Aspects of Inference in Natural Language

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Introduction

Native speakers of a natural language have certain intuitions about what can be inferred on the basis of a sentence uttered. For instance, consider the following example.

(1) The Japanese student disappeared.

Suppose that Lisa, who teaches at a university, utters (1) in a conversation. From this utterance, the hearer can naturally infer the propositions indicated in (2), (3), and (4), respectively.

(2) Someone disappeared.

(3) There is a Japanese student (in Lisa's class).

(4) The Japanese student *in Lisa's class* disappeared *from her class*.

Here three kinds of inferences and, parallel with them, three kinds of propositions inferred, can be distinguished. The proposition in (2) is an *entailment* of (1), and the proposition in (3) is a *presupposition* of (1). Regarding the proposition in (4), there seems to be no established terminology among philosophers and linguists; here we call it an *enrichment* of (1). (In this Introduction, we use the term "enrichment" in a broad sense, to describe data as shown in (4). In Chapter 3, we will introduce more fine-grained notions such as *saturation* and *free enrichment*, which can be regarded as subclasses of enrichment in this broad sense.)

This dissertation is divided into three chapters. Each chapter is devoted to discussing these three types of inference: entailment (Chapter 1), presupposition (Chapter 2), and enrichment (Chapter 3), respectively. Before entering into the details, we will first briefly explain the main theme of each chapter.

Entailment relations are of central importance in the enterprise of the formal semantics of natural language. The problem of determining whether one sentence intuitively entails another—in the sense that one could not accept the first without also being committed to the second—would require vast amounts of world knowledge. However, an important class of entailments seems to follow general patterns that arise from the way various "logical" expressions combine with other expressions to make up complex sentences. Formal semantics has largely been concerned with characterizing and formalizing such logical and structural aspects of entailment relations.

In modern logic, entailment relations are characterized from two viewpoints: the model-theoretic one and the proof-theoretic one. However, since Montague's invention of the field, most approaches within natural language semantics are based on model-theoretic conceptions. Thus, in linguistics, "formal semantics" usually means "model-theoretic semantics." Accordingly, the notion of the validity of inferences is only characterized in model-theoretic terms, and little is known about relevant proof-theoretic notions such as provability and proof as applicable to natural language inferences. The central aim of Chapter 1 is to fill this gap, by offering a simple inference system for a syllogistic fragment of natural language, a fragment containing quantificational sentences that is of fundamental importance in formalization of reasoning with natural language sentences.

Presuppositions have received a great deal of attention, both from philosophers and from linguists. Presuppositions are distinguished from entailments in that they *survive* in a certain environments. For an illustration, consider the following sentences.

(5) a. The Japanese student didn't disappear.

b. Did the Japanese student disappear?

c. If the Japanese student disappeared, Lisa would be surprised.

From an utterance of any one of (5a)–(5c), the hearer can naturally infer the proposition in (3), i.e., that there is a Japanese student in Lisa's class. By contrast, the proposition in (2), which is an entailment of (1), cannot be inferred from any one of (5a)–(5c). In general, among propositions which can be naturally inferred from an utterance of sentence S, those that can survive even when S is negated, questioned, or supposed are presuppositions of the utterance of S. Thus, entailments and presuppositions are primarily distinguished in terms of inferences.

Chapter 2 of this dissertation is an attempt to analyze presuppositions and their interaction with entailments from a *proof-theoretical* viewpoint. We propose a formal framework based on natural deduction systems, and explore the possibility this offers to analyze some presupposition phenomena. Our framework is intended to be an alternative to *dynamic* frameworks, such as dynamic semantics and discourse representation theory, which are currently standard within formal approaches to presuppositions.

The presupposition in (3) is triggered by the occurrence of the definite description *the Japanese student* in sentence (1). It is generally true of definite descriptions that appear in argument position as in (1), that they license such inferences, i.e., the so-called *existence presuppositions*. Chapter 2 is focused on existence presuppositions triggered by definite descriptions.

We often say that one sentence entails another sentence. But we will assume that an entailment relation is a relation between *propositions* (or, more generally, that it is a relation between a set of propositions and a proposition). When we say that (1) entails (2), where (1) is the sentence a speaker utters, it should be taken as meaning that the proposition expressed by an utterance of (1) entails the proposition indicated in (2). The same point applies to the case of presuppositions. When we say that (1) presupposes (3), where (1) is the sentence a speaker utters, it should be taken as meaning that the proposition expressed by an utterance of (1) presupposes the proposition indicated in (2). In this way, we can separate two questions, namely, the question of what proposition is expressed by the utterance of sentence S on a given occasion, and the question of what entailment or presupposition relations the proposition expressed by S has with other propositions.

To be more precise, by uttering the sentence in (1), Lisa communicates to the hearer the proposition that the Japanese student *in her class* disappeared *from her class*, i.e., the proposition indicated in (4). That is to say, the proposition expressed by Lisa's utterance, which has entailment or presupposition relations with other propositions, is not the "minimal" proposition that the Japanese student disappeared, but rather the pragmatically *enriched* one.

How can the enriched proposition as in (4) be inferred from the utterance of (1)? This question is the main topic of Chapter 3. Our approach to this question is based on a *Contextualist* viewpoint. In particular we will build on the framework of Relevance Theory, which is one of the most developed versions of Contextualism. As is well known, Relevance Theory is based on a representational or deductive perspective on inferences in natural language. The main aim of Chapter 3 is to show that there is a semantic constraint on how the process of enrichment works, and to explore the consequences of admitting such a constraint in light of the recent debates in the semantics-pragmatics interface.

Throughout this dissertation, we emphasize the role of intermediate representations in analyzing inferences in natural language. Currently, many semanticists and philosophers of language favor the "direct" approach to natural language semantics over the "translational" approach, claiming that intermediate levels of representation are essentially redundant. As is well known, Richard Montague, in his paper "English as a formal language" (1970), showed how a fragment of English can be model-theoretically interpreted without first translating it into an intermediate logical representation. Also, David Lewis, in a seminal paper (Lewis 1970), argued against the translational approach to natural language semantics on philosophical grounds.

Despite these facts, there seem to be methodological advantages to employing intermediate representations, whether they are formal representations (i.e., formulas in a logical language) as used in Chapter 1 and Chapter 2, or informal representations (i.e., disambiguated natural language sentences) as used in Chapter 3. Formal representations, such as formulas of first-order logic, are essential when we use a formal proof system to make a certain prediction about natural language inferences in a precise way. Furthermore, intermediate representations make it possible to describe semantic phenomena we are concerned with in a clear and transparent way. The fact that intermediate representations are redundant and theoretically eliminable does not mean that they are useless in theorizing about natural language inferences.

Overview for Chapter 1

Chapter 1 is concerned with entailment relations in a syllogistic fragment of natural languages, a fragment that has been widely discussed in the context of studies of natural language inferences. We introduce an inference system, called the *Generalized Syllogistic inference system* GS. The system is based upon two primitive relations between terms, inclusion and exclusion. We show the completeness and normalization theorem of GS, and then provide a characterization of the structure of normal proofs. Based on this result, we show that inferences in a syllogistic fragment of natural language are faithfully translated into inferences in GS.

As stated above, most approaches to entailment relations in natural languages are solely based on the model-theoretic conception. Thus, the notion of the validity of inferences is only characterized in model-theoretic terms, and only a few attempts have addressed the relevant proof-theoretic notions that are applicable to natural language inferences.

Indeed, there are several enterprises to fill this gap. One is the study of syllogisms from a modern logical viewpoint, which was started by Lukasiewicz (1951) and given natural deduction formulations by Corcoran (1972) and Smiley (1973), among others. In connection with this, more recently, natural logic, including the so-called calculus of monotonicity proposed in van Benthem (1986) and Sanchez (1991), has been developed by some linguists and logicians. The main aim of these studies is to characterize the natural language inferences of the forms as closely as possible to surface forms: that is to say, to construct an inference system whose syntax closely mirrors that of natural languages.

Our approach is consonant with these approaches, in that a proof system plays a dominant role, but one essential difference is that we *decompose* syllogistic inferences and the categorical statements constituting them in terms of more basic relations (i.e., inclusion and exclusion relations), whereas most approaches in modern logical studies of syllogisms and natural logic take surface forms as primitive and do not attempt to reduce them into more primitive forms. As a consequence, the inference systems proposed in these approaches have some additional axioms or inference rules concerned with *negation* other than those concerned with inclusion and exclusion relations.

Another important difference is that our system is closely related to an inference system for Euler diagrams. The field of *diagrammatic logic* was initiated by philosophers and logicians in the 1990s. It is well known that Euler diagrams can be used to represent not only Aristotelian categorical statements but also the syllogistic reasoning based on them. However, the exact formulation of inference rules operating on Euler diagrams was not clear until recently. Mineshima, Okada, and Takemura (2012a) made a first attempt to give a complete inference system to Euler diagrammatic reasoning. The system is called the "Generalized Diagrammatic Syllogistic inference system," abbreviated as GDS. We will show the correspondence between the inference system GS and the diagrammatic inference system GDS. Thus, the system GS can serve as a bridge between natural logic and diagrammatic logic. This opens an interesting possibility to connect studies of natural language inferences with studies of visual/diagrammatic inferences, thereby making it possible to compare these two kinds of inferences from a unified and rigorous logical (proof-theoretical) viewpoint.

Overview for Chapter 2

Chapter 2 is concerned with a proof-theoretic analysis of presuppositions in natural language. We focus on the interpretation of definite descriptions. We introduce a natural deduction based framework for dealing with existence presuppositions of definite descriptions. We show that our framework can handle a large class of phenomena, as discussed in the literature, and at the same time avoid some problems inherent in the standard dynamic approaches to presuppositions.

As is well known, the notion of presupposition as currently studied in philosophy and linguistics can be traced back to some of Frege's writings, and has received special attention since Strawson's classic paper "On Referring" (1950). However, it was only relatively recently that systematic theories of presuppositions appeared in connection with the development of formal semantics of natural language. In particular, several new approaches to the so-called projection problems of presuppositions have been launched since the 1980s. These new approaches emphasize that presupposition phenomena motivate the so-called *dynamic* conception of meaning, according to which the meaning of a sentence should be regarded as *context change potentials* or *context update conditions*, rather than as traditional truth-conditions.

There are two influential approaches along this line: Discourse Representation Theory and Dynamic Semantics. Discourse Representation Theory (DRT) was originally invented by Kamp (1981), and was augmented with a mechanism to handle presuppositions by van der Sandt (1992). The idea underlying Dynamic Sematics goes back to Stalnaker (1974) and Karttunen (1974), but its explicit formulation within dynamic semantics was done by the classical work of Heim (1983). Both approaches emphasize that standard logical systems are not suited to deal with presupposition phenomena. The main issue here is how to represent *contexts* that interact with the asserted content of an utterance in a complicated way. In particular, the problem is how to handle *local* contexts, i.e., contexts that are updated in the middle of a sentence, rather than given prior to an utterance.

In contrast to dynamic approaches, we opt to preserve the standard logical systems. The basic idea is to use the natural deduction systems developed in the tradition of Gentzen's proof theory; we apply them to analyses of presupposition phenomena in natural language. Our proposal is based on the natural deduction system of first-order ε-calculus and on constructive type theory. We will concentrate on the existence presupposition of definite descriptions, since there is a well-established proof system (i.e., ε-calculus) that deals with descriptions in the mathematical domain. Furthermore, the existence presupposition serves as a good test case for a more general theory of presuppositions.

Overview for Chapter 3

Recent studies in the semantics and pragmatics of natural language have shown that there is a considerable gap between the linguistic meaning of a sentence and the proposition expressed by an utterance of that sentence (i.e., "what is said" in Gricean terms, or "explicature" in relevance-theoretic terms). This raises the question: What kinds of pragmatic tasks are involved in the determination of the proposition expressed by an utterance? There are two influential approaches to this question, which we call "Indexicalism" and "Contextualism." Contextualism holds that purely pragmatic processes called "free enrichment" are involved in the derivation of the proposition expressed. Indexicalism, on the other hand, denies the existence of such processes, and maintains that no pragmatic processes are allowed to affect the proposition expressed by an utterance unless the linguistic meaning of the sentence itself so demands.

It is well known that the current relevance theory takes the position of Contextualism. However, in our view, the standard version of Contextualism is in fact very radical in that it holds that there are almost no linguistic factors or constraints involved in the way the process of free enrichment works. In this chapter, we will argue that both Indexicalism and the standard version of Contextualism are mistaken in their conception of the way linguistic semantics is related to the pragmatic processes involved in the determination of the proposition expressed.

Based on a close analysis of predicational copular sentences, we will show that there is an interesting constraint on the applicability of free enrichment, and argue that the existence of such a constraint poses serious problems to both Indexicalism and the standard version of Contextualism. More specifically, we argue that free enrichment is blocked for property concepts, i.e., those concepts that are expressed by property expressions such as predicate nominals and adjectives. We will then propose a new version of Contextualism that is compatible with the claim that there is a semantic constraint on free enrichment. We also argue that this constraint is based on a difference in semantic function between what we call "object-directed concepts" and "property concepts." In our framework, the so-called "over-generation" problems against Contextualism pointed out by Stanley (2002, 2005) can be avoided without stipulating any *ad hoc* mechanism. We also discuss and reject Hall's (2008) pragmatic account of the over-generation problem. The distinctive features of our conception of free enrichment will be made clear by comparing it with Hall's (2008) pragmatic approach within the standard relevance theory.

Chapter 1

A Syllogistic Inference System with Inclusion and Exclusion

1. Introduction to Chapter 1

Entailment relations are of central importance in the enterprise of natural language semantics. In modern logic, entailment relations are characterized from two viewpoints: model-theoretic and the proof-theoretic. Since the work of Richard Montague (Montague 1974), however, most approaches to formalizing entailment relations in natural languages have been based solely on model-theoretic conceptions. Thus, the notion of validity is only characterized in model-theoretic terms, and few attempts have been made at the relevant proof-theoretic notions, such as provability and proof, as applied to natural language inferences. the aim of this chapter is to offer a simple inference system for *syllogistic* fragment of natural language, thereby making a connection between proof theory and natural language semantics.

As is well known, there is an influential approach applying proof theory to natural language *syntax*, which started with Lambek calculus (Lambek 1958) and has been more fully explored in the recent literature of categorial grammar such as Type-Logical Grammar (cf. Jäger 2005; Barker and Jacobson 2007). But our interest here is in applying proof-theoretic methods to natural language *semantics*. The goal is to represent and analyze, in proof-theoretic terms, the relevant semantic notions such as entailments, rather than syntactic structures of sentences,

Indeed, there have been several prior attempts in this direction. One is the study of syllogisms from a modern logical viewpoint which was started by Łukasiewicz (1957) and then given their natural deduction formulations by Corcoran (1972, 1974) and Smiley (1974), among others. In connection with this, the so-called *monotonicity calculus*, originally proposed by van Benthem (1986) and Sánchez Valencia (1991), has been developed by some linguists and logicians.[1] There have also been recent important developments in the logic and AI literature, including the program of natural logic, which extend syllogistic logics to cover more expressive fragments of natural language (cf. Nishihara and Morita 1988; Nishihara, Morita, and Iwata 1990; Moss 2008, 2010a, 2010c; Pratt-Hartmann and Moss 2009; Francez, Dyckhoff, and Ben-Avi 2010). One of the main aims of these studies is to characterize natural language inferences with forms as close as possible to their surface forms.[2]

Our approach agrees with these approaches in that a proof system plays a role in developing natural language semantics. However, an essential difference is that we *decompose* syllogistic inferences and categorical statements constituting them in terms of two primitive relations, i.e., inclusion and exclusion relations, whereas the approaches we just mentioned take surface forms as primitive and do not attempt to reduce them into more primitive forms. In our approach, syllogistic inferences are characterized in a reductive way, in terms of the structure of proofs in our underlying inference system. We call this system “Generalized Syllogistic inference system”, abbreviated as GS.[3]

It should be noted that the traditional approaches to formalization of categorical syllogisms, including Łukasiewicz (1957), Corcoran (1972, 1974), Smiley (1973), Westerståhl (1989) and Moss (2008), among others, essentially rely on the axioms and inference rules of full (classical) propositional logic, in particular, the rule of *reductio ad absurdum*. By contrast, a remarkable feature of GS is that it is “logic free” in the sense that only atomic formulas and their conjunction appear in proofs; essential steps in a GS-proof

[1] For recent overviews, see van Eijck (2007), van Benthem (2008) and Moss (2010b).

[2] One important origin of these studies is the semantics of generalized quantifiers started by Barwise and Cooper (1981). Some connections between generalized quantifiers and syllogistic forms of inferences are investigated by van Eijck (1985) and Westerståhl (1989).

