, to obtain the conclusion diagram $\mathcal{V}+A$, add circle A to $\mathcal{V}$ observing the partial-overlapping rule, i.e., each conjunctive or negative region (as well as shaded one) of $\mathcal{V}$ is split into two regions.

Superposition of diagrams: Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be Venn diagrams that contain the same circles. Then, to obtain the conclusion diagram $\mathcal{V}_1 + \mathcal{V}_2$, construct a primary diagram with the same circles as $\mathcal{V}_1$. Then, shade all minimal regions that are shaded in $\mathcal{V}_1$ or $\mathcal{V}_2$.

Erasure of shading Let $\mathcal{V}$ be a Venn diagram in which a minimal region z is shaded. Then, to obtain the conclusion diagram $\mathcal{V} - \{z\}$, erase the shading of z.

Erasure of a circle Let $\mathcal{V}$ be a Venn diagram that contains a circle A. Then, to obtain the conclusion diagram $\mathcal{V} - A$, circle A must be erased from $\mathcal{V}$ and any shading remaining in only a part of a minimal region after the erasure of A should also be erased.

The notion of *Venn diagrammatic proof* is defined inductively as tree structures consisting of the above inference rules (see Example 2.5 below). Note in particular that each leaf of a proof-tree is a premise diagram or a primary diagram (i.e., an axiom).

Example 2.5 Fig. 2 is a Venn diagrammatic proof of $\mathcal{V}_3$ from premises $\mathcal{V}_1$ and $\mathcal{V}_2$.

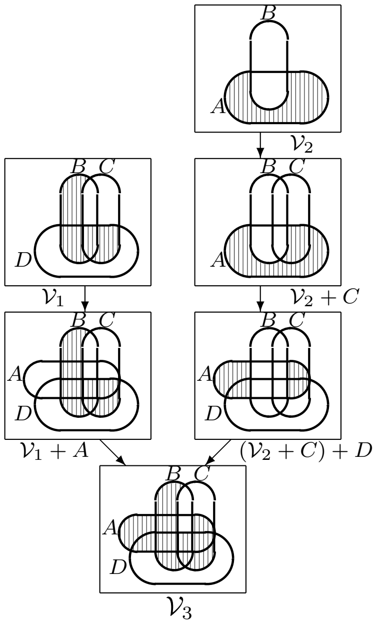


Fig. 2 Venn diagrammatic proof

3 Translation of Venn diagrammatic system

In this section, after a brief review of the resolution calculus in Section 3.1, we define a translation of the Venn diagrammatic inference system V into resolution calculus in Section 3.2. In particular, we translate the rule for Superposition (instead of the rule for Erasure of a circle as in [6]) into a combination of the resolution principle.

3.1 Resolution calculus

Propositional and first-order resolution were introduced by Robinson "for use as a basic theoretical instrument of the computer theorem-proving program" [7]. The "resolution principle" is very powerful in that it forms by itself a complete system of propositional and first-order logic. The efficiency of the rule makes it easy to implement decision procedures involved in establishing the provability of given formulas. The resolution principle gives the theoretical basis of logic programming such as Prolog. For details refer to [2], for example.

A **literal** is either an atom A or its negation $\overline{A}$. In the context of resolution, we use the "overbar" symbol for negation instead of the usual $\neg$. We denote literals by $L, L_1, L_2, \ldots$. If L is a literal of the form $\overline{A}$, then $\bar{L}$ denotes the unnegated literal A. A **clause** is a finite set of literals, and is denoted by $r, x, y, z, \ldots$. If a clause r is $\{L_1, \ldots, L_n\}$, it is usually expressed as $L_1 \cdots L_n$. A set of clauses is called a **clause set**, and is denoted by $\Gamma, \Delta, \Sigma, \ldots$. In particular, the singleton $\{r\}$ of a clause r is often denoted by r. Let r_1 and r_2 be clauses such that $L \in r_1$ and $\bar{L} \in r_2$. The **resolution principle** is defined by the following operation to derive the clause $(r_1 \setminus \{L\}) \cup (r_2 \setminus \{\bar{L}\})$, called the **resolvent** of r_1 and r_2:

$$\frac{r_1 \qquad\qquad r_2}{(r_1 \setminus \{L\}) \cup (r_2 \setminus \{\bar{L}\})}\ res, L \quad .$$

A **resolution derivation** is then defined as the process of deriving a clause r from a given clause set Γ by applications of the resolution principle.

For any non-empty clause set $\Gamma = \{r_1, \ldots, r_m\}$, we denote by $d(\Gamma)$, the disjunctive normal form (DNF) formula $(\wedge r_1) \vee \cdots \vee (\wedge r_m)$.

To investigate the translation of the Venn diagrammatic system V, we introduce a calculus VR over clause sets based on the resolution principle.

