Transcendental syntax: three lectures

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Introduction

These lectures (approximately 3×2 hours) are intented for a rather general audience, since they are supposed to bring together three areas: mathematical logic, informatics and philosophy in an a rather intertwisted way.

The starting point is the refusal of the realistic gang semantics/axiomatics, typical of XIX$^{\text{th}}$ century logic, still in favour among, say, analytic philosophers. The idea being of replacing it with a knitting between analytic (raw, untyped) and synthetic (formatted, typed). Analytic and synthetic in turn split into explicit/implicit: the analytic thus remifies into *constat* (result) vs. *performance* (program); the synthetic into *usine* (factory, *a posteriori*) and *usage* (use, *a priori*).

The four blocks constat/performance/usine/usage are all distinct. The distinction, already present in Richard's paradox (1905) is the main achievement of Gödel's incompleteness (1931) (usine $\neq$ usage) whose analytical counterpart is to be found in Turing's undecidability (1936) (constat $\neq$ performance). None of them is primitive in the sense that it can explain the others: we rather observe a sort of knitting, a tight mutual dependency: typically, the knitting usine/usage is known as *cut-elimination*.

1 What is an answer?

The major twist of BHK ($\sim$ 1930) — proofs as functions, programs — is existentialist. Before, logic was seen as dealing with a sort of preexisting truth and proofs were just the mean to reach this goal. With BHK the mean becomes the answer to the question asked by the goal.

In order to get rid of the semantic prejudice, it is necessary to start with answers, without any relation to questions, in order terms to forget the sense.

By doing so, we enter into the deep meaning of computation, a meaningless activity; but also beyond criticism. The word is *analytic* or *raw*.

Analytic splits into constat and performance: like the ↲ key, which can open a new line (constat) or launch a program (performance). The constat is incremental, it never erases, never forgets, like in the subformula property; the performance is destructive, hasardous.

The first subliminal slogan of scientism is « on can answer all questions »; it is typical of, say, AI. In concrete terms, it claims that performance can be reduced to constat; namely, the existence of a sort of a sort of *Table of Results* yielding the answers to all questions. The idea is so preposterous that one (indeed, Turing, 1936) can refute it without bothering about questions. The refutation can be seen as the writing of an « anti-book » in an electronic version of Borges' Babel Library: this virtual book will never display.

A typical analytic statement is of the form $27 + 37 = 64$; in order to be beyond discussion, everything must be displayed on the board, typically the program for addition. If, by mistake, I provide the program for multiplication, I get $27 + 37 = 999$ which is thus analytic as well. Moreover, the board itself should be put on the board: when performing the addition, I must make sure that I respect the instructions.

Due to this requirement, absolute analyticity seems out of reach: one should content one self with tendencies: « this is more analytic than... ». Paradigms like pure λ-calculus are thus only partly analytic, due to their external performative process — rewriting. However, it contains all the ingredients of anayticity: a distinction constat/performance (constative = normal) and a knitting, Church-Rosser. By the way, the failure of π-calculi is due to the fact that they are not analytic, even in this lax sense.

In order to reach real analyticity, the performance should be internal, i.e., perform itself. Unification (Herbrand, 1930) enables us to combine *stars* (sort of clauses) in *constellations* (sort of logic programs). The paradigm, familiar from logic programming, was fumbled by PROLOG who tried to mix it with syntheticity. It surfaced again with *Geometry of Interaction* which is devoted to the purely analytic (= logic-free) part of cut-elimination.

We reach here the hard analytic rock. The distinction constat/performance appears as subjective: am I sastified with my answer or should I proceed further? The answer is a matter of colours: uncoloured for constative, couloured for « to be matched with the complementary colour ». The major analytic knitting corresponds to Church-Rosser: with two pairs of colours, the three possible performances (blue/yellow, then green/magenta; green/magenta, then blue/yellow; blue/yellow, green/magenta simultaneously) amount to the same. Church-Rosser is responsible for one of the major synthtiic kinittings: compositionality, i.e., the transitivity of implication.

2 What is a question?

The question gives a sense to the answer; if the latter is analytic, the former is *synthetic*: the word is *format*. Syntheticity should not be apprehended through the language: this would lead to its interpretentation, eventually to the compendium of prejudices know as semantics.

A good starting point is the *Ouija* board of spirits: put a finger on the right place, the board *beeps*; if we forget the spook supposedly in charge of the beep, it is hardly more irrational than an *Ipad*. And completely rational if we forget what is written on the board: « beep » means « touché », i.e., a purely locative fact that we put our finger on a sensitive place.