[3] An extended discussion of GS is found in Mineshima, Okada and Takemura (2012b). Recently, exclusion relations have also been investigated in the context of textual inference in natural language processing. In particular, MacCartney and Manning (2008) generalized the so-called *monotonicity calculus* (see van Benthem 1986; Sànchez Valencia 1991) by incorporating exclusion relations.

do little more than inferring an atomic formula from other atomic formulas given as premises. Our treatment of categorical syllogisms within GS shows that traditional categorical syllogisms can be reconstructed in such a simple way.

Another important characteristic is that our system is closely related to the inference system for Euler diagrams (called GDS) by Mineshima, Okada, and Takemura (2012a), which is an attempt to provide a complete inference system for Euler diagrams defined in terms of relations between objects such as circles (contours) and points.[4] There is a close connection between the two types of inferences, that is, natural language inferences and diagrammatic inferences, in that there are certain restrictions on nested occurrences of operators involved, such as negation, implication, and quantifiers; see van Benthem 2008 for some discussion. Although restrictions on nested structures may cause a loss of expressive power, the resulting system could become a more efficient and human-oriented tool in communication and reasoning. In view of this connection, it seems to be interesting to develop an underlying proof system for these two types of inferences. Indeed, the system GS serves as such a system; it provides a bridge between natural logic and diagrammatic logic, or more specifically, between linguistic syllogistic inferences and Euler-style diagrammatic inferences. This opens an interesting possibility to connect studies of natural language inferences with studies of visual/diagrammatic inference, thereby making it possible to compare these two kinds inferences in a unified and rigorous logical (proof-theoretical) viewpoint.

Introduction to GS

Let us informally explain by examples how categorical sentences and syllogisms composed of them can be represented in the system GS. In our approach, the categorical sentences in syllogisms are analyzed in terms of two primitive relations: inclusion ($\sqsubset$) and exclusion ($\vdash\!\dashv$).[5] To begin with, a categorical sentence of the form

[4]See also Mineshima, Okada, Sato and Takemura (2008) and Mineshima, Okada, and Takemura (2009) for earlier proposals.

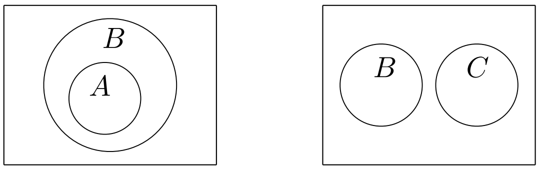
[5]Our notation of $\vdash\!\dashv$ derives from Gergonne (1817). For the purpose of the abstract

(1) All A are B

is represented as $A \sqsubset B$, and a categorical sentence of the form

(2) No B are C

is represented as $B \mathrel{\vdash\!\dashv} C$. Here A, B and C are general terms denoting sets of individuals, and the symbols $\sqsubset$ and $\mathrel{\vdash\!\dashv}$ are semantically interpreted as the subset and disjointness relations, respectively. Figure 1.1 below shows the well known correspondence between universal categorical sentences and Euler diagrams.



D_1: All A are B D_2: No B are C

$A \sqsubset B$ $A \mathrel{\vdash\!\dashv} B$

Fig. 1.1 Correspondence between universal sentences and Euler diagrams.

It should be emphasized here that categorical sentences such as (1) and (2) are a special case of *quantificational* sentences in natural language. According to the standard treatment in logic textbooks, such sentences have been analyzed using representations in first-order predicate logic, which essentially involve quantification over individuals. In the field of natural language semantics, by contrast, quantifiers are analyzed as denoting *relations* between sets, i.e., what is called *generalized quantifiers* (Barwise and Cooper 1981). Thus, a sentence of the form All A are B can be analyzed as having

representation of Euler diagrams, Gergonne introduced some symbols for binary relations, which are widely called "Gergonne relations," i.e., exclusion ($A \mathrel{\vdash\!\dashv} B$), identity ($A \mathsf{I} B$), overlap ($A \mathsf{X} B$), proper containment ($A \mathsf{C} B$), and proper inclusion (A Ɔ B) in his notation. See Faris (1955) for analyses of Gergonne relations from a modern standpoint. In Section 5.1, we will briefly discuss Gergonne's approach to representing Euler diagrams, in comparison with ours.

a logical form $All(A, B)$, rather than as having the first-order representation $\forall x(Ax \rightarrow Bx)$. $All(A, B)$ means that $\mathbf{A} \subseteq \mathbf{B}$, where $\mathbf{A}$ and $\mathbf{B}$ are sets denoted by terms A and B, respectively. Similarly, No A are B can be analyzed as having a relational logical form $No(A, B)$, expressing that $\mathbf{A} \cap \mathbf{B} = \emptyset$. The point is that the semantic primitives of quantificational sentences can be regarded as the relations between sets, such as subset relation and disjointness relation. The relational representations of universal categorical sentences in our approach such as $A \sqsubset B$ and $A \vdash\!\dashv B$, are consistent with, and could be regarded as a proof-theoretical counterpart of, such a semantic conception of quantified sentences developed in generalized quantified theory. Note that the modern reconstructions of Aristotelian categorical syllogisms (Łukasiewicz, 1958; Corcoran, 1972; Smiley, 1973) and recent development of natural logic (e.g. Moss 2008; Pratt-Hartmann and Moss 2009) take as a primitive logical form the relational structure of a quantified sentence, such as the one schematically represented as *Q(A,B)*. As we will see below, such a relational approach suggests that syllogistic inferences can be formulated as a certain kind of *relational* inferences in a perspicuous way, without reference to first-order quantifiers and individuals terms. We will push forward this research tradition, with a detailed analysis of the structure of proofs built from inclusion and exclusion.

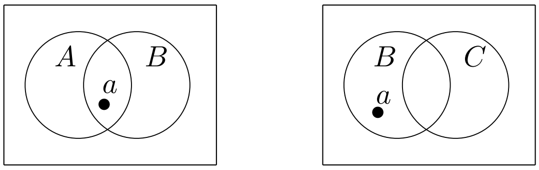
Our approach differs from the previous studies on Aristotelian syllogisms and natural logic in the treatment of existential sentences. To see it, consider existential sentences of the forms

(3) Some A are B

and

(4) Some A are not B.

These sentences are diagrammatically represented by D_3 and D_4, respectively, in Figure 1.2. These diagrams suggest that Some A are B is decomposed into two primitive assertions, $a \sqsubset A$ and $a \sqsubset B$, and Some A are not B into $a \sqsubset A$ and $a \vdash\!\dashv B$, where a is a singular term denoting a witness of the existential sentence. In our system, singular terms are semantically interpreted as a singleton set, which enables us to treat $\sqsubset$ as the subset relation



D_3: Some A are B D_4: Some A are not B

$a \sqsubset A, a \sqsubset B$ $a \sqsubset A, a \mathbin{\vdash\!\dashv} B$

Fig. 1.2 Correspondence between existential sentences and Euler diagrams.

in a uniform way.

Although the syntax of GS is quite simple, it is sufficient to represent a categorical syllogism (more generally, a chain of categorical syllogisms). We represent proofs in GS in a tree-form, as is usual in Gentzen-style natural deduction systems. For example, a syllogism called Celarent

$$\frac{\begin{array}{c}\text{All } A \text{ are } B\\ \text{No } B \text{ are } C\end{array}}{\text{No } A \text{ are } C}$$

is represented in GS simply as

$$\frac{B \mathbin{\vdash\!\dashv} C \quad A \sqsubset G}{A \mathbin{\vdash\!\dashv} C}\,(\sqsubset).$$

As an example involving an existential sentence, consider a syllogism Darii:

$$\frac{\begin{array}{c}\text{Some } A \text{ are } B\\ \text{All } B \text{ are } C\end{array}}{\text{Some } A \text{ are } C}$$

This syllogism is simulated as:

$$\cfrac{\cfrac{\begin{array}{c}\text{Some } A \text{ are } B\\ a \sqsubset A,\ a \sqsubset B\end{array}}{a \sqsubset A} \qquad \cfrac{\cfrac{\begin{array}{c}\text{Some } A \text{ are } B\\ a \sqsubset A,\ a \sqsubset B\end{array}}{a \sqsubset B} \quad \begin{array}{c}\text{All } B \text{ are } C\\ B \sqsubset C\end{array}}{a \sqsubset C}\,(\sqsubset)}{\begin{array}{c}a \sqsubset A,\ a \sqsubset C\\ \text{Some } A \text{ are } C\end{array}}$$

Here, in order to make clear the translation between categorical sentences and formulas of GS, we attach a categorical sentence with each assumption and conclusion. The crucial inference rules here are the ones concerned with the transitivity of inclusion and exclusion relations, labeled as ($\sqsubset$) and ($\vdash\!\dashv$), respectively. A detailed translation procedure will be given in Section 3.2.

It should be noted that these two inference rules can also account for syllogistic inferences involving a categorical sentences with a proper name. Thus, a syllogism involving a proper name *Socrates*

Socrates is a Greek.
All Greeks are mortal.
―――――――――――
Socrates is mortal.

is simply interpreted in GS as

$$\frac{s \sqsubset G \quad G \sqsubset M}{s \sqsubset M}.$$

Here the singular term s stands for Socrates, the general term G for Greek, and the general term M for man. This example shows that in GS, both a singular term such as Socrates and a general term such as man and Greek can appear equally on the left and right sides of an inclusion relation $\sqsubset$ or an exclusion relation $\vdash\!\dashv$. As noted before, in the semantics of GS, singular terms are interpreted as denoting a singleton set, and thus they can be regarded as special cases of general terms. In this respect, our treatment of singular terms is similar to the Leibnizian view on singular terms revived by Fred Sommers, where a particular proposition Socrates is mortal is represented as a kind of general proposition Some Socrates is mortal, or equivalently, as Every Socrates is mortal.[6] Note that in GS the translations of these two propositions are collapsed into a formula $s \sqsubset M$. Note also that by allowing a singular term to appear on both the right and left sides of $\sqsubset$ and $\vdash\!\dashv$, we can represent a sentence involving only as in (5), an identity sentence as in (6), and a sentence for non-identity as in (7).

[6] See Sommers (1982: chap.1) and Englebretsen (1981).

(5)	Only Socrates is wise	$W \sqsubset s$
(6)	Cicero is Tully	$c \sqsubset t$
(7)	Cicero is not Tully	$c \vdash\!\dashv t$

Here for simplicity, we take the positive component—known as the *prejacent*—of (5), i.e., the proposition that Socrates is wise, to be part of the entailment.

The structure of Chapter 1

The outline of this chapter is as follows.

In Section 2, we define the semantics and proof theory of GS and proves a completeness theorem. We also show a normalization theorem for our system GS in an analogous way to a standard natural deduction system. Then we characterize the structure of *normal proof* in GS.

In Section 3, we present systems of categorical syllogisms with and without existential import, called CS and CS$^+$, respectively. We show the faithful embeddability of CS and CS$^+$ into GS using the notion of normal proofs in GS. The fragment of GS corresponding to categorical syllogism is called a *syllogistic fragment* of GS.

To relate GS to a well established proof system, in Section 4 we show that the syllogistic fragment of GS can be embedded into a propositional fragment of minimal logic (called ML), where by minimal logic we mean intuitionistic logic minus the absurdity rule. The basic idea is to translate a GS-formula $A \sqsubset B$ into $A \to B$ in propositional logic, and $A \vdash\!\dashv B$ into $A \to \neg B$. Thus, the inclusion and exclusion relations between terms are treated like two kinds of *implication* between propositions.

In Section 5, we introduce an inference system for Euler diagrams, termed as GDS. The main result of this section is to show the faithful embeddability of GDS into GS and thereby to establish the relationships between the linguistic and diagrammatic inference systems we are concerned with in this chapter.

In Section 6, we consider an extension of GS with *intersection*. This extended system of GS allows a term of the form $A \sqcap B$ which denotes an intersection of the denotations of A and B. In this system, we can represent categorical sentences with modifying phrases such as relative clauses. The main result is a completeness theorem for this extended system.

The relationships between GS and other inference systems are summarized in Figure 1.3.

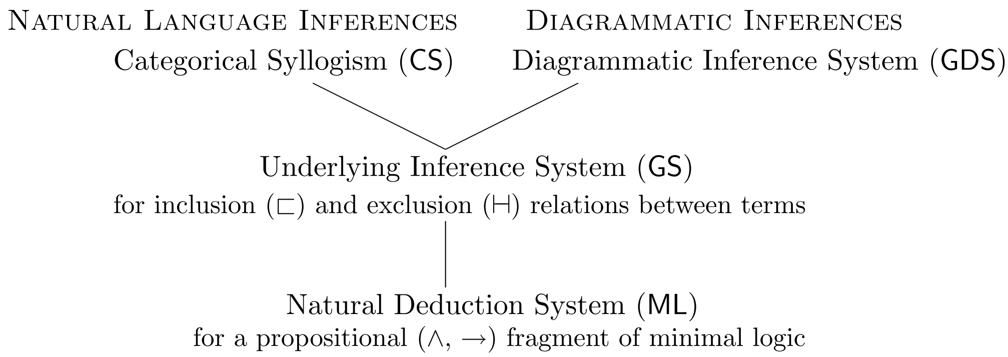


Fig. 1.3 Relationships between GS and other inference systems

2. Generalized syllogistic inference system GS

In this section, we introduce the proof theory and semantics of GS. In Section 2.1, we present the proof theory of GS, and show a normalization theorem, based upon which, we investigate the structures of normal proofs in GS. In Section 2.2, we present the semantics of GS, and prove a completeness theorem.

We first present the language of GS. Throughout this chapter, the binary relation symbol $\equiv$ is used to denote syntactic identity.

Definition 2.1 The language of GS contains singular terms, denoted by $a, b, c, \ldots$, general terms, denoted by $A, B, C, \ldots$, and relation symbols between terms, $\sqsubset$ (inclusion) and $\vdash\!\dashv$ (exclusion). Formulas of GS are defined as follows.

(i) If s and t are terms, then $s \sqsubset t$ and $s \vdash\!\dashv t$ are formulas. These are called *atomic* formulas.

(ii) If $P_1, \ldots, P_n$ are atomic formulas ($n \geq 1$), then $\{P_1, \ldots, P_n\}$ is a formula. A singleton $\{P\}$ is identified with P.

We call a formula of the form $s \sqsubset t$ an *inclusion formula* ($\sqsubset$-formula) and a formula of the form $s \vdash\!\dashv t$ an *exclusion formula* ($\vdash\!\dashv$-formula). For all terms s, t, $s \vdash\!\dashv t \equiv t \vdash\!\dashv s$, i.e., $s \vdash\!\dashv t$ and $t \vdash\!\dashv s$ are syntactically identical.

Notation. We use syntactic variables (possibly with subscripts) $s, t, u, \ldots$ to denote terms, $P, Q, \ldots$ to denote atomic formulas, $\mathcal{P}, \mathcal{Q}, \ldots$ to denote formulas, and $\Gamma, \Delta, \ldots$ to denote a set of formulas.

Remark 2.2 To keep the proof theory of GS as simple as possible, we disregard the order of terms in exclusion formulas and identity $s \vdash\dashv t$ with $t \vdash\dashv s$ at syntactic level. Alternatively, we could treat $s \vdash\dashv t$ and $t \vdash\dashv s$ as distinct formulas and adopt an inference rule that derives $t \vdash\dashv s$ from $s \vdash\dashv t$. Here, we prefer the current approach, because it considerably simplifies the proof structures in GS. Additionaly, as we will see later, it will make the translation of the diagrammatic inference system GDS into GS much simpler.

The language of GS is summarized in Table 2.1.

	Variables	Form
Term	$s, t, u, \ldots$	
Singular term	$a, b, c, \ldots$	
General term	$A, B, C, \ldots$	
Atomic formula	$P, Q, \ldots$	$s \sqsubset t \mid s \vdash\dashv t$
Formula	$\mathcal{P}, \mathcal{Q}, \ldots$	$\{P_1, \ldots, P_n\}$
A set of formulas	$\Gamma, \Delta, \ldots$	

Table 2.1 Terms and formuas of GS

Our definition of formulas differs from the standard one in that we regard a set of atomic formulas as a formula. This makes comparisons of GS with categorical syllogisms and Euler diagrams easier. As we will see in later sections, both a categorical sentence and an Euler diagram can be interpreted as a GS-formula. The intended meaning of a formula of the form

$$\{P_1, \ldots, P_n\}$$

is the conjunction

$$P_1 \wedge \ldots \wedge P_n$$

of atomic formulas. See also Definition 2.13 of semantics below.

2.1 The proof theory of GS

We present the proof theory of GS. Proofs of GS are given in tree form. The axiom and inference rules are given below.