Definition 3.1 (VR) The resolution calculus for Venn diagrams VR consists of the following rules over clause sets.

$$\frac{\{rL\} \quad \{r\bar{L}\}}{\{r\}}\ res, L \qquad \frac{\Gamma}{\Gamma L}\ intro, L \qquad \frac{\Gamma_1 \quad \cdots \quad \Gamma_n}{\Gamma_1 \cup \cdots \cup \Gamma_n}\ sup \qquad \frac{\Gamma \cup \Delta}{\Gamma}\ er$$

Here, ΓL (or equivalently $L\Gamma$) signifies $\{r_1 L, \ldots, r_n L\}$ when $\Gamma = \{r_1, \ldots, r_n\}$.
We assume the empty clause $\emptyset$ is an axiom of VR.

Note that the above resolution principle in VR is slightly restricted, since r is shared in the premises. However, by using the *intro* rule, the usual resolution principle is easily simulated by the above rule.

By $Res(\Gamma)$, we denote the set of clauses that is derivable from a clause set Γ by applications of only the above resolution principle in VR.

Instead of proving the soundness of VR, we explain the inference rules of VR in terms of those of natural deduction under our interpretation of clauses. We interpret each clause set, say $\Gamma = \{r_1, \ldots, r_n\}$, as the negation of the disjunctive normal form formula $\neg d(\Gamma) = \neg((\wedge r_1) \vee \cdots \vee (\wedge r_n))$ (cf. the semantics of Venn diagrams). In particular, the empty clause set $\emptyset$ is interpreted as the constant $\top$ which expresses the truth. Then, using this interpretation and the usual equivalence of formulas, the *intro* rule corresponds to the disjunction introduction rule of natural deduction, since, when $\Gamma = \{r_1, \ldots, r_n\}$, $\neg((\wedge r_1) \vee \cdots \vee (\wedge r_n))$ implies $\neg((\wedge r_1) \vee \cdots \vee (\wedge r_n)) \vee \neg L$, which is equivalent to $\neg((\wedge r_1 \wedge L) \vee \cdots \vee (\wedge r_n \wedge L))$. The *sup* rule corresponds to the n-ary conjunction introduction rule since $\neg d(\Gamma_1), \ldots, \neg d(\Gamma_n)$ implies $\neg d(\Gamma_1) \wedge \cdots \wedge \neg d(\Gamma_n)$, i.e., $\neg(d(\Gamma_1) \vee \cdots \vee d(\Gamma_n))$. The *er* rule corresponds to the conjunction elimination rule since $\neg(d(\Gamma) \vee d(\Delta))$, which is equivalent to $\neg d(\Gamma) \wedge \neg d(\Delta)$, implies $\neg d(\Gamma)$.

3.2 Translation of V

We present a translation of the Venn diagrammatic inference system V into the resolution calculus VR. Our translation here is different from and more suitable than that in [6] for comparing free rides in Venn and Euler diagrammatic systems.

Translation between logical systems is one of the basic methods in proof theory, and can be applied to various systems for a variety of reasons. Compared with semantic interpretation, a logic translation reveals various properties of inference rules in a logical system. For example, it enables us to study logical connectives and inference rules of the original system in terms of those of the translated system.

In general, diagrams correspond to formulas in symbolic logic, and diagram manipulations correspond to applications of inference rules in a certain logical system. In particular, our Venn diagrams abstractly specified in terms of conjunctive regions correspond to clause sets, i.e., disjunctive normal form formulas, and the inference rules of the Venn diagrammatic system V correspond to inference rules over clause sets.

We first define a translation of a Venn diagram into a clause set.

Definition 3.2 (Translation of Venn diagrams) Let $\mathcal{V}$ be a Venn diagram such that $shc(\mathcal{V}) = \{r_1, \ldots, r_n\}$.

- Each shaded conjunctive region r_i such that $X_1 \ldots X_n \overline{Y_1} \ldots \overline{Y_m}$ is translated into a clause $r_i^\bullet$ such that $X_1 \ldots X_n \overline{Y_1} \ldots \overline{Y_m}$.
- The Venn diagram $\mathcal{V}$ is translated into the clause set $\mathcal{V}^\bullet = \{r_1^\bullet, \ldots, r_n^\bullet\}$.

  In particular, each primary diagram is translated as $\emptyset$.

To avoid notational complications, we denote simply by r (instead of $r^\bullet$) the translation of a region r.

Example 3.3 Venn diagrams $\mathcal{V}_1$ and $\mathcal{V}_2$ in Example 2.5 are translated as follows: $\mathcal{V}_1^\bullet = \{\overline{B}CD, BCD, B\overline{C}D, B\overline{C}\,\overline{D}, CD, BD, B\overline{C}\}$ and $\mathcal{V}_2^\bullet = \{A\overline{B}\}$.

Based on the above translation of Venn diagrams into clause sets, inference rules for V are translated into resolution calculus VR.

Definition 3.4 (Translation of V) Rules for V are translated as follows.