A mere *beep* is not enough; the board indeed provides a locative witness, to which I can react, thus initiating a sort of dialectics. When reading of a DVD in an unknown language, the dialectics command/menu will eventually determine the sense. A witness is no proof, it may be mistaken, but w.r.t. what, since we placed ourselves beyond truth and falsity?

Finally, the board opposes a single question — e.g., a DVD player — to its multiple answers, e.g., all possible players; the tests are provided by various DVD. The symmetry answer/test leads to *linear* negation, close to Hegel's contradictory foundations. Compared to usual negation which *refutes*, the hegelian (linear) negation *recuses*: « la question ne sera pas posée ».

A question should be presented through a sampling of tests, e.g., a selection of DVD: this is l'*usine*, the factory tests, a.k.a. gabarit. This mode is an experimental mode, based on performance; we can also style it *a posteriori*, since it never anticipates. It is analytic but in its spirit: the sampling at work in the tests is highly problematic, why these and not those?

Because « meaning is use »; the sampling is supposed to anticipate further uses: these form l'*usage*, a.k.a. *a priori*. This the *implicit*, or implicative, contents of answers, at work when in the reduction of a question to another: an answer to the second question becomes an answer to the original one. However, there is no reason to believe that this indirect answer will pass the factory tests. This is the problem of cut-elimination and certainty.

Richard's paradox (1905) deals with the synthetic, the format. Either we don't format enough and get an inconsistency — roughly an inadequation usine/usage — or we format too much and miss some answers, this is incompleteness; nothing in between. The format is, like the family, protective (against inconsistency) et coercive (against our freedom).

The second subliminal slogan of scientism « one can can compare everything » negates syntheticity. Populists — down with the system —, analytic philosophers — down with concepts —, declarative programming —down with algorithmics—, are samples of a medicine which kills the patient.

3 What conveys certainty?

Hilbert's *formalist* Programme (∼1925) is a kind of scientist reading of Kant. W.r.t. Russell's *logicism* (analytic philosophy) and its naive scientism, Hilbert acknowledges the doubts as to the adequation usine/usage, i.e., presuppositions; but he wants to « fix » kantism by means of a novel idea, the justification of presuppositions through *consistency*. Even flawed, formalism is, together with intuitionism, the origin of XX$^{\text{th}}$ century logic (after 1930).

Consistency is typical of the obsolete axiomatic/semantic paradigm, axiomatic in this case, since Hilbert clearly understood the irrelevance of semantics. The problem is that axioms + *Modus Ponens* have strictly no *form*: all proofs have the same tree-like shape, hence we cannot extract anything from the form, unless we start to interprete nodes, i.e., enter the prejudiced idea of truth. Incompletenees is the infirmation, not of formalism at large, but of the scientist, narrow-minded form proposed by Hilbert.

Following Hilbert's steps, Gentzen made (1934) a major beakthrough : we no longer answer pure questions, but imbricated ones; typically, $\vdash M, F$ denotes the imbrication of a man and a fly, neither man nor fly, not to speak of both! By replacing the implication $A \Rightarrow B$ with the imbrication $\vdash \sim A, B$, it is possible to jailbreak from the uniformity of the arborescence.

The result, the *proof-nets* of linear logic imbricate several trees through their leaves; these trees are incremental, i.e., with a constative analytic substrate. Furthermore, the meaning can be reconstructed through their shape, i.e., through the various ways of travelling through them. Indeed, proof-nets are the first individuation of l'usine in logic, if we except a glimpse in Herbrand's theorem, they also explain what *cut-free* means : l'usine.

L'usage thus becomes desimbrication, i.e., the way of recovering the man from the mix « man + fly » — or the fly for Brigitte Bardot. This is cut-elimination, i.e., the adequation usine/usage which can reach the paradoxical status described in Gödel's theorem. This is because the sampling done by l'usine is not satisfactory with no possibility of completion. Incomplete samplings are in the style of Popper's *falsifiability*, the « so far so good ». This explanation, swell for medicine, cannot account for predictive science.

Science can remain predictive power by a *derealist* leap: the answer, called *épure* now embodies a sort of *ad hoc* format — the mould — a short-cut saving infinitely many verifications : it is thus a mix analytic/synthetic. Èpures do not convey apodictic certainty because of a conflict of interests : the question (synthetic) is a sort of cop while the answer is rather in a role of thief; now, the thief encompasses a cop — the mould which splits in two parts right vs. duties —, and who tells us that the two actually match? This conflict is the very heart of our legimate doubts and our reasonable certainty.