Definition 2.3 (Axiom and inference rules of GS)
Axiom (ax): $s \sqsubset s$.
Inference rules:

$$\frac{s \sqsubset t \quad t \sqsubset u}{s \sqsubset u}\ (\sqsubset) \qquad \frac{s \sqsubset t \quad t \vdash\!\dashv u}{s \vdash\!\dashv u}\ (\vdash\!\dashv) \qquad \frac{s \sqsubset a}{a \sqsubset s}\ (\mathsf{C})$$

$$\frac{\mathcal{P} \quad \mathcal{Q}}{\mathcal{P} \cup \mathcal{Q}}\ (+) \qquad \frac{\mathcal{P}}{\mathcal{P}'}\ (-)$$

where $\mathcal{P} \neq \mathcal{Q}$ in (+), and $\mathcal{P}' \subset \mathcal{P}$ in (−).[1]

We call the term t in ($\sqsubset$) and ($\vdash\!\dashv$) a *middle term*. An inspection of the rules shows that except for (−), the rules that eliminate a term appearing in a premise are ($\sqsubset$) and ($\vdash\!\dashv$), and the only term eliminated is the middle term t. Note also that no rule introduces a new term in the conclusion.

By our definition the lower formula of (+) and the upper formula of (−) are non-atomic, whereas an upper formula of (+) and the lower formula of (−) may be atomic.

The (C) rule allows us to infer $a \sqsubset A$ ("a is A") from $A \sqsubset a$ ("Only a is A") and $a \sqsubset b$ ("a is b") from $b \sqsubset a$ ("b is a").

As stated above, by the formula $\{P_1, \ldots, P_n\}$ we mean the conjunction $P_1 \wedge \cdots \wedge P_n$, so that the (+) and (−) rules are generalizations of $\wedge$-introduction and $\wedge$-elimination rules in Gentzen's natural deduction system.

Next we define inductively the notion of a proof in GS. A proof π in GS is a formula-tree: the last formula of the tree is called the *conclusion* of π and the topmost formulas of the tree are called the *assumptions* of π.

Definition 2.4 (Proof) A *proof* in GS of a formula $\mathcal{P}$ (conclusion) from a set of formulas Γ (assumptions) is defined inductively as follows:

1. A formula $\mathcal{P}$ is a proof of $\mathcal{P}$ from $\{\mathcal{P}\}$.

2. An axiom $\mathcal{P}$, i.e., a formula of the form $s \sqsubset s$, is a proof of $\mathcal{P}$ from the empty set.

[1]Throughout this chapter, we use $\subset$ to denote a proper subset relation, i.e., $\subsetneq$.

3. Let π_1 be a proof of $\mathcal{P}_1$ from Γ_1, and π_2 be a proof of $\mathcal{P}_2$ from Γ_2. If $\mathcal{Q}$ is obtained by an application of $(\sqsubset)$, $(\vdash\!\dashv)$ or $(+)$ to $\mathcal{P}_1$ and $\mathcal{P}_2$, then the following formula-tree (i) is a proof of $\mathcal{Q}$ from $\Gamma_1 \cup \Gamma_2$.

4. Let π be a proof of $\mathcal{P}$ from Γ. If $\mathcal{Q}$ is obtained by an application of $(-)$ or (C) to $\mathcal{P}$, then the following formula-tree (ii) is a proof of $\mathcal{Q}$ from Γ.

$$\text{(i)} \quad \frac{\begin{matrix}\Gamma_1 \\ \vdots\ \pi_1 \\ \mathcal{P}_1\end{matrix} \qquad \begin{matrix}\Gamma_2 \\ \vdots\ \pi_2 \\ \mathcal{P}_2\end{matrix}}{\mathcal{Q}} \qquad\qquad \text{(ii)} \quad \frac{\begin{matrix}\Gamma \\ \vdots\ \pi \\ \mathcal{P}\end{matrix}}{\mathcal{Q}}$$

For each application α of inference rule I in a proof π, we say that a formula appearing on the upper side of α is a *premise* of α, and that the formula appearing on the lower side of α is the *conclusion* of α. Sometimes, instead of saying "a premise (conclusion) of an application of the rule I", we just say "a premise (conclusion) of the rule I" for the sake of brevity.

The *length* of a proof π is defined as the number of applications of inference rules in π.

Definition 2.5 (Provability) A formula $\mathcal{Q}$ is *provable* from Γ in GS, written as $\Gamma \vdash \mathcal{Q}$, if there is a proof in GS of $\mathcal{Q}$ from Γ or a subset of Γ.

Notation. For the sake of brevity, when a formula $\{P_1, \dots, P_n\}$ appears in a proof tree, we commonly omit the brackets and abbreviate it as $P_1, \dots, P_n$.

Example 2.6 The following proof establishes that

$$\{a \sqsubset A, a \sqsubset B\}\,,\ A \sqsubset C,\ B \sqsubset D,\ D \sqsubset E,\ E \vdash\!\dashv F\ \vdash\ a \vdash\!\dashv F.$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \sqsubset A, a \sqsubset B}{a \sqsubset A}\,(-) \quad A \sqsubset C}{a \sqsubset C}\,(\sqsubset) \qquad \dfrac{\dfrac{a \sqsubset A, a \sqsubset B}{a \sqsubset B}\,(-) \quad B \sqsubset D}{a \sqsubset D}\,(\sqsubset)}{a \sqsubset C, a \sqsubset D}\,(+)}{a \sqsubset D}\,(-) \qquad \dfrac{D \sqsubset E \quad E \vdash\!\dashv F}{D \vdash\!\dashv F}\,(\sqsubset)}{a \vdash\!\dashv F}\,(\vdash\!\dashv)$$

One of the important results of Gentzen-style natural deduction (Gentzen 1934) is the *normalization theorem*, according to which every proof can be converted to a normal form (cf. Prawitz 1965, 1971; von Plato 2008). A proof in normal form does not make a *detour*, that is, it is never the case that an elimination rule immediately follows an introduction rule. Given that the (+) and (−) rules are generalizations of the ∧-introduction and ∧-elimination rules of Gentzen's natural deduction, we can give a similar rewriting procedure of a GS-proof into an appropriate form of a *normal* GS*-proof*. In a normal GS-proof, it is never the case that the (−) rule immediately follows the (+) rule.

Definition 2.7 (Normal GS-proof) A formula $\mathcal{P}$ is a *cut formula* in π if $\mathcal{P}$ is the conclusion of an application of (+) and the premise of an application of (−). Suppose that a proof π contains a cut formula as shown on the left below. We can then transform π into the form on the right:

$$
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{\mathcal{P}_1} \quad \overset{\vdots\,\pi_2}{\mathcal{P}_2}}{\mathcal{P}_1 \cup \mathcal{P}_2}\,(+)}{\mathcal{Q}}\,(-)
\quad \rhd \quad
\begin{cases}
\dfrac{\overset{\vdots\,\pi_i}{\mathcal{P}_i}}{\mathcal{Q}}\,(-) & \text{when } \mathcal{Q} \subseteq \mathcal{P}_i \text{ for } i = 1 \text{ or } 2 \\[2ex]
\dfrac{\dfrac{\overset{\vdots\,\pi_1}{\mathcal{P}_1}}{\mathcal{Q}_1}\,(-) \quad \dfrac{\overset{\vdots\,\pi_2}{\mathcal{P}_2}}{\mathcal{Q}_2}\,(-)}{\mathcal{Q}}\,(+) & \begin{array}{c}\text{when } \mathcal{Q} = \mathcal{Q}_1 \cup \mathcal{Q}_2 \text{ such that} \\ \mathcal{Q}_1 \subseteq \mathcal{P}_1 \text{ and } \mathcal{Q}_2 \subseteq \mathcal{P}_2\end{array}
\end{cases}
$$

where in the resulting figures, if $\mathcal{P}_i = \mathcal{Q}$ (resp. $\mathcal{P}_i = \mathcal{Q}_i$) for $i = 1, 2$, it should be understood that the (−) rule is not applied and $\dfrac{\mathcal{P}_i}{\mathcal{Q}}$ (resp. $\dfrac{\mathcal{P}_i}{\mathcal{Q}_i}$) is replaced by $\mathcal{P}_i$.

We say that a GS-proof π is in the *normal form* if it contains no cut formulas.

Note that if $\mathcal{P}_1$ and $\mathcal{P}_2$ are atomic, it holds that either $\mathcal{Q} = \mathcal{P}_1$ or $\mathcal{Q} = \mathcal{P}_2$, and the transformation procedure is essentially the same as the one for conjunction in Gentzen's natural deduction system. Thus, in a similar way to normalization in a natural deduction system (see Prawitz 1965,1971;

Troelstra and Schwichtenberg 2000), we obtain the normalization theorem of GS.

Theorem 2.8 (Normalization) *Every proof π in* GS *can be transformed into a normal proof with the same conclusion as π.*

Proof. Let π be a proof in GS of $\mathcal{P}$ from Γ. We define the degree of a formula $\{\mathcal{P}_1, \ldots, \mathcal{P}_n\}$ as n. Let $\mathcal{Q}$ be a cut formula in π with a maximal degree, say k, in π. We apply the transformation procedure as stated in Definition 2.7 to $\mathcal{Q}$. It is easily seen that the resulting figure is still a proof in GS with conclusion $\mathcal{P}$. Note that a new cut formula may arise from this transformation, but the degree of this cut formula is lower than k. So by an application of the transformation procedure, the number of cut formulas of degree k is reduced by one. Hence, by a finite number of applications of the transformation procedure, we obtain a normal proof of $\mathcal{P}$. ■

Example 2.9 Consider the proof in Example 2.6, which contains a cut formula. By the transformation procedure, it is rewritten as the following normal proof.

$$\dfrac{\dfrac{\dfrac{a \sqsubset A, a \sqsubset B}{a \sqsubset B}\,(-) \quad B \sqsubset D}{a \sqsubset D}\,(\sqsubset) \quad \dfrac{D \sqsubset E \quad E \vdash\!\dashv F}{D \vdash\!\dashv F}\,(\vdash\!\dashv)}{a \vdash\!\dashv F}\,(\vdash\!\dashv)$$

A normal proof in GS has a specific structure. We say that a sequence $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ of formula occurrences in a proof π is a *path* in π if (1) $\mathcal{P}_1$ is an assumption (top-formula) of π, (2) $\mathcal{P}_i$ occurs immediately above $\mathcal{P}_{i+1}$ for each $i < n$, and (3) $\mathcal{P}_n$ is the conclusion (end-formula) of π.

In general, a path in a normal proof consists of three parts: (i) the *deletion* part in which the premises are decomposed into atomic formulas by applications of the $(-)$ rule, (ii) the *transitive* part in which atomic formulas are derived by applications of the $(\sqsubset)$ or $(\vdash\!\dashv)$ rules, and (iii) the *addition* part in which the atomic formulas derived in the transitive part are combined by applications of the $(+)$ rule. More precisely, we can characterize the structure of a normal proof in the following way.

Corollary 2.10 (The structure of normal proofs) Let π be a normal proof in GS and $\beta = \mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n$ be a path in π. Then, β can be divided into three parts (where each part is possibly empty):

1. the deletion part, $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{i-1}$, in which each formula is the premise of $(-)$ and hence non-atomic;
2. the transitive part, $\mathcal{P}_i, \mathcal{P}_{i+1}, \ldots, \mathcal{P}_{i+k}$, in which each formula is atomic; more specifically, $\mathcal{P}_i$ is an axiom or a premise of $(\sqsubset)$, $(\vdash\!\dashv)$, or (C) and the other formulas $\mathcal{P}_j$, except the last one (i.e., $i+1 \leq j < i+k$), are premises of $(\sqsubset)$, $(\vdash\!\dashv)$, or (C);
3. the addition part, $\mathcal{P}_{i+k}, \mathcal{P}_{i+k+1}, \ldots, \mathcal{P}_n$, in which each formula except the last one is a premise of $(+)$.

Furthermore, the transitive part is separated into two parts, i.e., the *inclusion* part consisting of applications of $(\sqsubset)$ or (C) and the *exclusion* part consisting of applications of $(\vdash\!\dashv)$. The inclusion part precedes the exclusion part. More precisely, there is a formula occurrence $\mathcal{P}_m$ which separates the transitive part into two (possibly empty) parts, so that:

(i) each formula $\mathcal{P}_j$ in the inclusion part ($i \leq j < m$), is an axiom or a premise of $(\sqsubset)$ or (C);

(ii) $\mathcal{P}_m$, provided that $m \neq i + k$, is a premise of $(\vdash\!\dashv)$;

(iii) each formula $\mathcal{P}_j$ in the exclusion part, except the last one ($m < j < i + k$), is a premise of $(\vdash\!\dashv)$.

Proof. We first show that in β each formula that is a premise of $(-)$ precedes each formula that is a premise of $(\sqsubset)$ or $(\vdash\!\dashv)$. Suppose for contradiction that in β there is a formula $\mathcal{Q}$ that is a premise of $(-)$ and which succeeds a formula that is a premise of $(\sqsubset)$ or $(\vdash\!\dashv)$. Since a premise of $(-)$ should be non-atomic, there must be an intervening formula that is a premise of $(+)$ and immediately precedes $\mathcal{Q}$. But then $\mathcal{Q}$ is a cut formula in π, contradicting the assumption that π is normal.

Next, it can be seen that in β each formula that is a premise of $(\sqsubset)$ or $(\vdash\!\dashv)$ precedes each formula that is a premise of $(+)$. Otherwise, there is a formula that is a premise of (+) and which precedes a formula that is a premise of ($\sqsubset$) or ($\vdash\!\dashv$). But a premise of ($\sqsubset$) or ($\vdash\!\dashv$) should be atomic; so there must be a formula that is a premise of ($-$) and which immediately follows the premise of (+). Again, this is impossible since π is normal.

Finally, by inspection of the ($\sqsubset$), ($\vdash\!\dashv$), and (C) rules, it is easily verified that in the transitive part of β, each formula that is an axiom or a premise of ($\sqsubset$) or (C) precedes each formula that is a premise of ($\vdash\!\dashv$). ■

The following are immediate consequences of Theorem 2.8 and Corollary 2.10.

Corollary 2.11

1. *Let P be an atomic formula. If $\Gamma \vdash P$, then $\Gamma \vdash P$ without using* (+).
2. *Let Γ be a set of atomic formulas and let P be an atomic formula. If $\Gamma \vdash P$, then $\Gamma \vdash P$ without using* (+) *and* ($-$).

Proof. By Theorem 2.8 and Corollary 2.10. ■

2.2 The semantics of GS

Now we introduce the set-theoretical semantics of GS.

Definition 2.12 A *model* M is a pair (U, I) where U is a non-empty set (the domain of M) and I is an interpretation function assigning to each term s a non-empty subset of U; in particular, $I(a)$ is a singleton for all singular term a.

Definition 2.13 We define the satisfaction relation $\models$ by:

$$M \models s \sqsubset t \text{ if } I(s) \subseteq I(t);$$
$$M \models s \vdash\!\dashv t \text{ if } I(s) \cap I(t) = \emptyset;$$
$$M \models \{P_1, \ldots, P_n\} \text{ if } M \models P_i \text{ for each } P_i \text{ with } 1 \leq i \leq n.$$

Note that if s is a singular term, $I(s)$ is a singleton, hence $I(s) \subseteq I(t)$ is equivalent to $I(s) \in I(t)$ and $I(s) \cap (t) = \emptyset$ is equivalent to $I(s) \notin I(t)$.

We say that (1) a model M satisfies a set of formulas Γ, written as $M \models \Gamma$, if $M \models \mathcal{P}$ for every formula $\mathcal{P}$ in Γ; (2) a set of formulas Γ is *semantically consistent* if there is a model M that satisfies Γ; (3) $\mathcal{P}$ is a *semantically valid consequence* of Γ, written as $\Gamma \models \mathcal{P}$, if every model satisfying Γ also satisfies $\mathcal{P}$.

Theorem 2.14 (Soundness) *If* $\Gamma \vdash \mathcal{P}$ *in* GS*, then* $\Gamma \models \mathcal{P}$*.*

Proof. By a straightforward induction on the length of the proof from Γ to $\mathcal{P}$. The axiom clearly holds in each model, and if the premises of an inference rule hold in a model, so does the conclusion. ■

When a set of formulas is semantically inconsistent, any formula $\mathcal{P}$ is a semantically valid consequence thereof. In general, however, there is no proof in GS of $\mathcal{P}$ from such an inconsistent set, since GS does not have an inference rule corresponding to the absurdity rule (i.e. *Ex Falso Quodlibet*) of a natural deduction system. Accordingly, for the completeness of GS, we impose a model existence condition for the set of assumptions.

Theorem 2.15 (Completeness) *Let* Γ *be a semantically consistent set of formulas of* GS*. If* $\Gamma \models \mathcal{P}$*, then* $\Gamma \vdash \mathcal{P}$ *in* GS*.*

It is obvious that the soundness theorem (Theorem 2.14) also holds under the condition of the semantic consistency of given assumptions.