Introduction of a circle: The conclusion diagram $\mathcal{V}+A$ is specified in terms of the set of shaded conjunctive regions as follows:

$$shc(\mathcal{V} + A) = shc(\mathcal{V}) \cup \{rA \mid r \in shc(\mathcal{V})\} \cup \{r\overline{A} \mid r \in shc(\mathcal{V})\}.$$

Then this rule is translated as follows.

$$\dfrac{\mathcal{V}^\bullet \quad \dfrac{\mathcal{V}^\bullet}{A\mathcal{V}^\bullet}\,intro, A \quad \dfrac{\mathcal{V}^\bullet}{\mathcal{V}^\bullet\overline{A}}\,intro, \overline{A}}{\mathcal{V}^\bullet \cup A\mathcal{V}^\bullet \cup \mathcal{V}^\bullet\overline{A}}\,sup$$

Superposition of diagrams: The conclusion diagram $\mathcal{V}_1 + \mathcal{V}_2$ is specified in terms of the set of shaded conjunctive regions as follows. Let n be the length, i.e., the number of names of each minimal region of $\mathcal{V}_1$, which is uniquely determined. Let $\Gamma_0 = \{r \mid rL \in shm(\mathcal{V}_1), r\overline{L} \in shm(\mathcal{V}_2)\}$ and $\Gamma_i = \{r \mid rL, r\overline{L} \in \Gamma_{i-1}\}$.

$$shc(\mathcal{V}_1 + \mathcal{V}_2) = shc(\mathcal{V}_1) \cup shc(\mathcal{V}_2) \cup \bigcup_{i=0}^{n} \Gamma_i$$

Then this rule is translated as shown below.

$$\dfrac{\mathcal{V}_1^\bullet \quad \mathcal{V}_2^\bullet \quad Res(\mathcal{V}_1^\bullet \cup \mathcal{V}_2^\bullet)}{\mathcal{V}_1^\bullet \cup \mathcal{V}_2^\bullet \cup Res(\mathcal{V}_1^\bullet \cup \mathcal{V}_2^\bullet)}\ sup$$

Here, $Res(\mathcal{V}_1^\bullet \cup \mathcal{V}_2^\bullet)$ denotes the set of clauses derived from $\mathcal{V}_1^\bullet \cup \mathcal{V}_2^\bullet$ by applying the resolution principle.

Erasure of shading The conclusion diagram $\mathcal{V} - \{z\}$ is specified as follows:

$$shc(\mathcal{V} - \{z\}) = shc(\mathcal{V}) \setminus (\{z\} \cup \{r \in shc(\mathcal{V}) \mid \exists r'.rr' = z\})\ .$$

Then this rule is translated as shown below. Let $\mathcal{V}^\bullet$ be the disjoint union $\{z\} \cup \Gamma_1 \cup \Gamma_2$, where for all $r \in \Gamma_1$, there exists r' such that $rr' = z$, and for all $x \in \Gamma_2$, there is no r' such that $xr' = z$.

$$\dfrac{\{z\} \cup \Gamma_1 \cup \Gamma_2}{\Gamma_2}\ er$$

Erasure of a circle The conclusion diagram $\mathcal{V} - A$ is specified as follows:

$$shc(\mathcal{V} - A) = shc(\mathcal{V}) \setminus \{r \in shc(\mathcal{V}) \mid A \in |r|\}\ .$$

Then this rule is translated as shown below. Let $\mathcal{V}^\bullet = \Gamma_1 \cup \Gamma_2$ in which $A \in |r|$ for all $r \in \Gamma_1$, and $A \notin |x|$ for all $x \in \Gamma_2$.

$$\dfrac{\Gamma_1 \cup \Gamma_2}{\Gamma_2}\ er$$

Example 3.5 The proof in Fig. 2 for Example 2.5 is translated as π in Fig. 3.

$$\pi_1 \left\{ \dfrac{\mathcal{V}_1^\bullet \qquad \dfrac{\mathcal{V}_1^\bullet}{\mathcal{V}_1^\bullet A}\ intro, A \qquad \dfrac{\mathcal{V}_1^\bullet}{\mathcal{V}_1^\bullet \overline{A}}\ intro, \overline{A}}{(\mathcal{V}_1 + A)^\bullet = \mathcal{V}_1^\bullet \cup \left\{ \begin{array}{l} A\overline{B}CD,\ ABCD,\ AB\overline{C}D,\ AB\overline{C}\overline{D},\ ACD,\ ABD,\ AB\overline{C}, \\ \overline{A}\overline{B}CD,\ \overline{A}BCD,\ \overline{A}B\overline{C}D,\ \overline{A}B\overline{C}\overline{D},\ \overline{A}CD,\ \overline{A}BD,\ \overline{A}B\overline{C} \end{array} \right\}}\ sup \right.$$