The rest of this section is devoted to proving Theorem 2.15. We start with showing some useful results on the semantically consistent set of formulas in GS.

Lemma 2.16 *Let* Γ *be a semantically consistent set of formulas. Then neither of the following holds in* GS *for any term* s *and* t*:*

(1) $\Gamma \vdash s \vdash\!\dashv s$.

(2) $\Gamma \vdash s \sqsubset t$ *and* $\Gamma \vdash s \vdash\!\dashv t$.

(3) There is a term u such that $\Gamma \vdash s \vDash t$, $\Gamma \vdash u \sqsubset s$ and $\Gamma \vdash u \sqsubset t$.

Proof. For (1), assume to the contrary that $\Gamma \vdash s \vDash s$ holds. Since Γ is semantically consistent, there is a model $M = (U, I)$ which satisfies Γ. By the soundness theorem, we have $\Gamma \models s \vDash s$, so $M \models s \vDash s$, i.e., $I(s) = \emptyset$. However, $I(s) \neq \emptyset$ by Definition 2.12, hence this is a contradiction. (2) and (3) are reduced to (1) using the ($\vDash$) rule. ■

In order to show Theorem 2.15, we construct a syntactic model, called *canonical model.* In GS, an equality relation between terms can be expressed by a formula of the form $\{s \sqsubset t, t \sqsubset s\}$. So we first define an equivalence relation $\sim$ on a set of GS-terms.

Definition 2.17 Let Γ be a set of formulas in GS. We define a binary relation $\sim$ on the set of GS-terms by:

1. when s and t are general terms, $s \sim t$ if and only if s and t are the same terms, i.e., $s \equiv t$;
2. when at least one of s and t is a singular term, $s \sim t$ if and only if $\Gamma \vdash \{s \sqsubset t, t \sqsubset s\}$.

The following are immediate consequence of the definition.

Lemma 2.18 *Let Γ be a set of GS-formulas.*

1. *$\sim$ is an equivalence relation on the set of GS-terms.*
2. *If $s_1 \sim s_2$ and $t_1 \sim t_2$, then $\Gamma \vdash s_1 \Box t_1$ if and only if $\Gamma \vdash s_2 \Box t_2$ for $\Box \in \{\sqsubset, \vDash\}$.*

We will introduce several kinds of canonical models. The first one is the following.

Definition 2.19 (Canonical model M_Γ) Let Γ be a semantically consistent set of of formulas of GS. Let $\overline{s}$ be the equivalence class of s, i.e., $\overline{s} = \{t \mid s \sim t\}$. Let $M_\Gamma = (U_\Gamma, I_\Gamma)$ be defined as follows:

1. $U_\Gamma = \{\overline{s} \mid s \text{ is a term}\}$

2. For any term t, $I_\Gamma(t) = \{\overline{s} \mid \Gamma \vdash s \sqsubset t \text{ in } \mathsf{GS}\}$

Note that the well-definedness of I_Γ follows from Lemma 2.18(2).

The following lemma confirms that M_Γ satisfies the condition of GS-models in Definition 2.12.

Lemma 2.20 *Let $M_\Gamma = (U_\Gamma, I_\Gamma)$ be as described in Definition 2.19.*

(a) For every term t, $I_\Gamma(t) \neq \emptyset$.

(b) For every singular term a, $I_\Gamma(a)$ is a singleton.

Proof. (a) is immediate by the fact that $s \sqsubset s$ is an axiom in GS. For (b), suppose that $\overline{s} \in I_\Gamma(a)$ and $\overline{u} \in I_\Gamma(a)$. Then by definition $\Gamma \vdash s \sqsubset a$ and $\Gamma \vdash u \sqsubset a$, so by using the (C) rule, we have $\Gamma \vdash a \sqsubset s$ and $\Gamma \vdash a \sqsubset u$. This implies $s \sim a$ and $u \sim a$. Hence $\overline{s} = \overline{a} = \overline{u}$. ■

Lemma 2.21 *If Γ is a semantically consistent set of formulas of* GS*, then M_Γ is a model of Γ.*

Proof. Let Γ be a semantically consistent set of formulas, and suppose $\mathcal{P} \in \Gamma$. We show that $M_\Gamma \models \mathcal{P}$. We may assume that $\mathcal{P}$ is of the form $\{P_1, \ldots, P_n\}$ where $P_1, \ldots, P_n$ are atomic formulas (when $n = 1$, $\mathcal{P} \equiv P_1$). We claim that $M_\Gamma \models P_i$ for each P_i with $1 \leq i \leq n$. This implies that $M_\Gamma \models \mathcal{P}$ by definition. Note that by the assumption that $\mathcal{P} \in \Gamma$, we have $\Gamma \vdash P_i$ (when $n = 1$ this is immediate; when $n > 2$, apply the $(-)$ rule to $\mathcal{P}$). We have two cases according to whether P_i is of the form $s \sqsubset t$ or $s \vdash\!\dashv t$.

(Case 1) P_i is of the form $s \sqsubset t$. We show that $M_\Gamma \models s \sqsubset t$, i.e., $I_\Gamma(s) \subseteq I_\Gamma(t)$. Let $\overline{u} \in I_\Gamma(s)$. By the definition of $I_\Gamma(s)$ we have $\Gamma \vdash u \sqsubset s$. Since we have $\Gamma \vdash s \sqsubset t$, using the $(\sqsubset)$ rule we obtain $\Gamma \vdash u \sqsubset t$, that is, $\overline{u} \in I_\Gamma(t)$.

(Case 2) P_i is of the form $s \vdash\!\dashv t$. We show that $M_\Gamma \models s \vdash\!\dashv t$, i.e., $I_\Gamma(s) \cap I_\Gamma(t) = \emptyset$. Assume for contradiction that $\overline{u} \in I_\Gamma(s)$ and $\overline{u} \in I_\Gamma(t)$ for some $\overline{u}$. Then we have $\Gamma \vdash u \sqsubset s$ and $\Gamma \vdash u \sqsubset t$. By Lemma 2.16(3), this contradicts the assumption that $\Gamma \vdash u \vdash\!\dashv t$, as required. ■

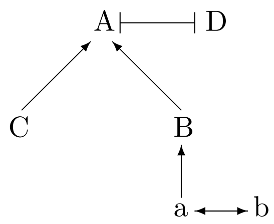
Example 2.22 (Canonical model M_Γ) Consider:

$$\Gamma = \{B \sqsubset A, C \sqsubset A, A \vdash\!\dashv D, a \sqsubset B, b \sqsubset a\}.$$

In the canonical model M_Γ, we have:

$$I_\Gamma(A) = \{\overline{A}, \overline{B}, \overline{C}, \overline{a}\}, I_\Gamma(B) = \{\overline{B}, \overline{a}\}, I_\Gamma(C) = \{\overline{C}\},$$
$$I_\Gamma(D) = \{\overline{D}\}, I_\Gamma(a) = \{\overline{a}\}, I_\Gamma(b) = \{\overline{a}\}.$$

Note that by the (C) rule we have $\Gamma \vdash \{a \sqsubset b, b \sqsubset a\}$, i.e., $a \sim b$, hence $\overline{a} = \overline{b}$. The construction of M_Γ may be illustrated by the following kind of directed graph.[2]



Here each node represents a term appearing in Γ, and two types of edges, $\rightarrow$ and $\vdash\!\dashv$, represent the relations between terms defined by: $s \rightarrow t$ if $\Gamma \vdash s \sqsubset t$, and $s \vdash\!\dashv t$ if $\Gamma \vdash s \vdash\!\dashv t$. Note that the $\rightarrow$-relation is a preorder, i.e., a reflexive and transitive relation. $I_\Gamma(t)$ corresponds to the downset $\downarrow t := \{\overline{s} \mid s \rightarrow t\}$. In this graph representation, we omit the reflexive and transitive edges for $\rightarrow$. We also leave implicit the $\vdash\!\dashv$-edges to be drawn between D and all the other terms, except that between D and A, because they are inferable.

In the canonical model M_Γ, $I_\Gamma(s) \subseteq I_\Gamma(t)$ implies $\Gamma \vdash s \sqsubset t$, but $I_\Gamma(s) \cap I_\Gamma(t) = \emptyset$ does not imply $\Gamma \vdash s \vdash\!\dashv t$. For instance, consider the canonical model in Example 2.22. Here we have $I_\Gamma(C) \cap I_\Gamma(B) = \emptyset$ but $\Gamma \nvdash B \vdash\!\dashv C$. That is to say, in the canonical model M_Γ, the semantic validity of $\vdash\!\dashv$-relation does not imply the provability of $\vdash\!\dashv$-relation, and hence such a model is not enough to establish completeness.

We say that s is indeterminate with respect to t given Γ, written as $s \bowtie_\Gamma t$, when the following holds:

$$\Gamma \nvdash s \sqsubset t \text{ and } \Gamma \nvdash t \sqsubset s \text{ and } \Gamma \nvdash s \vdash\!\dashv t.$$

[2] This kind of directed graph is introduced in Mineshima, Okada and Takemura (2009) for the purpose of the abstract representation of Euler diagrams.

The problem with the canonical model M_Γ is that we have $I_\Gamma(s) \cap I_\Gamma(t) = \emptyset$ even when $s \bowtie_\Gamma t$ holds. So let us consider a modified canonical model M'_Γ where the interpretation of any pair of terms s and t that are indeterminate has an intersection, i.e, $I'_\Gamma(s) \cap I'_\Gamma(t) \neq \emptyset$. Unfortunately, this is again not a satisfactory model. Let us consider the set of formulas Γ in Example 2.22. Here we have $C \bowtie_\Gamma a$, thus we would have some $\overline{u} \in I'_\Gamma(C) \cap I'_\Gamma(a)$. But then, since $I'_\Gamma(a)$ must be a singleton by definition 2.12, we would have $I'_\Gamma(a) = \{\overline{u}\}$, hence $I'_\Gamma(a) \subseteq I'_\Gamma(C)$, i.e., $M'_\Gamma \models a \sqsubset C$, contrary to what we want.

In order to overcome this difficulty, we define a slightly extended notion of model, which we call *quasi-model* of GS.

Definition 2.23 (Quasi-model) A quasi-model of GS is just as described in Definition 2.12 except that the condition for singular terms is deleted, i.e., for every singular term a, $I^*(a)$ may not be a singleton. The satisfaction relation $\models$ is defined for quasi-models in the same way as in Definition 2.13.

We define a *canonical quasi-model* for a set of formulas Γ.

Definition 2.24 (Canonical quasi-model M^*_Γ) Let Γ be a semantically consistent set of formulas of GS. Let $\overline{s} = \{t \mid s \sim t\}$. A canonical quasi-model $M^*_\Gamma = (U^*_\Gamma, I^*_\Gamma)$ for Γ is defined as follows:

1. $U^*_\Gamma = \{\overline{s} \mid s \text{ is a term}\} \cup \big\{\{\overline{s}, \overline{t}\} \mid s \text{ and } t \text{ are terms}\big\}$
2. For all term t, $I^*_\Gamma(t) = \{\overline{s} \mid \Gamma \vdash s \sqsubset t\} \cup \{\{\overline{s}, \overline{u}\} \mid \Gamma \vdash u \sqsubset t \text{ and } s \bowtie_\Gamma u\}$

Note that by the clause 2, $I^*_\Gamma(a)$ may not be a singleton for a singular term a.

Notation. We use $x, y, \ldots$ as a variable over elements in the domain U^*_Γ.

Example 2.25 (Canonical quasi-model) Consider again

$$\Gamma = \{B \sqsubset A, C \sqsubset A, A \vDash D, a \sqsubset B, b \sqsubset a\}.$$

In the canonical quasi-model M^*_Γ, we have:

$$\begin{aligned}
I^*_\Gamma(A) &= \left\{\overline{A}, \overline{B}, \overline{C}, \overline{a}, \left\{\overline{C}, \overline{B}\right\}, \left\{\overline{C}, \overline{a}\right\}\right\},\\
I^*_\Gamma(B) &= \left\{\overline{B}, \left\{\overline{C}, \overline{B}\right\}, \left\{\overline{C}, \overline{a}\right\}\right\},\\
I^*_\Gamma(C) &= \left\{\overline{C}, \left\{\overline{C}, \overline{B}\right\}, \left\{\overline{C}, \overline{a}\right\}\right\},\\
I^*_\Gamma(D) &= \left\{\overline{D}\right\},\\
I^*_\Gamma(a) &= \left\{\overline{a}, \left\{\overline{C}, \overline{a}\right\}\right\},\\
I^*_\Gamma(b) &= \left\{\overline{a}, \left\{\overline{C}, \overline{a}\right\}\right\}.
\end{aligned}$$

Here C is indeterminate with respect to B given Γ. Thus, we have $\left\{\overline{C}, \overline{B}\right\} \in I^*_\Gamma(s)$ for every term s such that $\Gamma \vdash C \sqsubset s$ or $\Gamma \vdash B \sqsubset s$. Similarly, $\left\{\overline{C}, \overline{a}\right\}$ is included in any $I^*_\Gamma(s)$ such that $\Gamma \vdash C \sqsubset s$ or $\Gamma \vdash a \sqsubset s$.

Lemma 2.26 *If* Γ *is a semantically consistent set of formulas of* GS*, then* M^*_Γ *is a model of* Γ*.*

Proof. As in Lemma 2.21, it is sufficient to prove the following:

1. If $\Gamma \vdash s \sqsubset t$, then $I^*_\Gamma(s) \subseteq I^*_\Gamma(t)$.
2. If $\Gamma \vdash s \mathrel{\vdash\!\dashv} t$, then $I^*_\Gamma(s) \cap I^*_\Gamma(t) \neq \emptyset$.

1. Let $x \in I^*_\Gamma(s)$. When $x \equiv \overline{u}$ for a term u, the proof is the same as Case 1 in the proof of Lemma 2.21. Otherwise, by the definition of $I_\Gamma(s)$, we have $x \equiv \{\overline{u}, \overline{v}\}$ for some u and v such that $\Gamma \vdash u \sqsubset s$ and $u \bowtie_\Gamma v$. Since $\Gamma \vdash s \sqsubset t$, we have $\Gamma \vdash u \sqsubset t$ using the ($\sqsubset$) rule. Hence by the definition of $I^*_\Gamma(t)$, we have $\{\overline{u}, \overline{v}\} \in I^*_\Gamma(t)$.

2. Assume to the contrary that there exists an x such that $x \in I^*_\Gamma(s)$ and $x \in I^*_\Gamma(t)$. When $x \equiv \overline{u}$ for some term u, the proof is the same as Case 2 of the proof of Lemma 2.26. Otherwise, we have $x \equiv \{\overline{u}, \overline{v}\}$ for some terms u and v. We may assume without loss of generality that $\Gamma \vdash u \sqsubset s$ and $u \bowtie_\Gamma v$. Since $\{\overline{u}, \overline{v}\} \in I_\Gamma(t)$, we have either $\Gamma \vdash u \sqsubset t$ or $\Gamma \vdash v \sqsubset t$. Given $\Gamma \vdash u \sqsubset s$, the former case contradicts $\Gamma \vdash s \mathrel{\vdash\!\dashv} t$ by Lemma 2.16(3). In the latter case, together with $\Gamma \vdash s \mathrel{\vdash\!\dashv} t$, we get $\Gamma \vdash s \mathrel{\vdash\!\dashv} v$ by the ($\vdash\!\dashv$) rule. Since $\Gamma \vdash u \sqsubset s$, we obtain $\Gamma \vdash u \mathrel{\vdash\!\dashv} v$ by the ($\vdash\!\dashv$) rule. This is a contradiction to $u \bowtie_\Gamma v$, as required. ■

In order to pass from a model M_Γ to a quasi-model M^*_Γ, it is desired to have: for any formula $\mathcal{P}$, if $M_\Gamma \models \mathcal{P}$, then $M^*_\Gamma \models \mathcal{P}$. Unfortunately, this claim does not generally hold. For example, when $s \bowtie_\Gamma t$ holds, we have $M_\Gamma \models s \sqcap t$ but $M^*_\Gamma \not\models s \sqcap t$, since $\{\bar{s}, \bar{t}\} \in I^*_\Gamma(s) \cap I^*_\Gamma(t)$. However, under the assumption that $\Gamma \models \mathcal{P}$, the claim holds. In order to prove it, we need the following lemma.