Here, $\mathcal{V}_1^\bullet = \{\overline{B}CD, BCD, B\overline{C}D, B\overline{C}\overline{D}, CD, BD, B\overline{C}\}$

$$\pi_2 \left\{ \dfrac{\dfrac{\mathcal{V}_2^\bullet = \{A\overline{B}\} \quad \dfrac{\mathcal{V}_2^\bullet}{\mathcal{V}_2^\bullet C}\ intro, C \quad \dfrac{\mathcal{V}_2^\bullet}{\mathcal{V}_2^\bullet \overline{C}}\ intro, \overline{C}}{(\mathcal{V}_2 + C)^\bullet = \mathcal{V}_2^\bullet \cup \{A\overline{B}C, A\overline{B}\overline{C}\}}\ sup \qquad \dfrac{(\mathcal{V}_2 + C)^\bullet}{(\mathcal{V}_2 + C)^\bullet D}\ intro, D \quad \dfrac{(\mathcal{V}_2 + C)^\bullet}{(\mathcal{V}_2 + C)^\bullet \overline{D}}\ intro, \overline{D}}{((\mathcal{V}_2 + C) + D)^\bullet = (\mathcal{V}_2 + C)^\bullet \cup \{A\overline{B}CD, A\overline{B}C\overline{D},\ \ A\overline{B}\overline{C}D, A\overline{B}\overline{C}\overline{D},\ \ A\overline{B}D, A\overline{B}\overline{D}\}}\ sup \right.$$

$$\pi \left\{ \dfrac{\begin{array}{c} \vdots\ \pi_1 \\ (\mathcal{V}_1{+}A)^\bullet \end{array} \quad \begin{array}{c} \vdots\ \pi_2 \\ ((\mathcal{V}_2{+}C){+}D)^\bullet \end{array} \quad \dfrac{\dfrac{(\mathcal{V}_1 A)^\bullet}{AB\overline{C}D} \quad \dfrac{(\mathcal{V}_2 CD)^\bullet}{A\overline{B}\overline{C}D}}{A\overline{C}D} \quad \dfrac{\dfrac{(\mathcal{V}_1 A)^\bullet}{AB\overline{C}\overline{D}} \quad \dfrac{(\mathcal{V}_2 CD)^\bullet}{A\overline{B}\overline{C}\overline{D}}}{A\overline{C}\overline{D}} \quad \dfrac{\dfrac{(\mathcal{V}_1 A)^\bullet}{ABD} \quad \dfrac{(\mathcal{V}_2 CD)^\bullet}{A\overline{B}D}}{AD} \quad \dfrac{\dfrac{(\mathcal{V}_1 A)^\bullet}{AB\overline{C}} \quad \dfrac{(\mathcal{V}_2 CD)^\bullet}{A\overline{B}\overline{C}}}{A\overline{C}}}{(\mathcal{V}_1 + A)^\bullet \cup ((\mathcal{V}_2 + C) + D)^\bullet \cup \{A\overline{C}D, A\overline{C}\overline{D}, AD, A\overline{C}\}}\ sup \right.$$

Fig. 3 Translation of the Venn diagrammatic proof in Fig. 2

In order to justify the above translation, we show the following lemma.

Lemma 3.6 *Let r be a region of a given Venn diagram $\mathcal{V}$, and let X be a circle of $\mathcal{V}$ such that $X \notin |r|$. $r \in shc(\mathcal{V})$ if and only if $rX, r\overline{X} \in shc(\mathcal{V})$.*

Proof. $\Leftarrow$) When $rX, r\overline{X} \in shc(\mathcal{V})$, we have $r = rX \cup r\overline{X} \in shc(\mathcal{V})$.
$\Rightarrow$) Let $r \in shc(\mathcal{V})$. Since $X \notin |r|$, r is still shaded after the erasure of circle X, i.e., $r \in shc(\mathcal{V} - X)$. Then, since each region is split into two parts by the introduction of X to $\mathcal{V} - X$, we have $rX, r\overline{X} \in shc((\mathcal{V} - X) + X)$.

Assume to the contrary that $rX \notin shc(\mathcal{V})$ or $r\overline{X} \notin shc(\mathcal{V})$. If $rX \notin shc(\mathcal{V})$, since unshaded regions are kept by the erasure and introduction of circle X, we have $rX \notin shc((\mathcal{V}-X)+X)$, which contradicts $rX \in shc((\mathcal{V} - X) + X)$. The same applies to the case $r\overline{X} \notin shc(\mathcal{V})$. Therefore, we have $rX, r\overline{X} \in shc(\mathcal{V})$. ■

By the above Lemma 3.6, we have the following lemma.

Lemma 3.7 *Let $\mathcal{V}$ be a Venn diagram, and r be a region of $\mathcal{V}$. $r \in shc(\mathcal{V})$ if and only if $r \in Res(shm(\mathcal{V}))$.*

Proof. $\Leftarrow$) Let $r \in Res(shm(\mathcal{V}))$. We show $r \in shc(\mathcal{V})$ by induction on the length n of a given derivation of r. The base step $n = 0$ is immediate, since we have $r \in shm(\mathcal{V}) \subseteq shc(\mathcal{V})$. In the induction step $n > 0$, r is obtained by the resolution principle from rX and $r\overline{X}$ for some X. By the induction hypothesis, we have $rX, r\overline{X} \in shc(\mathcal{V})$, and hence, by Lemma 3.6, we have $r \in shc(\mathcal{V})$.
$\Rightarrow$) Let $r \in shc(\mathcal{V})$. When r is a minimal region, we immediately have $r \in Res(shm(\mathcal{V}))$. When r is not a minimal region, by Lemma 3.6, for some $X_1 \notin |r|$, we have $rX_1, r\overline{X_1} \in shc(\mathcal{V})$.

On the one hand, when rX_1 and $r\overline{X_1}$ are minimal regions, we have $r \in Res(shm(\mathcal{V}))$ by the resolution principle. On the other hand, when rX_1 and $r\overline{X_1}$ are not minimal, again by Lemma 3.6, for some $X_2 \notin |rX_1|$, we have $rX_1X_2, rX_1\overline{X_2} \in shc(\mathcal{V})$ and $r\overline{X_1}X_2, r\overline{X_1}\,\overline{X_2} \in shc(\mathcal{V})$. Since the length of a minimal region in $\mathcal{V}$ is finite, we are able to reduce these case to the case of minimal regions by repeated applications of Lemma 3.6. In this way, we have $r \in Res(shm(\mathcal{V}))$. ■

Based on Lemmas 3.6 and 3.7, the soundness of the translation is easily obtained.

Theorem 3.8 (Soundness of translation of VR) *If $\mathcal{V}$ is provable from $\mathcal{V}_1, \ldots, \mathcal{V}_n$ in the Venn diagrammatic system* V*, then $\mathcal{V}^\bullet$ is derivable from $\mathcal{V}_1^\bullet, \ldots, \mathcal{V}_n^\bullet$ in resolution calculus* VR.

4 Free ride in Venn diagrammatic system

We now investigate the free ride property of our Venn diagrammatic system V. In Takemura [12], a translation of Euler diagrammatic system of Mineshima-Okada-Takemura [5] into a natural deduction system is given, and it is shown that the soundness theorem of the translation implies the occurrence of free ride (in the sense of [9, 1]) between the systems. In this way, free ride is formalized in the proof-theoretical framework through the logic translation. Thus, the soundness of our translation in Section 3.2 shows that free ride occurs between the Venn diagrammatic system V and the resolution calculus VR. See Takemura [12].

Our formalization of free ride in the proof-theoretical framework makes it possible to analyze free ride in more detail than that in the semantic framework. In the translation (Definition 3.4) of the Venn diagrammatic system V into the resolution calculus VR, it is revealed that which conjunctive regions are shaded in the conclusion diagram after the application of an inference rule. Some of such regions are not specified in the given premise diagrams or in the description of the diagrammatic operation of the rule, and they are automatically represented by the application of the rule. By slightly extending the notion of free ride of Shimojima [9], let us call the regions "free rides" those that are automatically represented in a diagram after a manipulation of given diagrams. Our free rides may also be called "emergent objects."

For example, when we carry out the operation of Superposition (cf. Definition 2.4) as illustrated in Fig. 2, we only need to consider shaded minimal regions in both premise diagrams; we do not need to take the other regions into account. Hence, the conjunctive regions $A\overline{C}D, A\overline{C}\,\overline{D}, AD, A\overline{C}$ in Fig. 3 are automatically represented by the application of Superposition, and they are free rides of this application of Superposition.

In this way, for our Venn diagrammatic system V, we are able to define the free rides of an application of each inference rule in terms of shaded conjunctive regions. In general, the free rides of an application of an inference rule are defined as the regions obtained by subtracting from the shaded conjunctive regions of the conclusion, the shaded conjunctive regions of premises and those described in the operation of the rule. Thus, Introduction of a circle, Erasure of shading, and Erasure of a circle have no free rides, and in particular, the free rides of Superposition are defined as follows.

Definition 4.1 (Free rides of superposition) In an application S of Superposition between $\mathcal{V}_1$ and $\mathcal{V}_2$, the following set of conjunctive regions are called the *free rides* of S:

$$shc(\mathcal{V}_1+\mathcal{V}_2)\setminus\Big(shc(\mathcal{V}_1)\cup shc(\mathcal{V}_2)\Big)\ .$$

Based on the above definition of free rides, we are able to investigate what kind of and how many inference steps are required in terms of the resolution calculus, to derive free rides. We here investigate the number of free rides occurring in an application of Superposition.

First, we calculate the number of conjunctive regions $\mathsf{cr}(\mathcal{V})$ in a given diagram $\mathcal{V}$ consisting of n circles as follows:

$$\mathsf{cr}(\mathcal{V})=\sum_{i=1}^{n}{}_n\mathrm{C}_i\times 2^i-\sum_{i=1}^{n}{}_n\mathrm{C}_i\ .$$

Here, ${}_n\mathrm{C}_n\times 2^n$ $(=2^n)$ is the number of usual (i.e., including negative) minimal regions of $\mathcal{V}$ (cf. truth table), and $\sum_{i=1}^{n}{}_n\mathrm{C}_i$ is the number of negative regions. Thus, for a Venn diagram $\mathcal{V}_3$ (resp. $\mathcal{V}_4$) consisting of 3 (resp. 4) circles, we have $\mathsf{cr}(\mathcal{V}_3)=19$, i.e.,

$$\{A,B,C,AB,AC,A\overline{B},A\overline{C},B\overline{C},\overline{A}B,BC,\overline{A}C,\overline{B}C,ABC,A\overline{B}C,A\overline{B}\,\overline{C},AB\overline{C},\overline{A}B\overline{C},\overline{A}BC,\overline{A}\,\overline{B}C\}$$

(resp. $\mathsf{cr}(\mathcal{V}_4)=65$). (Cf. Example 2.3 for the case of $n=2$.)