Lemma 2.27 *Let Γ be a semantically consistent set of formulas, and let u and t be terms in* GS. *If $\Gamma \not\vdash u \sqcap v$ holds, there is a model M in* GS *such that $M \models \Gamma$ and $M \not\models u \sqcap v$.*

Proof. We construct a canonical model $M_{u,v}$ whose domain consists of the set of terms in GS and whose interpretation function $I_{u,v}$ is defined as follows:
(I) When there is a singular term b such that $b \sim v$: for any term t,

1. $I_{u,v}(t) = \{\overline{u}\}$ if $v \sim t$.
2. $I_{u,v}(t) = I_\Gamma(t) \cup \{\overline{u}\}$ if $\Gamma \vdash v \sqsubset t$ and $\Gamma \not\vdash t \sqsubset v$.
3. $I_{u,v}(t) = I_\Gamma(t)$, otherwise.

(II) When there is no singular term b such that $b \sim v$: for any term t,

1. $I_{u,v}(t) = I_\Gamma(t) \cup \{\overline{u}\}$ if $\Gamma \vdash v \sqsubset t$.
2. $I_{u,v}(t) = I_\Gamma(t)$, otherwise.

Note that in (I) t may be a singular term or a general term, while in (II), t must be a general term.

It is easily seen that $M_{u,v}$ is a GS-model; in particular, $I_{u,v}(a)$ is a singleton for any singular term a. Note that non-singleton interpretations may arise in (I.2) and (II.1), but in both cases, t cannot be a singular term by definition.

It can also be easily verified that $M_{u,v} \not\models u \sqcap v$. In the case of (I), $I_{u,v}(v) = \{\overline{u}\}$ (by Clause I.1) and $I_{u,v}(u) = \{\overline{u}\}$ (by Clause I.3). In the case of (II), $\overline{u} \in I_{u,v}(v)$ (by Clause II.1) and $\overline{u} \in I_{u,v}(u)$ (by Clause II.2). Hence in both cases, we have $I_{u,v}(u) \cap I_{u,v}(v) \neq \emptyset$.

Now we show that $M_{u,v} \models \Gamma$. It is sufficient to prove the following:

(1) If $\Gamma \vdash s \sqsubset t$, then $M_{u,v} \models s \sqsubset t$.
(2) If $\Gamma \vdash s \sqcap t$, then $M_{u,v} \models s \sqcap t$.

For (1), suppose $\Gamma \vdash s \sqsubset t$. We show $I_{u,v}(s) \subseteq I_{u,v}(t)$.

We first consider the case of canonical model $M_{u,v}$ in (I), so assume that there is a singular term b such that $b \sim v$. We distinguish the cases according to the relation holding between v and s with respect to Γ.

(a) When $v \sim s$ holds, $I_{u,v}(s) = \{\overline{u}\}$ (by Clause I.1). Given $\Gamma \vdash s \sqsubset t$, we obtain $\Gamma \vdash v \sqsubset t$ by using the (+) rule. So we have $\overline{u} \in I_{u,v}(t)$, whether $\Gamma \vdash t \sqsubset v$ holds or not. Hence $I_{u,v}(s) \subseteq I_{u,v}(t)$.

(b) When $\Gamma \vdash v \sqsubset s$ and $\Gamma \not\vdash s \sqsubset v$, we have $I_{u,v}(s) = I_\Gamma(s) \cup \{\overline{u}\}$ (by Clause I.2), and given $\Gamma \vdash s \sqsubset t$, we have $\Gamma \vdash v \sqsubset t$. Now $\Gamma \not\vdash t \sqsubset v$; otherwise, we would obtain $\Gamma \vdash s \sqsubset v$, contradicting our assumption. So $I_{u,v}(t) = I_\Gamma(t) \cup \{\overline{u}\}$. Since $I_\Gamma(s) \subseteq I_\Gamma(t)$ by Lemma 2.21, we have $I_{u,v}(s) \subseteq I_{u,v}(s)$, as desired.

(c) Otherwise (i.e., $s \not\sim v$ and $\Gamma \not\vdash v \sqsubset s$), we have $I_{u,v}(s) = I_\Gamma(s)$. We also have $t \not\sim v$, since otherwise, we would have $\Gamma \vdash s \sqsubset v$; but then, given that there is a singular term b such that $b \sim v$, we could prove $\Gamma \vdash v \sqsubset s$, which would be a contradiction. Hence, $I_{u,v}(s) = I_\Gamma(s) \subseteq I_\Gamma(t) \subseteq I_{u,v}(t)$.

Next we consider the case of (II), so let us assume that there is no singular term b such that $b \sim v$. When $\Gamma \vdash v \sqsubset s$, the proof is the same as in (b) above. Otherwise, we have $I_{u,v}(s) = I_\Gamma(s)$. Note that $I_\Gamma(t) \subseteq I_{u,v}(t)$ by definition, and $I_\Gamma(s) \subseteq I_\Gamma(t)$ by Lemma 2.21. Hence $I_{u,v}(s) \subseteq I_{u,v}(s)$. This completes the proof of the claim (1).

For the claim (2), suppose $\Gamma \vdash s \mathrel{\vdash\!\dashv} t$. We show $M_{u,v} \models s \mathrel{\vdash\!\dashv} t$, that is, $I_{u,v}(s) \cap I_{u,v}(t) = \emptyset$.

In the case of (I), assume that there is a singular term b such that $b \sim v$. We distinguish the following cases:

(a′) When $s \sim v$ holds, $I_{u,v}(s) = \{\overline{u}\}$. Given $\Gamma \vdash s \mathrel{\vdash\!\dashv} t$, we have $\Gamma \vdash v \mathrel{\vdash\!\dashv} t$. Then by Lemma 2.16, we have $\Gamma \not\vdash v \sqsubset t$, so $I_{u,v}(t) = I_\Gamma(t)$ (by Clause I.3). Thus, if $I_{u,v}(s) \cap I_{u,v}(t) \neq \emptyset$, then $\overline{u} \in I_\Gamma(t)$, i.e., $\Gamma \vdash u \sqsubset t$. Hence we have $\Gamma \vdash u \mathrel{\vdash\!\dashv} v$, which is a contradiction to our assumption.

(b′) When $\Gamma \vdash v \sqsubset s$, we have $I_{u,v}(s) = I_\Gamma(s) \cup \{\overline{u}\}$, and also, $I_{u,v}(t) = I_\Gamma(t)$ by the same reasoning as in (a′). Suppose for contradiction that $\overline{r} \in I_{u,v}(s) \cap I_{u,v}(t)$ for some term r. Then we have $\Gamma \vdash r \sqsubset t$, and either $\Gamma \vdash r \sqsubset s$ or $\overline{r} = \overline{u}$. The former case contradicts $\Gamma \vdash s \vDash t$ by Lemma 2.16. In the latter case, we have $\overline{u} \in I_\Gamma(t)$, i.e., $\Gamma \vdash u \sqsubset t$. Then, since $\Gamma \vdash v \sqsubset s$ and $\Gamma \vdash s \vDash t$, we would get $\Gamma \vdash u \vDash v$, in contradiction to our assumption.

(c′) When $s \not\sim v$ and $\Gamma \nvdash v \sqsubset s$, we have $I_{u,v}(s) = I_\Gamma(s)$. The only case we need to consider is the case in which $t \not\sim v$ and $\Gamma \nvdash v \sqsubset t$ hold, so we may assume that $I_{u,v}(t) = I_\Gamma(t)$. But then the assertion follows from Lemma 2.26.

In the case of (II), let us assume that there is no singular term b such that $b \sim v$. It is sufficient to check the following cases: (i) $\Gamma \vdash v \sqsubset s$ and $\Gamma \vdash v \sqsubset t$; (ii) $\Gamma \vdash v \sqsubset s$ and $\Gamma \nvdash v \sqsubset t$; (iii) $\Gamma \nvdash v \sqsubset s$ and $\Gamma \nvdash v \sqsubset t$. Case (i) immediately contradicts $\Gamma \vdash s \vDash t$ by Lemma 2.16. In Case (ii), we have $I_{u,v}(s) = I_\Gamma(s) \cap \{\overline{u}\}$ and $I_{u,v}(t) = I_\Gamma(t)$. So the proof is the same as in (b′) above. In Case (iii), the proof is the same as (c′) above. This completes the proof of Lemma 2.27. ∎

Remark. When $v \sim a$ holds for some singular term a, $I(v)$ must be a singleton even when v is a general term. So in defining the canonical model $M_{u,v}$, we need to divide the cases of (I) and (II), depending on whether or not there is a singular term b such that $b \sim v$, rather than whether v is a singular term or a general term.

Lemma 2.28 *Let Γ be a semantically consistent set of formulas of* GS*, and let $\mathcal{P}$ be a formula. If $\Gamma \models \mathcal{P}$ and $M_\Gamma \models \mathcal{P}$, then $M^*_\Gamma \models \mathcal{P}$.*

Proof. It is enough to prove the claim for any atomic formula P. So we have two cases.

(Case 1) P is of the form $s \sqsubset t$. We show $M^*_\Gamma \models s \sqsubset t$, i.e., $I^*_\Gamma(s) \subseteq I^*_\Gamma(t)$. Let $x \in I^*_\Gamma(s)$.

1. When $x \equiv \overline{u}$ for a term u, by the definition of $I^*_\Gamma(s)$ we have $\Gamma \vdash u \sqsubset s$, so by the definition of $I_\Gamma(s)$ we have $\overline{u} \in I_\Gamma(s)$. By assumption we have $M_\Gamma \models s \sqsubset t$, i.e., $I_\Gamma(s) \subseteq I_\Gamma(t)$. So we have $\overline{u} \in I_\Gamma(t)$, i.e., $\Gamma \vdash u \sqsubset t$, hence by the definition of $I^*_\Gamma(t)$ we obtain $\overline{u} \in I^*_\Gamma(t)$.

2. Otherwise, by the definition of $I^*_\Gamma(s)$, we have $x \equiv \{\overline{u}, \overline{v}\}$ for terms u and v. We may assume without loss of generality that $\Gamma \vdash u \sqsubset s$ and $u \bowtie_\Gamma v$. Then $\overline{u} \in I_\Gamma(s)$, which implies $\overline{u} \in I_\Gamma(t)$ by our assumption. So $\Gamma \vdash u \sqsubset t$, hence $\{\overline{u}, \overline{v}\} \in I^*_\Gamma(t)$.

(Case 2) $\mathcal{P}$ is of the form $s \mapsfrom\!\!\mapsto t$. Note that we have $M_\Gamma \models s \mapsfrom\!\!\mapsto t$, that is, $I_\Gamma(s) \cap I_\Gamma(t) = \emptyset$. We show that $M^*_\Gamma \models s \mapsfrom\!\!\mapsto t$, i.e., $I^*_\Gamma(s) \cap I^*_\Gamma(t) = \emptyset$. Assume to the contrary that there exists some x such that $x \in I^*_\Gamma(s)$ and $x \in I^*_\Gamma(t)$.

1. When $x \equiv \overline{u}$ for a term u, by the definition of $I^*_\Gamma(s)$ and $I^*_\Gamma(t)$ we have $\Gamma \vdash u \sqsubset s$ and $\Gamma \vdash u \sqsubset t$. So $\overline{u} \in I_\Gamma(s)$ and $\overline{u} \in I_\Gamma(t)$, that is, $I_\Gamma(s) \cap I_\Gamma(t) \neq \emptyset$, which is a contradiction.

2. Otherwise, we have $x \equiv \{\overline{u}, \overline{v}\}$ for some terms u and v. There are four possible cases, of which the following two are representative:
(i) When $\Gamma \vdash u \sqsubset s$ and $\Gamma \vdash u \sqsubset t$, we have $\overline{u} \in I_\Gamma(s)$ and $\overline{u} \in I_\Gamma(t)$, which contradicts $I_\Gamma(s) \cap I_\Gamma(t) = \emptyset$.
(ii) When $\Gamma \vdash u \sqsubset s$ and $\Gamma \vdash v \sqsubset t$, by soundness we have $\Gamma \models u \sqsubset s$ and $\Gamma \models v \sqsubset t$. By assumption $\Gamma \models s \mapsfrom\!\!\mapsto t$, hence $\Gamma \models u \mapsfrom\!\!\mapsto v$. Since we have $u \bowtie_\Gamma v$, by Lemma 2.27 we can construct a canonical model $M_{u,v}$ such that $M_{u,v} \models \Gamma$ and $M_{u,v} \not\models u \mapsfrom\!\!\mapsto v$. Hence we have $M_{u,v} \models u \mapsfrom\!\!\mapsto v$ as well, which is a contradiction. ■

Finally, we proceed to prove the completeness of GS.

Proof of Theorem 2.15. Suppose $\Gamma \models \mathcal{P}$. Then using Lemma 2.21 we have $M_\Gamma \models \mathcal{P}$, and so by Lemma 2.28 we have $M^*_\Gamma \models \mathcal{P}$. We may assume that $\mathcal{P}$ has the form $\{P_1, \ldots, P_n\}$ where $P_1, \ldots, P_n$ are atomic formulas. Thus we have $M^*_\Gamma \models P_i$ for each P_i $(1 \leq i \leq n)$. We claim that $\Gamma \vdash P_i$. It then follows that $\Gamma \vdash \{P_1, \ldots, P_n\}$ by repeated applications of the (+) rule, which completes the proof of Theorem 2.15.

(Case 1) When P_i is of the form $s \sqsubset t$, we already have $I^*_\Gamma(s) \subseteq I^*_\Gamma(t)$. Since $\Gamma \vdash s \sqsubset s$ by Axiom (ax), we have $s \in I^*_\Gamma(s)$. Since s is a term, this implies that $\Gamma \vdash s \sqsubset t$.

(Case 2) When $\mathcal{P}_i$ is of the form $s \vDash t$, we have $I^*_\Gamma(s) \cap I^*_\Gamma(t) = \emptyset$. Assume for contradiction that $\Gamma \nvdash s \vDash t$. We claim that $\Gamma \nvdash s \sqsubset t$. If $\Gamma \vdash s \sqsubset t$, we have $\Gamma \models s \sqsubset t$ by soundness, so $M_\Gamma \models s \sqsubset t$ by Lemma 2.21. It then follows that $M^*_\Gamma \models s \sqsubset t$ by Lemma 2.28, hence $I^*_\Gamma(s) \subseteq I^*_\Gamma(t)$. But this is a contradiction to $I^*_\Gamma(s) \cap I^*_\Gamma(t) = \emptyset$, since $I^*_\Gamma(s) \neq \emptyset$ under our semantics. In the same way, we have $\Gamma \nvdash t \sqsubset s$, thus $s \bowtie_\Gamma t$. We also have $\Gamma \vdash s \sqsubset s$ by Axiom (ax). Hence by the definition of $I^*_\Gamma(s)$, we have $\{\overline{s}, \overline{t}\} \in I^*_\Gamma(s)$. By the same reasoning, we have $\{\overline{s}, \overline{t}\} \in I^*_\Gamma(t)$, hence $I^*_\Gamma(s) \cap I^*_\Gamma(t) \neq \emptyset$. This is a contradiction, as required. ■

3. GS and categorical syllogisms

In this section, we show that categorical syllogisms correspond to a fragment of GS using proof-theoretic methods. We first consider a system of categorical syllogisms without existential import (CS), and then consider the case of a system with existential import (CS^+).

3.1 An inference system CS for categorical syllogism

We introduce a simple formulation of an inference system of categorical syllogisms, called CS. First we present the language of CS. The terms of CS are only general terms, denoted by $A, B, C, \ldots$ as in GS. Categorical sentences in CS are defined as follows.

Definition 3.1 (Categorical sentences) A categorical sentence has one of the following forms: All A are B, No A are B, Some A are B, or Some A are not B, where A and B are distinct general terms. We call All A are B and No A are B *universal* sentences, and Some A are B and Some A are not B *existential* sentences. We use variables $S, T, R, \ldots$ to denote categorical sentences.

A categorical syllogism (or simply a *syllogism* for short) is composed of two premises and one conclusion. We write it in tree form, for example,

$$\frac{\textsf{All } B \textsf{ are } C \quad \textsf{Some } A \textsf{ are } B}{\textsf{Some } A \textsf{ are } C}\ .$$

The validity of some patterns of syllogisms depends on the so-called *existential import* of the subject term of universal sentences, which allows the derivation of Some A are B from All A are B. For simplicity, we first exclude such patterns and only consider categorical syllogisms that are valid without existential import. In Section 3.3, we will introduce a system CS^+ of categorical syllogisms with existential imports. All valid patterns of categorical syllogisms are listed in Table 3.1 below, where $\vdash^*$ indicates the patterns whose validity depends on the existential import of a subject term A. These patterns are provable in CS^+.

As is well known, in *Prior Analytics*, Aristotle showed that all valid syllogisms are reduced to syllogisms of the first figure, which he calls *perfect* syllogisms (see Parsons 2008 for a detailed historical overview of Aristotle's syllogistic). They consists of four patterns: Barbara, Celarent, Darii, and Ferio under the traditional mnemonic names. In modern formulations of categorical syllogisms, such as Łukasiewicz's (1957) axiomatic system or various natural deduction systems (cf. Corcoran 1972, 1974; Smiley 1974; Martin 1997), the syllogisms of the first figure are taken as axioms or inference rules; all other valid syllogisms are then derivied using classical logical rules, including the rule of *Reductio ad absurdum.* Here instead, we regard two additional valid patterns, i.e., Baroco and Bocardo, as primitive inference rules of CS. We also need conversion rules for Some A are B and No A are B.

Definition 3.2 (Inference rules) The inference rules of CS are the following.

$$\frac{\text{All } A \text{ are } B \quad \text{All } B \text{ are } C}{\text{All } A \text{ are } C}\ \text{Barbara} \qquad \frac{\text{Some } A \text{ are not } B \quad \text{All } C \text{ are } B}{\text{Some } A \text{ are not } C}\ \text{Baroco}$$

$$\frac{\text{All } A \text{ are } B \quad \text{No } B \text{ are } C}{\text{No } A \text{ are } C}\ \text{Celarent} \qquad \frac{\text{All } B \text{ are } A \quad \text{Some } B \text{ are not } C}{\text{Some } A \text{ are not } C}\ \text{Bocardo}$$

$$\frac{\text{Some } A \text{ are } B \quad \text{All } B \text{ are } C}{\text{Some } A \text{ are } C}\ \text{Darii} \qquad \frac{\text{Some } B \text{ are } A}{\text{Some } A \text{ are } B}\ \text{conv1}$$

$$\frac{\text{Some } A \text{ are } B \quad \text{No } B \text{ are } C}{\text{Some } A \text{ are not } C}\ \text{Ferio} \qquad \frac{\text{No } B \text{ are } A}{\text{No } A \text{ are } B}\ \text{conv2}$$

It is easy to check that all other valid patterns of categorical syllogisms (with no existential import) are derived from these rules.

We consider not only a single syllogism but also a *chain* of syllogisms with multiple premises (i.e., what is traditionally called *sorites*), where the conclusion of a syllogism can be a premise for the subsequent one, forming a sequence of syllogisms. Accordingly, a proof of CS is given in an analogous way to that of GS: it is a tree beginning with categorical sentences and proceeding by one of the inference rules in Definition 3.2.

Definition 3.3 (Proof) Proofs of CS are defined inductively as follows.

1. A categorical sentence S is a proof from premise S to conclusion S.

2. Let π_1 be a proof of S from $S_1, \ldots, S_n$, and let π_2 be a proof of T from $T_1, \ldots, T_n$. If a categorical sentence R is derived by an application of Barbara, Celarent, Darii, Ferio, Baroco, or Bocardo to S and T, then the following tree is a proof of R from $S_1, \ldots, S_n, T_1, \ldots, T_n$.

$$\dfrac{\begin{array}{c} S_1 \quad \cdots \quad S_n \\ \vdots\ \pi_1 \\ S \end{array} \qquad \begin{array}{c} T_1 \quad \cdots \quad T_m \\ \vdots\ \pi_2 \\ T \end{array}}{R}$$

3. Let π be a proof of S from $S_1, \ldots, S_n$. If a categorical sentence T is derived by an application of conv1 or conv2 to S, then the following tree is a proof of T from $S_1, \ldots, S_n$.

$$\dfrac{\begin{array}{c} S_1 \quad \cdots \quad S_n \\ \vdots\ \pi \\ S \end{array}}{T}$$

Definition 3.4 (Provability) A categorical sentence S is provable from $S_1, \ldots, S_n$, written as $S_1, \ldots, S_n \vdash S$, if there is a proof of S in CS from $S_1, \ldots, S_k$ with $1 \leq k \leq n$

Note that by this definition of provability, premises $S_1, \ldots, S_n$ may contain a superfluous one, as in GS.

Figure	Mnemonic	Form
I	Barbara	All B are C, All A are B $\vdash$ All A are C
	——	All B are C, All A are B $\vdash^*$ Some A are C
	Celarent	No B are C, All A are B $\vdash$ No A are C
	——	No B are C, All A are B $\vdash^*$ Some A are not C
	Darii	All B are C, Some A are B $\vdash$ Some A are C
	Ferio	No B are C, Some A are B $\vdash$ Some A are not C
II	Cesare	No C are B, All A are B $\vdash$ No A are C
	——	No C are B, All A are B $\vdash^*$ Some A are not C
	Camestres	All C are B, No A are B $\vdash$ No A are C
	——	All C are B, No A are B $\vdash^*$ Some A are not C
	Festino	No C are B, Some A are B $\vdash$ Some A are not C
	Baroco	All C are B, Some A are not B $\vdash$ Some A are not C
III	Darapti	All B are C, All B are A $\vdash^*$ Some A are C
	Felapton	No B are C, All B are A $\vdash^*$ Some A are not C
	Disamis	Some B are C, All B are A $\vdash$ Some A are C
	Datisi	All B are C, Some B are A $\vdash$ Some A are C
	Bocardo	Some B are not C, All B are A $\vdash$ Some A are not C
	Ferison	No B are C, Some B are A $\vdash$ Some A are not C
IV	Bramantip	All C are B, All B are A $\vdash^*$ Some A are C
	Camenes	All C are B, No B are A $\vdash$ No A are C
	——	All C are B, No B are A $\vdash^*$ Some A are not C
	Dimaris	Some C are B, All B are A $\vdash$ Some A are C
	Fesapo	No C are B, All B are A $\vdash^*$ Some A are not C
	Fresison	No C are B, Some B are A $\vdash$ Some A are not C

Table 3.1 List of all valid patterns of categorical syllogisms. The relation $\vdash^*$ indicates the patterns whose validity depends on existential import.

Example 3.5 The proof

$$\cfrac{\cfrac{\cfrac{\text{Some } A \text{ are } B \quad \text{All } B \text{ are } C}{\text{Some } A \text{ are } C}\,\text{Darii} \quad \text{No } C \text{ are } D}{\text{Some } A \text{ are not } D}\,\text{Ferio} \quad \text{All } E \text{ are } D}{\text{Some } A \text{ are not } E}\,\text{Baroco}$$

establishes that in CS we have:

$$\text{Some } A \text{ are } B, \text{All } B \text{ are } C, \text{No } C \text{ are } D, \text{All } E \text{ are } D \vdash \text{Some } A \text{ are not } E.$$

3.2 The relation between GS and CS

We present a sound and faithful embedding of CS into GS. We provide a translation procedure between proofs in CS and those in GS. In this way, we establish the embedding in a purely syntactic way, i.e., without appealing to semantic notions.

Theorem 3.6 (Soundness) *Every proof in* CS *of S from $S_1, \ldots, S_n$ can be translated into a proof in* GS *of S° from $S_1^\circ, \ldots, S_n^\circ$, where the translation $(\cdot)^\circ$ from a categorical sentence in* CS *into a formula in* GS *is defined by:*

$$\begin{aligned}
(\text{All } A \text{ are } B)^\circ &= A \sqsubset B, \\
(\text{No } A \text{ are } B)^\circ &= A \mapsto B, \\
(\text{Some } A \text{ are } B)^\circ &= \{c \sqsubset A, c \sqsubset B\}, \\
(\text{Some } A \text{ are not } B)^\circ &= \{d \sqsubset A, d \mapsto B\}.
\end{aligned}$$

Here we impose the following restrictions: (i) every premise S_i° $(1 \leq i \leq n)$ is assigned a different singular term, and (ii) any singular term appearing in the conclusion S° also appears in one of the premises $S_1^\circ, \ldots S_n^\circ$.

Proof. By induction on the length of a proof in CS. The base case is immediate: if S is a proof in CS, so is S° in GS. For the induction step, it suffices to show that each inference rule of CS is translated into a combination of inference rules of GS.

1. $\cfrac{\text{All } A \text{ are } B \quad \text{All } B \text{ are } C}{\text{All } A \text{ are } C}\,\text{Barbara}$ is translated into $\cfrac{A \sqsubset B \quad B \sqsubset C}{A \sqsubset C}\,(\sqsubset)$

2. $\dfrac{\text{All } A \text{ are } B \quad \text{No } B \text{ are } C}{\text{No } A \text{ are } C}$ Celarent is translated into $\dfrac{A \sqsubset B \quad B \mathrel{\vdash\!\!\dashv} C}{A \mathrel{\vdash\!\!\dashv} C}\,(\mathrel{\vdash\!\!\dashv})$

3. $\dfrac{\text{Some } C \text{ are } B \quad \text{All } B \text{ are } A}{\text{Some } C \text{ are } A}$ Darii and $\dfrac{\text{All } B \text{ are } A \quad \text{Some } B \text{ are not } C}{\text{Some } A \text{ are not } C}$ Bocardo

are translated into
$$\dfrac{\dfrac{a \sqsubset B, a \Box C}{a \Box C}\,(-) \qquad \dfrac{\dfrac{a \sqsubset B, a \Box C}{a \sqsubset B}\,(-) \quad B \sqsubset A}{a \sqsubset A}\,(\sqsubset)}{a \Box C, a \sqsubset A}\,(+)$$

where $\Box$ is $\sqsubset$ in Darii and $\mathrel{\vdash\!\!\dashv}$ in Bocardo.

4. $\dfrac{\text{Some } A \text{ are } B \quad \text{No } B \text{ are } C}{\text{Some } A \text{ are not } C}$ Ferio and $\dfrac{\text{Some } A \text{ are not } B \quad \text{All } C \text{ are } B}{\text{Some } A \text{ are not } C}$ Baroco

are translated into
$$\dfrac{\dfrac{a \sqsubset A, a \Box_1 B}{a \sqsubset A}\,(-) \qquad \dfrac{\dfrac{a \sqsubset A, a \Box_1 B}{a \Box_1 B}\,(-) \quad C \Box_2 B}{a \mathrel{\vdash\!\!\dashv} C}}{a \sqsubset A, a \mathrel{\vdash\!\!\dashv} C}\,(+)$$

where $\Box_1$ is $\sqsubset$ in Ferio and $\mathrel{\vdash\!\!\dashv}$ in Baroco, and $\Box_2$ is $\mathrel{\vdash\!\!\dashv}$ in Ferio and $\sqsubset$ in Baroco.

5. Since the order of terms in Some A are B and No A are B is irrelevant in GS, conv1 and conv2 are collapsed into a single formula:

$\dfrac{\text{Some } B \text{ are } A}{\text{Some } A \text{ are } B}$ conv1 is translated into $\{a \sqsubset A, a \sqsubset B\}$.

$\dfrac{\text{No } B \text{ are } A}{\text{No } A \text{ are } B}$ conv2 is translated into $A \mathrel{\vdash\!\!\dashv} B$. ∎

Example 3.7 A proof in CS

$$\dfrac{\text{No } C \text{ are } D \qquad \dfrac{\text{Some } B \text{ are } C \quad \text{All } B \text{ are } A}{\text{Some } A \text{ are } C}\,\text{Disamis}}{\text{Some } A \text{ are not } D}\,\text{Ferio}$$

is translated in GS as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \sqsubset B, a \sqsubset C}{a \sqsubset B} \quad B \sqsubset A}{a \sqsubset A} \quad \dfrac{a \sqsubset B, a \sqsubset C}{a \sqsubset C}}{a \sqsubset A, a \sqsubset C}\,(+)}{a \sqsubset A}\,(-) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \sqsubset B, a \sqsubset C}{a \sqsubset B} \quad B \sqsubset A}{a \sqsubset A} \quad \dfrac{a \sqsubset B, a \sqsubset C}{a \sqsubset C}}{a \sqsubset A, a \sqsubset C}\,(+)}{a \sqsubset C}\,(-) \quad C \mathrel{\vdash\!\!\dashv} D}{a \mathrel{\vdash\!\!\dashv} D}}{a \sqsubset A, a \mathrel{\vdash\!\!\dashv} D}$$

Note that this proof is not in normal form. As indicated by (+) and (−), two cut formulas appear in the proof.

When we translate a categorical sentence into a formula in GS, we need to assign different singular terms to different existential sentences. However, we do not need to be concerned about such a clash of singular terms when we consider valid syllogisms in CS, since as we show in Lemma 3.9, at most one existential sentence can appear in the premises of valid syllogisms in CS.

As noted in Section 1, a remarkable feature of GS is that it is *logic free* in that only atomic formulas (i.e., a formula of the form $s \sqsubset t$ or $s \mapsto t$) and their conjunction appear in a proof. Essential steps in a GS-proof consist of applications of ($\sqsubset$) and ($\mapsto$), which do nothing more than inferring an atomic formula from other atomic formulas in given premises. In this respect, our treatment of categorical syllogisms differs from other approaches in the literature. Thus, in the seminal work of Łukasiewicz (1957), categorical syllogisms are reconstructed using the axioms of the full (classical) propositional logic. Also in the work of Corcoran (1972, 1974) and Smiley (1973), where categorical syllogisms are formalized in natural deduction systems, rather than in the Frege-Hilbert style axiomatization, negation and the inference rule of *reductio ad absurdum* play an essential role. In recent developments of natural logic, Westerståhl (1989) and Moss (2008) also rely on the axioms of propositional logic in their formalization of inferences in the syllogistic fragment of natural language.

The rest of this subsection is devoted to proving the converse direction of Theorem 3.6, that is, the *faithfulness* of the translation $(\cdot)^\circ$ from CS to GS. For this purpose, we introduce the notion of syllogistic formulas in GS.

Definition 3.8 (Syllogistic formula) We call formulas in GS of the form $A \sqsubset B$ or $A \mapsto B$ *universal*, and formulas of the form $\{c \sqsubset A, c \sqsubset B\}$ or $\{c \sqsubset A, c \mapsto B\}$ *existential*. They are collectively called *syllogistic* formulas in GS. We also say that they are in the syllogistic fragment of GS.

We show some useful results on the normal proofs in the syllogistic fragment of GS.

Lemma 3.9 Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be syllogistic formulas in GS such that the singular terms appearing in the $\mathcal{P}_i$ $(1 \leq i \leq n)$ differ from each other. Let π be a normal proof in GS of a syllogistic formula $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$.

1. If the conclusion $\mathcal{P}$ is universal, then the assumptions $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are also universal. More specifically:
   (i) If $\mathcal{P}$ is an $\sqsubset$-formula, then all the assumptions are $\sqsubset$-formulas.
   (ii) If $\mathcal{P}$ is an $\vdash\!\dashv$-formula, then one of the assumptions is an $\vdash\!\dashv$-formula and the other assumptions are all $\sqsubset$-formulas.
2. If the conclusion $\mathcal{P}$ is existential, then one of the assumptions is existential and the other assumptions are all universal.

Proof. Claims 1(i) and 1(ii) are immediate from Corollary 2.10 by inspection of the $(\sqsubset)$ and $(\vdash\!\dashv)$ rules. Note that a formula of the form $A \sqsubset a$, where a is a singular term, does not appear in proofs of the syllogistic fragment of GS. For Claim 2, it is easily verified that in each inference rule I of GS, every term occurring in the conclusion of an application of I also occurs in some premise of that application of I, and hence occurs in some assumption of the proof as well. Thus, if the conclusion $\mathcal{P}$ of π contains a singular term, say c, then so does one of the assumptions (which must be an existential formula). Now suppose for contradiction that the assumptions have another existential formula, which contains a singular term different from c, say d. (We may assume that it has the form $\{d \sqsubset A, d \,\Box\, B\}$ where $\Box$ is $\sqsubset$ or $\vdash\!\dashv$.) Since d does not appear in the conclusion $\mathcal{P}$, there should be a formula of the form $C \sqsubset d$ in π which is a premise of $(\sqsubset)$ or $(\vdash\!\dashv)$. But as noted above, this is impossible in the syllogistic fragment of GS. ■

The set of general terms appearing in the premises and conclusion of a proof in CS has a specific form. For example, we have Some A_1 are A_2, No A_2 are A_3 $\not\vdash$ Some A_2 are not A_3 in CS. Here, term A_2 shared by the two premises is not eliminated in the conclusion. By contrast, we have (Some A_1 are $A_2)^\circ$, (No A_2 are $A_3)^\circ$ $\vdash$ (Some A_2 are not $A_3)^\circ$ in GS with a suitable choice of singular terms for existential sentences. To avoid such a discrepancy, we appeal to the notion of cyclicity.

Definition 3.10 (Cyclicity) Let $\mathcal{P}, \mathcal{P}_1, \dots, \mathcal{P}_n$ be syllogistic formulas in GS. We denote by $gtm(\mathcal{P})$ a set of general terms appearing in $\mathcal{P}$. We say that a sequence $\mathcal{P}_1, \dots, \mathcal{P}_n, \mathcal{P}$ of syllogistic formulas is a *cycle* if $gtm(\mathcal{P}_1) = \{A_1, A_2\}, \dots, gtm(\mathcal{P}_n) = \{A_n, A_{n+1}\}, gtm(\mathcal{P}) = \{A_{n+1}, A_1\}$, where $A_i \not\equiv A_j$ for all $1 \leq i, j \leq n+1$.