Next we calculate the number of free rides in the worst case, where the maximum number of free rides occur. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be Venn diagrams such that (1) $m(\mathcal{V}_1)=m(\mathcal{V}_2)$ (i.e., $\mathcal{V}_1$ and $\mathcal{V}_2$ have the same minimal regions); (2) $shc(\mathcal{V}_i)\setminus shm(\mathcal{V}_i)=\emptyset$ for each $i=1,2$ (i.e., the shaded regions are only minimal regions); and (3) $r\in shc(\mathcal{V}_1)$ if and only if $r\notin shc(\mathcal{V}_2)$, for any conjunctive region r of $\mathcal{V}_1$ (i.e., $\mathcal{V}_1$ and $\mathcal{V}_2$ are dual diagrams with respect to shading). Thus, after an application of Superposition, all conjunctive regions are shaded in $\mathcal{V}_1+\mathcal{V}_2$. The following Fig. 4 illustrates such $\mathcal{V}_1$ and $\mathcal{V}_2$ consisting of three circles.

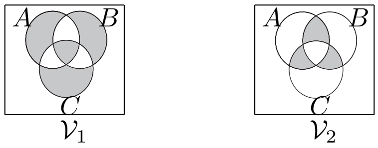


Fig. 4

Then, the number of free rides in $\mathcal{V}_1 + \mathcal{V}_2$, which consists of n circles, is as follows.

$$\mathsf{fr}(\mathcal{V}_1 + \mathcal{V}_2) = \mathsf{cr}(\mathcal{V}_1 + \mathcal{V}_2) - ({}_n\mathrm{C}_n \times 2^n - {}_n\mathrm{C}_n)$$

Here, ${}_n\mathrm{C}_n$ is the number of negative minimal regions, and ${}_n\mathrm{C}_n \times 2^n - {}_n\mathrm{C}_n$ is the number of conjunctive minimal regions. Thus, for example, if $\mathcal{V}_1$ and $\mathcal{V}_2$ consist of 2 (resp. 3, and 4) circles, $\mathsf{fr}(\mathcal{V}_1 + \mathcal{V}_2) = 5 - 3 = 2$ (resp. $19 - 7 = 12$ and $64 - 14 = 50$). In this way, if $n \geq 3$, we have $2^n < \mathsf{fr}(\mathcal{V}_1 + \mathcal{V}_2)$, and hence, the number of free rides increases exponentially with the number of circles n.

Furthermore, if we consider negative and disjunctive regions as well, the number of free rides is estimated as follows. Since a (general) region is a non-empty union of certain minimal regions, and since the number of minimal regions $\mathsf{mr}(\mathcal{V})$ of a Venn diagram consisting of n circles is 2^n, the number of regions $\mathsf{r}(\mathcal{V})$ is $2^{2^n} - 1$. Then, the number of free rides in the worst case is calculated by subtracting the number of minimal regions from the total number of regions:

$$\mathsf{fr}(\mathcal{V}_1 + \mathcal{V}_2) = \mathsf{r}(\mathcal{V}_1 + \mathcal{V}_2) - \mathsf{mr}(\mathcal{V}_1 + \mathcal{V}_2) \ .$$

Thus, for example, if $\mathcal{V}_1 + \mathcal{V}_2$ consists of 2 (resp. 3) circles, we have $\mathsf{fr}(\mathcal{V}_1 + \mathcal{V}_2) = 11$ (resp. $= 248$).

5 A comparison of Venn diagrams and Euler diagrams

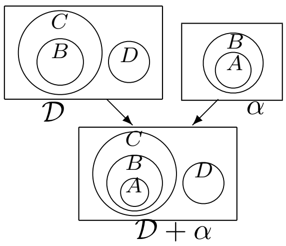
We try to compare Venn and Euler diagrams with respect to free rides. In Section 5.1, we briefly review the analysis, given in Takemura [12], on free rides in our Euler diagrammatic system. Then, in Section 5.2, we discuss differences between the Venn and Euler diagrammatic systems, and discuss our future work.

5.1 Free rides in Euler diagrammatic system

While the most basic and essential components of Venn diagrams are regions and shading of them, those of Euler diagrams are topological (inclusion and exclusion) relations between circles and points. Based on this idea, Mineshima-Okada-Takemura [6] distinguished a "region-based" framework for Venn diagrams, where a diagram is specified in terms of shaded regions, and a "relation-based" framework for Euler diagrams, where a diagram is specified in terms of inclusion and exclusion relations. Then Mineshima-Okada-Takemura investigated a relation-based Euler diagrammatic inference system in [5].