If a sequence $\sigma = \mathcal{P}_1, \dots, \mathcal{P}_n, \mathcal{P}$ is a cycle, σ contains $n+1$ general terms, and each general term has exactly two occurrences in σ.

The normalization procedure in GS preserves the cyclicity of the assumptions and conclusion of a proof. That is, we have:

Lemma 3.11 *Let $\mathcal{P}, \mathcal{P}_1, \dots, \mathcal{P}_n$ be syllogistic formula in* GS*. Let π be a proof in* GS *of $\mathcal{P}$ from $\mathcal{P}_1, \dots, \mathcal{P}_n$ such that the sequence $\mathcal{P}_1, \dots, \mathcal{P}_n, \mathcal{P}$ is a cycle. If π' is normal proof of π, then π' has the same assumptions as π.*

Proof. Suppose for contradiction that there is a formula $\mathcal{P}_i$ that is among the assumptions $\mathcal{P}_1, \dots, \mathcal{P}_n$ of π but not among the assumptions of π'. Let $gtm(\mathcal{P}_i) = \{A_i, A_j\}$. We first observe that neither A_i nor A_j appears in the conclusion $\mathcal{P}$ of π'. Otherwise, A_i [resp. A_j] would appear in the conclusion but not in any assumption of π'; this is clearly impossible since no inference rules in GS introduce a new term in their conclusion. Accordingly, we may assume, without loss of generality, that some formulas $\mathcal{P}_k$ with $A_i \in gtm(\mathcal{P}_k)$ and $\mathcal{P}_m$ with $A_j \in gtm(\mathcal{P}_m)$ are among the assumptions of π', and no other assumptions in π' contain A_i nor A_j. By Lemma 3.9 at least one of $\mathcal{P}_k$ and $\mathcal{P}_m$ must be a universal formula. Suppose, without loss of generality, that $\mathcal{P}_k$ is a universal formula. Then, since A_i does not appear in the conclusion of π', A_i must be eliminated in π' as a middle term in an application of ($\sqsubset$) or ($\vdash\!\dashv$). But then, there must be another universal formula containing A_i among the assumptions of π'. This is a contradiction, as required. ■

Now, we explain by examples how to translate a proof in GS into a proof in CS. Let π be a normal proof in GS from $S_1^\circ, \dots, S_n^\circ$ to S° such that the sequence $S_1^\circ, \dots, S_n^\circ, S^\circ$ is a cycle and all singular terms appearing in S_i° $(1 \leq i \leq n)$ are different from each other. If the conclusion S° is a universal formula, i.e., a formula of the form $A \sqsubset B$ or $A \vdash\!\dashv B$, by Lemma 3.9(1) the assumptions $S_1^\circ, \ldots, S_n^\circ$ are all universal formulas, and π is composed only of applications of $(\sqsubset)$ and $(\vdash\!\dashv)$. Thus, it is immediate to translate π into a proof in CS. For example,

$$\dfrac{A_5 \sqsubset A_4 \qquad \dfrac{\dfrac{A_1 \sqsubset A_2 \quad A_2 \sqsubset A_3}{A_1 \sqsubset A_3}\,(\sqsubset) \qquad A_3 \vdash\!\dashv A_4}{A_1 \vdash\!\dashv A_4}\,(\vdash\!\dashv)}{A_1 \vdash\!\dashv A_5}\,(\vdash\!\dashv)$$

is translated into:

$$\dfrac{\textsf{All } A_5 \textsf{ are } A_4 \qquad \dfrac{\dfrac{\dfrac{\textsf{All } A_1 \textsf{ are } A_2 \quad \textsf{All } A_2 \textsf{ are } A_3}{\textsf{All } A_1 \textsf{ are } A_3}\,\textsf{Barbara} \qquad \textsf{No } A_3 \textsf{ are } A_4}{\textsf{No } A_1 \textsf{ are } A_4}\,\textsf{Celarent}}{\textsf{No } A_4 \textsf{ are } A_1}\,\textsf{conv2}}{\textsf{No } A_5 \textsf{ are } A_1}\,\textsf{Celarent}$$

More problematic is the case in which the conclusion S° is an existential formula, namely, a formula of the form $\{c \sqsubset A_1, c \,\Box\, A_2\}$, where $\Box$ is $\sqsubset$ or $\vdash\!\dashv$. In this case, translating π into a CS-proof is not trivial since there are applications of $(+)$ and $(-)$ in π, and accordingly, there may be applications of the $(\sqsubset)$ or $(\vdash\!\dashv)$ rule to *non*-syllogistic formulas of the form $c \sqsubset B$ or $c \vdash\!\dashv B$ in π.

Example 3.12 Consider:

$$\dfrac{\dfrac{\dfrac{c \sqsubset A_2, c \sqsubset A_3}{c \sqsubset A_2}\,(-) \qquad A_2 \sqsubset A_1}{c \sqsubset A_1}\,(\sqsubset) \qquad \dfrac{\dfrac{\dfrac{c \sqsubset A_2, c \sqsubset A_3}{c \sqsubset A_3}\,(-) \qquad A_3 \vdash\!\dashv A_4}{c \vdash\!\dashv A_4}\,(\vdash\!\dashv) \qquad A_5 \sqsubset A_4}{c \vdash\!\dashv A_5}\,(\vdash\!\dashv)}{c \sqsubset A_1, c \vdash\!\dashv A_5}\,(+)$$

This proof involves applications of $(\sqsubset)$ and $(\vdash\!\dashv)$ to *non*-syllogistic formulas, although all the premises and the conclusion are syllogistic.

To translate this kind of proofs into one in CS, we introduce derived rules whose premises and conclusion are restricted to syllogistic formulas.

Lemma 3.13 (Derived rules) The following are derived rules in GS.

$$\frac{a \sqsubset A, a \sqsubset B \quad B \sqsubset C}{a \sqsubset A, a \sqsubset C}\ (\sqsubset_1) \qquad \frac{a \vDash A, a \sqsubset B \quad B \sqsubset C}{a \vDash A, a \sqsubset C}\ (\sqsubset_2)$$

$$\frac{a \sqsubset A, a \sqsubset B \quad B \vDash C}{a \sqsubset A, a \vDash C}\ (\vDash_1) \qquad \frac{a \sqsubset A, a \vDash B \quad C \sqsubset B}{a \sqsubset A, a \vDash C}\ (\vDash_2)$$

Proof. Note that each rule corresponds to a categorical syllogism: $(\sqsubset_1)$ to Darii, $(\sqsubset_2)$ to Bocardo, $(\vDash_1)$ to Ferio, and $(\vDash_2)$ to Baroco. So each can be justified as shown in the proof of Theorem 3.6. ∎

In each derived rule, we call a formula of the form $a \sqsubset A$ or $a \vDash A$, which appears both in a premise and the conclusion a *minor* premise of that rule.

Using the derived rules, the above example can be transformed into

$$\dfrac{\dfrac{\dfrac{a \sqsubset A_2, c \sqsubset A_3 \quad A_2 \sqsubset A_1}{c \sqsubset A_1, c \sqsubset A_3}\ (\sqsubset_1) \quad A_3 \vDash A_4}{c \sqsubset A_1, c \vDash A_4}\ (\vDash_1) \quad A_5 \sqsubset A_4}{c \sqsubset A_1, c \vDash A_5}\ (\vDash_2)$$

where only syllogistic formulas appear in each step. Note that since a sequence P_1, P_2 means the set $\{P_1, P_2\}$, the order of formulas in a sequence is immaterial in applying the derived rules.

Notation. In the sequel, by $s \,\square\, t$ we denote an atomic formula containing s and t, i.e., $s \sqsubset t$, $t \sqsubset s$, or $s \vDash t$. Note that different occurrences of $\square$ in a proof tree may denote different relations.

We say that a proof in GS is *syllogistic* if it begins with syllogistic formulas and proceeds by the rules $(\sqsubset)$, $(\vDash)$, $(\sqsubset_1)$, $(\sqsubset_2)$, $(\vDash_1)$, and $(\vDash_2)$. We prove the following lemma.

Lemma 3.14 Let $\mathcal{P}, \mathcal{P}_1, \ldots, \mathcal{P}_n$ be syllogistic formulas. Let π be a GS-proof of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$, such that (i) the sequence $\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{P}$ is a cycle, and (ii) the singular terms appearing in $\mathcal{P}_i$ $(1 \leq i \leq n)$ are different from each other. Then π can be transformed into a syllogistic proof of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$.

Proof. By Theorem 2.8, π can be transformed into a normal proof π'. By Lemma 3.11, π' has the same assumptions, $\mathcal{P}_1, \ldots, \mathcal{P}_n$, as π. We show that π' can be transformed into a syllogistic proof of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$. We divide the cases according to the form of $\mathcal{P}$.

If $\mathcal{P}$ is a universal formula, by Lemma 3.9(1), $\mathcal{P}_1, \ldots, \mathcal{P}_n$ are universal formulas as well, and π' consists only of applications of $(\sqsubset)$ and $(\vdash\!\dashv)$, and hence π' is already syllogistic.

If $\mathcal{P}$ is an existential formula, by Definition 3.8, $\mathcal{P}$ has the form $c \sqsubset A, c \Box B$ (where $A \not\equiv B$). By Corollary 2.10, there are paths $\alpha = \mathcal{P}_i, \mathcal{Q}_1, \ldots, \mathcal{Q}_k, \mathcal{P}$ and $\beta = \mathcal{P}_j, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{P}$ such that $\mathcal{P}_i$ and $\mathcal{P}_j$ are existential formulas, $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ (resp. $\mathcal{R}_1, \ldots, \mathcal{R}_m$) is the transitive part of α (resp. β), and the singular term c occurs in each formula in α and β. We see that $\mathcal{P}_i \equiv \mathcal{P}_j$. Otherwise, the assumptions contain two existential formulas, which is a contradiction to Lemma 3.9(2). Moreover, since a formula of the form $A \sqsubset c$ does not occur in a proof in the syllogistic fragment of GS, each formula in the transitive part is of the form $c \sqsubset A_l$ with $1 \leq l \leq k, A_k \equiv A$ (resp. $c \Box B_l$ with $1 \leq l \leq m, B_m \equiv B$).

It can also be seen that $\mathcal{Q}_1 \not\equiv \mathcal{R}_1$. Suppose for contradiction, that $\mathcal{Q}_1 \equiv \mathcal{R}_1 \equiv c \sqsubset A_1$ so that $c \sqsubset B_1$ occurs only in the assumption and not in the transitive part of any path in π'. By cyclicity there would be another assumption containing B_1, and no other assumptions can contain B_1. By Lemma 3.9(2), the assumption containing B_1 would be a universal formula. Thus it must be a premise of $(\sqsubset)$ or $(\vdash\!\dashv)$, and hence there must be another premise containing B_1 so as to eliminate it as a middle term. This is a contradiction.

Thus, π' has the following form:

$$\dfrac{\dfrac{c \sqsubset A_1, c \Box B_1}{c \sqsubset A_1}\,(-) \qquad \begin{matrix}\vdots\ \pi_1^1 \\ A_1 \sqsubset A_2\end{matrix}}{c \sqsubset A_2}\,(\sqsubset) \qquad \dfrac{\dfrac{c \sqsubset A_1, c \Box B_1}{c \Box B_1}\,(-) \qquad \begin{matrix}\vdots\ \pi_1^2 \\ B_1 \Box B_2\end{matrix}}{c \Box B_2}\,I_1$$

$$\dfrac{\dfrac{\begin{matrix}\ddots \\ c \sqsubset A_{k-1}\end{matrix} \qquad \begin{matrix}\vdots\ \pi_{k-1}^1 \\ A_{k-1} \sqsubset A\end{matrix}}{c \sqsubset A}\,(\sqsubset) \qquad \dfrac{\begin{matrix}\ddots \\ c \Box B_{m-1}\end{matrix} \qquad \begin{matrix}\vdots\ \pi_{m-1}^2 \\ B_{m-1} \Box B\end{matrix}}{c \Box B}\,I_{m-1}}{c \sqsubset A, c \Box B}\,(+)$$

Note that $\pi_1^1, \ldots, \pi_{k-1}^1$ and $\pi_1^2, \ldots, \pi_{m-1}^2$ are already syllogistic, since their end formulas are all universal. If $k = m = 1$, π' consists of a single formula $\mathcal{P}$, and our assertion is trivial. So we can assume that $k \geq 2$ or $m \geq 2$.

We transform the proof π' into a syllogistic form. The transformation consists of permuting applications of the $(-)$ rule, and thereby removing all applications of $(\sqsubset)$ and $(\mathrel{\vdash\!\dashv})$ whose premise and conclusion are non-syllogistic formulas. For this purpose, we introduce the following transformation rules.

$$\dfrac{\dfrac{a \sqsubset A, a \sqsubset B}{a \sqsubset B}(-) \quad B \sqsubset C}{a \sqsubset C}(\sqsubset) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, a \sqsubset B \quad B \sqsubset C}{a \sqsubset A, a \sqsubset C}(\sqsubset_1)}{a \sqsubset C}(-)$$

$$\dfrac{\dfrac{a \mathrel{\vdash\!\dashv} A, a \sqsubset B}{a \sqsubset B}(-) \quad B \sqsubset C}{a \sqsubset C}(\sqsubset) \quad \rhd \quad \dfrac{\dfrac{a \mathrel{\vdash\!\dashv} A, a \sqsubset B \quad B \sqsubset C}{a \mathrel{\vdash\!\dashv} A, a \sqsubset C}(\sqsubset_2)}{a \sqsubset C}(-)$$

$$\dfrac{\dfrac{a \sqsubset A, a \sqsubset B}{a \sqsubset B}(-) \quad B \mathrel{\vdash\!\dashv} C}{a \mathrel{\vdash\!\dashv} C}(\mathrel{\vdash\!\dashv}) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, a \sqsubset B \quad B \mathrel{\vdash\!\dashv} C}{a \sqsubset A, a \mathrel{\vdash\!\dashv} C}(\mathrel{\vdash\!\dashv}_1)}{a \mathrel{\vdash\!\dashv} C}(-)$$

$$\dfrac{\dfrac{a \sqsubset A, a \mathrel{\vdash\!\dashv} B}{a \mathrel{\vdash\!\dashv} B}(-) \quad C \sqsubset B}{a \mathrel{\vdash\!\dashv} C}(\mathrel{\vdash\!\dashv}) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, a \mathrel{\vdash\!\dashv} B \quad C \sqsubset B}{a \sqsubset A, a \mathrel{\vdash\!\dashv} C}(\mathrel{\vdash\!\dashv}_2)}{a \mathrel{\vdash\!\dashv} C}(-)$$

Starting with the topmost applications of $(\sqsubset)$ and $(\mathrel{\vdash\!\dashv})$ to non-syllogistic formulas (i.e., the topmost application of $(\sqsubset)$ in the path α on the left and the application of I_1 in the path β on the right), we repeatedly apply the transformation rules and permute the applications of $(-)$. Note that the resulting proof has the same premises and conclusion as π'.

Now we obtain:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{c \sqsubset A_1, c \,\square\, B_1 \quad \begin{matrix}\vdots\\ A_1 \sqsubset A_2\end{matrix}}{c \sqsubset A_2, c \,\square\, B_1}(\sqsubset_1)}{\ddots\; c \sqsubset A_{k-1}, c \,\square\, B_1} \quad \begin{matrix}\vdots\\ A_{k-1} \sqsubset A\end{matrix}}{c \sqsubset A, c \,\square\, B_1}(\sqsubset_1)}{c \sqsubset A}(-) \quad \dfrac{\dfrac{\dfrac{\dfrac{c \sqsubset A_1, c \,\square\, B_1 \quad \begin{matrix}\vdots\\ B_1 \sqsubset B_2\end{matrix}}{c \sqsubset A_1, c \,\square\, B_2}I_1^*}{\ddots\; c \sqsubset A_1, c \,\square\, B_{m-1}} \quad \begin{matrix}\vdots\\ B_{m-1} \,\square\, B\end{matrix}}{c \sqsubset A_1, c \,\square\, B}I_{m-1}^*}{c \,\square\, B}(-)}{c \sqsubset A, c \,\square\, B}(+)$$

We take a subproof ending with $c \sqsubset A, c \Box B_1$ on the left path and a subproof ending with $c \sqsubset A_1, c \Box B$ which depends on the assumption $c \sqsubset A_1, c \Box B_1$ on the right path. Call these π_1 and π_2, respectively, and note that they are syllogistic. Finally, we combine π_1 and π_2:

$$\begin{array}{c} c \sqsubset A_1, c \Box B_1 \\ \vdots \; \pi_1 \\ c \sqsubset A, c \Box B_1 \\ \vdots \; \pi_2[c \sqsubset A_1 := c \sqsubset A] \\ c \sqsubset A, c \Box B \end{array}$$

where by $\pi_2[c \sqsubset A_1 := c \sqsubset A]$ we denote the proof obtained from π_2 by substituting all occurrences of $c \sqsubset A_1$ with $c \sqsubset A$. Note that since $c \sqsubset A_1$ is a minor premise of each application I^* of the derived rules in π_2, the figure obtained by this substitution is still a proof in GS. Hence we have a syllogistic proof of $c \sqsubset A, c \Box B$. ■

Now we are in a position to show the faithfulness of the translation $(\cdot)^\circ$.