This inference system comprises two kinds of inference rules: Deletion and Unification. Essentially, Deletion allows us to delete a diagrammatic object from a given Euler diagram. Unification allows us to combine two Euler diagrams into one diagram, where the semantic information is equivalent to the conjunction of the two original diagrams. To characterize intuitive manipulations on diagrams as formal inference rules, the unification rules are defined by requiring that one of the unified diagrams be a *minimal diagram* that consists of two objects (circles and points). Each inference rule is described in terms of relations by specifying (i) premise diagrams; (ii) the constraints that the premise diagrams should satisfy; and (iii) diagrammatic operations to introduce a new object into, or to rearrange a configuration of objects of, one of the premise diagrams. See [5, 12] for a detailed description.

The following diagrammatic proof on the left in Fig. 5 is an application of our unification rule, in which, to obtain the unified diagram $\mathcal{D} + \alpha$, circle A in the minimal diagram α is added to diagram $\mathcal{D}$ so that "A is inside B" holds.



$$\dfrac{\mathcal{D}^\circ \quad \alpha^\circ \quad \dfrac{\dfrac{\dfrac{[A]^1 \quad \dfrac{\alpha^\circ}{A \to B}}{B} \quad \dfrac{\mathcal{D}^\circ}{B \to C}}{C}}{A \to C}\,1 \quad \dfrac{\dfrac{\dfrac{[A]^1 \quad \dfrac{\alpha^\circ}{A \to B}}{B} \quad \dfrac{\mathcal{D}^\circ}{B \to \neg D}}{\neg D}}{A \to \neg D}\,1}{\mathcal{D}^\circ \wedge \alpha^\circ \wedge \underbrace{(A \to C) \wedge (A \to \neg D)}_{\text{Free rides}}}$$

Fig. 5 Free rides of an Unification in the Euler diagrammatic proof that corresponds to the Venn diagrammatic proof in Fig. 2

While our region-based Venn diagrammatic system is naturally translated into resolution calculus as seen in Section 3.2, the relation-based Euler diagrammatic system is translated into a natural deduction system (cf. [6, 12]). An Euler diagram specified in terms of relations corresponds to a conjunction of implicational formulas, and an inference rule of our Euler diagrammatic system corresponds to natural deduction inference rules associated with the implicational connective. For example, the Euler diagram $\mathcal{D}$ (resp. α) on the left in Fig. 5 is translated into the conjunction of implicational formulas $\mathcal{D}^\circ = (B \to C) \wedge (B \to \neg D) \wedge (C \to \neg D) \wedge (B \to B) \wedge (C \to C) \wedge (D \to D)$ (resp. $\alpha^\circ = (A \to B) \wedge (A \to A) \wedge (B \to B)$). Then the application of Unification is translated into the natural deduction proof on the right in Fig. 5, in which natural deduction rules for the conjunction are generalized to those for n-ary conjunction. See [12] for a detailed description.

Note that, when we carry out the operation of Unification in Fig. 5, we only need to consider the relations between A and B, and not the other circles. Hence, the relations corresponding to $A \to C$ and $A \to \neg D$, i.e., "A is inside C" and "A is outside D," are automatically represented by this application of Unification, and we call these relations "free rides." In general, the free rides for an application of each Unification are defined in terms of relations that exist in the diagrams, as the relations obtained by subtracting, from the relations holding at the conclusion, the relations existing in the premises and the relations required to exist in the description of the operation of the rule (see [12]).

Let us calculate the number of free rides in the worst case. For an application of Unification between a (general) diagram $\mathcal{D}$ and a minimal diagram α, assume that $\mathcal{D}$ contains n circles $A_1, \ldots, A_n$ which are laid out in a line with respect to the inclusion relation so that A_1 is the smallest circle, and assume that "B is inside A_1" holds in α. Such a Unification $\mathcal{D} + \alpha$ is a worst case, where the maximum number of free rides occur, and it is easily seen that the number of free rides is $n - 1$, i.e., the relations "B is inside A_i" for $2 \leq i \leq n$. The application of Unification in Fig. 5 also illustrates a worst case, where $\mathcal{D}$ contains 3 circles, with a total of 2 free rides.

5.2 Discussion and future work

In order to compare Venn diagrams and Euler diagrams with respect to free rides, we consider the examples of translations given in Figs. 2 and 3, and Fig. 5.

On the one hand, in the application of Superposition of Venn diagrams in Fig. 2, free rides are the regions $A\overline{C}D, A\overline{C}\overline{D}, AD, A\overline{C}$, i.e., $\neg(A \wedge \neg C \wedge D), \neg(A \wedge \neg C \wedge \neg D), \neg(A \wedge D)$, and $\neg(A \wedge \neg C)$ as seen in Fig. 3. On the other hand, in the application of Unification of Euler diagrams in Fig. 5, free rides are $A \to C$ and $A \to \neg D$. In view of the equivalence between formulas $\neg(A \wedge \neg C)$ (resp. $\neg(A \wedge D)$) and $A \to C$ (resp. $A \to \neg D$), it is immediately seen that the Superposition of Venn diagrams has more free rides than the Unification of Euler diagrams. This generally holds, as we have already seen in the estimation of the number of free rides in Venn and Euler diagrammatic systems. In other words, an application of Superposition to Venn diagrams generally provides more pieces of information automatically than that of Unification of Euler diagrams.

However, this does not necessarily imply that Venn diagrams are more useful than Euler diagrams. In particular, from a cognitive viewpoint, it is often observed that Venn diagrams are harder to handle in actual reasoning than Euler diagrams, and there are some experimental results to support this claim, e.g., Sato-Mineshima-Takemura [8].

Although the translated inference steps in resolution calculus and natural deduction, are not simply regarded as cognitive/psychological processes to recognize free rides represented in diagrams, we may assume at least one step is required to recognize each of the free rides. Then, the number of free rides can be calculated as the number of steps required to recognize these free rides. Thus, we could be able to interpret our results as follows. On the one hand, in Venn diagrams the number of recognition steps for free rides increases exponentially with the number of circles contained in the given diagrams, whereas in Euler diagrams, it increases linearly. We consider this to be part of the reason that Euler diagrams are considered to be more tractable than Venn diagrams.

As future work, we need a more in-depth analysis of the estimation of free rides. The Venn diagrams used in our worst case analysis of the number of free rides do not correspond to Euler diagrams. It is better to evaluate free rides in an application of Superposition to Venn diagrams obtained by translating Euler diagrams. Although not fully investigated, we have made the following observations. In the simplest case where only conjunctive regions are considered, and moreover, at most three circles are contained in the given Venn diagrams (this case corresponds to the simplest one-step syllogism covering only universal sentences), the number of free rides in Superposition of such Venn diagrams is the same as that in Unification of the corresponding Euler diagrams. However, if we also consider negative or disjunctive regions, the number is greater in Superposition than that in Unification. Furthermore, in the case where the given Venn diagrams contain more than three circles, as shown in Figs. 3 and 5, the number of free rides is greater in Superposition than in Unification when considering only conjunctive regions. In view of these observations, we need to investigate how many free rides occur in the Superposition of Venn diagrams that are translated from Euler diagrams.

In our analysis of free rides, we assumed an ideal person who is able to recognize all shaded conjunctive regions fully and uniformly. However, in our actual reasoning, some regions may be more difficult to grasp than others. In fact, our conjunctive regions are not exactly equivalent to visually connected regions, which depend on a particular representation of a diagram. Our analysis may be considered as the one on the abstract or informational aspect of free ride. For the thorough analysis on free ride, we need further study on the cognitive or perceptual aspect of free ride.

References

[1] J. Barwise and J. Seligman, *Information Flow: The Logic of Distributed Systems*, Cambridge University Press, 1997.

[2] S. R. Buss, An Introduction to Proof Theory, in *Handbook Proof Theory*, S. R. Buss ed., Elsevier, Amsterdam, 1998.

[3] C. A. Gurr, J. Lee, and K. Stenning, Theories of diagrammatic reasoning: Distinguishing component problems, *Minds and Machines*, 8(4), 533-557, 1998.

[4] J. Howse, G. Stapleton, and J. Taylor, Spider Diagrams, *LMS Journal of Computation and Mathematics*, Volume 8, 145-194, London Mathematical Society, 2005.

[5] K. Mineshima, M. Okada, and R. Takemura, A Diagrammatic Inference System with Euler Circles, *Journal of Logic, Language and Information*, to appear.

[6] K. Mineshima, M. Okada, and R. Takemura, Two Types of Diagrammatic Inference Systems: Natural Deduction Style and Resolution Style, *Proc. of Diagrams 2010*, Lecture Notes In Artificial Intelligence, Springer, 99-114, 2010.

[7] J. A. Robinson, A Machine-Oriented Logic Based on the Resolution Principle, *Journal of the ACM*, Vol.12, No.1, 23-41, 1965.

[8] Y. Sato, K. Mineshima, and R. Takemura, The efficacy of Euler and Venn diagrams in deductive reasoning: empirical findings, *Proc. of Diagrams 2010*, Lecture Notes In Artificial Intelligence, Springer, 6-22, 2010.

[9] A. Shimojima, On the Efficacy of Representation, Ph.D. thesis, Indiana University, 1996.

[10] S.-J. Shin, *The Logical Status of Diagrams*, Cambridge University Press, 1994.

[11] G. Stapleton, A survey of reasoning systems based on Euler diagrams, *Proc. of Euler 2004*, Electronic Notes in Theoretical Computer Science, 134, 1, 127-151, 2005.

[12] R. Takemura, Proof theory for reasoning with Euler diagrams: a logic translation and normalization, *Studia Logica*, to appear.