Theorem 3.15 (Faithfulness) *Let $S, S_1, \ldots, S_n$ be categorical sentences in* CS*. Let π be a proof in* GS *of S° from $S_1^\circ, \ldots, S_n^\circ$, such that (i) the sequence $S_1^\circ, \ldots, S_n^\circ, S^\circ$ is a cycle, and (ii) all singular terms appearing in S_i° $(1 \leq i \leq n)$ are different from each other. Then π can be translated into a proof in* CS *of S from $S_1, \ldots, S_n$.*

Proof. By Lemma 3.14 π can be transformed to a syllogistic proof π' in GS, in which only syllogistic formulas appear. We show that π' can be translated into a proof in CS by induction on the length of π'. The base case is immediate: if π' consists only of a formula S°, S is a proof in CS. For the induction step, we divide the cases according to the last inference rule applied in π'.

1. The last inference of π' is $(\sqsubset)$. We have the following proof in GS.

$$\dfrac{\begin{matrix}\vdots\\ A \sqsubset B\end{matrix} \quad \begin{matrix}\vdots\\ B \sqsubset C\end{matrix}}{A \sqsubset C}\ (\sqsubset)$$

   By the induction hypothesis, we have a proof of All A are B and a proof of All B are C in CS. So we obtain a proof of All A are C by Barbara.

2. The last inference is $(\mathrel{\vdash\!\dashv})$, which is of the form:

$$\dfrac{\begin{matrix}\vdots\\ A \sqsubset B\end{matrix} \quad \begin{matrix}\vdots\\ B \mathrel{\vdash\!\dashv} C\end{matrix}}{A \mathrel{\vdash\!\dashv} C}\ (\mathrel{\vdash\!\dashv})$$

   Since the order of terms matters in categorical sentences, there are two possible translations for a premise $B \mathrel{\vdash\!\dashv} C$, that is,

$$(\textsf{No } B \textsf{ are } C)^\circ = B \mathrel{\vdash\!\dashv} C \quad \text{or} \quad (\textsf{No } C \textsf{ are } B)^\circ = B \mathrel{\vdash\!\dashv} C.$$

   Similarly the conclusion $A \mathrel{\vdash\!\dashv} C$ has two translations. So we need to consider four cases, of which we only show the following two cases:

   (i) $(\textsf{No } B \textsf{ are } C)^\circ = B \mathrel{\vdash\!\dashv} C$ and $(\textsf{No } A \textsf{ are } C)^\circ = A \mathrel{\vdash\!\dashv} C$

$$\dfrac{\begin{matrix}\vdots\ \text{(I.H.)}\\ \textsf{All } A \textsf{ are } B\end{matrix} \quad \begin{matrix}\vdots\ \text{(I.H.)}\\ \textsf{No } B \textsf{ are } C\end{matrix}}{\textsf{No } A \textsf{ are } C}\ \textsf{Celarent}$$

   (ii) $(\textsf{No } B \textsf{ are } C)^\circ = C \mathrel{\vdash\!\dashv} B$ and $(\textsf{No } A \textsf{ are } C)^\circ = A \mathrel{\vdash\!\dashv} C$.

$$\dfrac{\begin{matrix}\vdots\ \text{(I.H.)}\\ \textsf{All } A \textsf{ are } B\end{matrix} \quad \dfrac{\begin{matrix}\vdots\ \text{(I.H.)}\\ \textsf{No } C \textsf{ are } B\end{matrix}}{\textsf{No } B \textsf{ are } C}\ \textsf{conv2}}{\textsf{No } A \textsf{ are } C}\ \textsf{Celarent}$$

   For the cases where $(\textsf{No } C \textsf{ are } A)^\circ = A \mathrel{\vdash\!\dashv} C$, the desired proof is obtained by applying conv2 to the conclusion of (i) and (ii).

3. The last inference is $(\sqsubset_1)$. It has the form:

$$\frac{\begin{matrix}\vdots & \vdots\\ a \sqsubset A, a \sqsubset B & B \sqsubset C\end{matrix}}{a \sqsubset A, a \sqsubset C}\ (\sqsubset_1)$$

Premise $\{a \sqsubset A, a \sqsubset B\}$ and conclusion $\{a \sqsubset A, a \sqsubset C\}$ have two possible translations, respectively: i.e., for the former, (Some A are B)$^\circ$ and (Some B are A)$^\circ$. So we need to distinguish four cases, of which the most complex is the following one:

$$\frac{\dfrac{\dfrac{\vdots\ \text{(I.H.)}}{\text{Some } B \text{ are } A}}{\text{Some } A \text{ are } B}\ \mathsf{conv1} \qquad \dfrac{\vdots\ \text{(I.H.)}}{\text{All } B \text{ are } C}}{\dfrac{\text{Some } A \text{ are } C}{\text{Some } C \text{ are } A}\ \mathsf{conv1}}\ \mathsf{Darii}$$

4. The last inference is $(\sqsubset_2)$. It has the form:

$$\frac{\begin{matrix}\vdots & \vdots\\ a \mathrel{\vdash\!\dashv} A, a \sqsubset B & B \sqsubset C\end{matrix}}{a \mathrel{\vdash\!\dashv} A, a \sqsubset C}\ (\sqsubset_2)$$

By the induction hypothesis, we have a proof of Some B are not A and a proof of All B are C in CS. Hence by Bocardo we can obtain a proof of Some C are not A.

5. The last inference is $(\mathrel{\vdash\!\dashv}_1)$, which is of the form:

$$\frac{\begin{matrix}\vdots & \vdots\\ a \sqsubset A, a \sqsubset B & B \mathrel{\vdash\!\dashv} C\end{matrix}}{a \sqsubset A, a \mathrel{\vdash\!\dashv} C}\ (\mathrel{\vdash\!\dashv}_1)$$

Premise $\{a \sqsubset A, a \sqsubset B\}$ and $B \mathrel{\vdash\!\dashv} C$ each have two possible translations, so we have four cases. The basic case is the following:

$$\frac{\begin{matrix}\vdots\ \text{(I.H.)} & \vdots\ \text{(I.H.)}\\ \text{Some } A \text{ are } B & \text{No } B \text{ are } C\end{matrix}}{\text{Some } A \text{ are not } C}\ \mathsf{Ferio}$$

The other three proofs are obtained by applying conv2 to one or both of the premises.

6. The last inference is ($\vdash\!\dashv_2$). It has the form:

$$\dfrac{\begin{matrix}\vdots & & \vdots\\ a \sqsubset A, a \mathrel{\vdash\!\dashv} B & & C \sqsubset B\end{matrix}}{a \sqsubset A, a \mathrel{\vdash\!\dashv} C}\ (\vdash\!\dashv_2)$$

By the induction hypothesis, we have a proof of Some A are not B and a proof of All C are B in CS. Hence by Baroco we can obtain a proof of Some A are not C. This completes the proof. ■

3.3 Categorical syllogisms with existential import

In this section we consider an extension of our categorical syllogism to include the existential import. There is a long standing debate on the status of existential import of quantified sentences in natural language.[1] Here we consider a system in which the existential import is simply treated as an entailment relation and inference rules traditionally called *subaltern* are admitted. Thus CS is extended with the following rules.

$$\dfrac{\text{All } B \text{ are } A}{\text{Some } A \text{ are } B}\ \text{subalt1} \qquad \dfrac{\text{No } B \text{ are } A}{\text{Some } A \text{ are not } B}\ \text{subalt2}$$

We call the resulting system CS^+. By admitting subalt1 and subalt2, we have an additional eight valid patterns of categorical syllogisms, and hence overall twenty four valid patterns of categorical syllogisms. See Table 3.1 in Section 3.1 for the list of all the valid patterns.

As in CS, we consider a *chain* of syllogisms with existential import. The notions of proof and provability are extended to CS^+ in an obvious way.

Example 3.16 In CS^+, Darapti

$$\text{All } B \text{ are } C, \text{All } B \text{ are } A \vdash \text{Some } A \text{ are not } C$$

of the third figure is derived in the following way.

[1] See Geurts (2007) for a recent overview and discussion.

$$\dfrac{\textsf{All } B \textsf{ are } C \quad \dfrac{\textsf{All } B \textsf{ are } A}{\textsf{Some } A \textsf{ are } B}\ \textsf{subalt1}}{\textsf{Some } A \textsf{ are } C}\ \textsf{Darii}$$

To interpret these extended syllogisms in GS, we need to provide different translations to universal sentences All A are B and No A are B in CS^+ from the ones in CS. We denote by $S^\bullet$ the translation of a categorical statement S in CS^+.

Theorem 3.17 (Soundness) *Let π be a proof in CS^+ of S from $S_1, \dots S_n$. Then π can be translated into a proof in* GS *of $S^\bullet$ from $S_1^\bullet, \dots S_n^\bullet$, where a categorical sentence S in CS^+ is translated into a formula $S^\bullet$ in* GS *as follows:*

$$\begin{aligned}
(\textsf{All } A \textsf{ are } B)^\bullet &= \{a \sqsubset A, A \sqsubset B\} \\
(\textsf{No } A \textsf{ are } B)^\bullet &= \{a \sqsubset A, b \sqsubset B, A \dashv\vdash B\} \textit{ with } a \textit{ and } b \textit{ different} \\
(\textsf{Some } A \textsf{ are } B)^\bullet &= \{c \sqsubset A,\ c \sqsubset B\} \\
(\textsf{Some } A \textsf{ are not } B)^\bullet &= \{d \sqsubset A,\ d \dashv\vdash B\}.
\end{aligned}$$

Here we impose the following restrictions: (i) every premise $S_i^\bullet$ $(1 \le i \le n)$ is assigned different singular terms, and (ii) any singular term appearing in the conclusion $S^\bullet$ also appears in one of the premises $S_1^\bullet, \dots S_n^\bullet$.

Proof. The proof is by induction on the length of π. Each inference rule in CS^+ can be translated into a combination of inference rules of GS in a similar way to that shown in Lemma 3.6.

1. Barbara is translated into:

$$\dfrac{\dfrac{a \sqsubset A, A \sqsubset B}{a \sqsubset A}\,(\sqsubset) \qquad \dfrac{\dfrac{a \sqsubset A, A \sqsubset B}{A \sqsubset B}\,(-) \quad \dfrac{b \sqsubset B, B \sqsubset C}{B \sqsubset C}\,(-)}{A \sqsubset C}\,(\sqsubset)}{a \sqsubset A, A \sqsubset C}\,(+)$$

2. Celarent is translated into:

$$\dfrac{\dfrac{\dfrac{a \sqsubset A, A \sqsubset B}{a \sqsubset A}\,(-) \quad \dfrac{b \sqsubset B, c \sqsubset C, B \dashv\vdash C}{c \sqsubset C}\,(-)}{a \sqsubset A, c \sqsubset C}\,(+) \qquad \dfrac{\dfrac{a \sqsubset A, A \sqsubset B}{A \sqsubset B}\,(-) \quad \dfrac{b \sqsubset B, c \sqsubset C, B \dashv\vdash C}{B \dashv\vdash C}\,(-)}{A \dashv\vdash C}\,(\dashv\vdash)}{a \sqsubset A, c \sqsubset C, A \dashv\vdash C}\,(+)$$

3. Darii and Bocardo are translated into:

$$\dfrac{\dfrac{a \sqsubset B, a \Box C}{a \Box C}\,(-) \quad \dfrac{\dfrac{a \sqsubset B, a \Box C}{a \sqsubset B}\,(-) \quad \dfrac{b \sqsubset B, B \sqsubset A}{B \sqsubset A}\,(-)}{a \sqsubset A}\,(\sqsubset)}{a \Box C, a \sqsubset A}\,(+)$$

where $\Box$ is $\sqsubset$ for Darii and $\vdash\!\dashv$ for Bocardo.

4. Ferio and Baroco are translated into:

$$\dfrac{\dfrac{a \sqsubset A, a \Box_1 B}{a \sqsubset A}\,(-) \quad \dfrac{\dfrac{a \sqsubset A, a \Box_1 B}{a \Box_1 B}\,(-) \quad \dfrac{\Gamma, C \Box_2 B}{C \Box_2 B}\,(-)}{a \mathrel{\vdash\!\dashv} C}\,(\vdash\!\dashv)}{a \sqsubset A, a \mathrel{\vdash\!\dashv} C}\,(+)$$

where for Ferio (Baroco), $\Box_1$ is $\sqsubset$ ($\vdash\!\dashv$), $\Box_2$ is $\vdash\!\dashv$ ($\sqsubset$), and Γ is $c \sqsubset C$ ($b \sqsubset B, c \sqsubset C$).

5. subalt1 is translated into:

$$\dfrac{\dfrac{\dfrac{a \sqsubset B, B \sqsubset A}{a \sqsubset B}\,(-) \quad \dfrac{a \sqsubset B, B \sqsubset A}{B \sqsubset A}\,(-)}{a \sqsubset A}\,(\sqsubset) \quad \dfrac{a \sqsubset B, B \sqsubset A}{a \sqsubset B}\,(-)}{a \sqsubset A, a \sqsubset B}\,(+)$$

6. subalt2 is translated into:

$$\dfrac{\dfrac{b \sqsubset B, a \sqsubset A, B \mathrel{\vdash\!\dashv} A}{a \sqsubset A}\,(-) \quad \dfrac{\dfrac{b \sqsubset B, a \sqsubset A, B \mathrel{\vdash\!\dashv} A}{a \sqsubset A}\,(-) \quad \dfrac{b \sqsubset B, a \sqsubset A, B \mathrel{\vdash\!\dashv} A}{B \mathrel{\vdash\!\dashv} A}\,(-)}{a \mathrel{\vdash\!\dashv} B}\,(\vdash\!\dashv)}{a \sqsubset A, a \mathrel{\vdash\!\dashv} B}\,(+)$$

■

We call a formula of the form $\{a \sqsubset A, A \sqsubset B\}$ an *A-formula* and a formula of the form $\{a \sqsubset A, b \sqsubset B, A \mathrel{\vdash\!\dashv} B\}$ an *N-formula* in GS. The formula of these forms and existential formulas given in Definition 3.8 are collectively called *E-syllogistic* formulas in GS.

For the faithfulness of the translation $(\cdot)^\bullet$ from CS^+ to GS, we introduce derived inference rules in GS in an analogous way to those in CS.

Lemma 3.18 (Derived rules) The following are derived rules in GS.

$$\frac{a \sqsubset A, A \sqsubset B \quad b \sqsubset B, B \sqsubset C}{a \sqsubset A, A \sqsubset C}\ (\sqsubset_1^+) \qquad \frac{a \sqsubset A, A \sqsubset B \quad b \sqsubset B, c \sqsubset C, B \vDash C}{a \sqsubset A, c \sqsubset C, A \vDash C}\ (\vDash_1^+)$$

$$\frac{a \sqsubset A, a \sqsubset B \quad b \sqsubset B, B \sqsubset C}{a \sqsubset A, a \sqsubset C}\ (\sqsubset_2^+) \qquad \frac{a \sqsubset A, a \sqsubset B \quad b \sqsubset B, c \sqsubset C, B \vDash C}{a \sqsubset A, a \vDash C}\ (\vDash_2^+)$$

$$\frac{a \vDash A, a \sqsubset B \quad b \sqsubset B, B \sqsubset C}{a \vDash A, a \sqsubset C}\ (\sqsubset_3^+) \qquad \frac{a \sqsubset A, a \vDash B \quad c \sqsubset C, C \sqsubset B}{a \sqsubset A, a \vDash C}\ (\vDash_3^+)$$

$$\frac{a \sqsubset A, A \sqsubset B}{a \sqsubset A, a \sqsubset B}\ (\sqsubset_4^+) \qquad \frac{a \sqsubset A, b \sqsubset B, A \vDash B}{a \sqsubset A, a \vDash B}\ (\vDash_4^+)$$

Proof. Observe that each rule corresponds to an inference rule in CS^+, i.e., $(\sqsubset_1^+)$ to Barbara, $(\sqsubset_2^+)$ to Darii, $(\sqsubset_3^+)$ to Bocardo, $(\sqsubset_4^+)$ to subalt1, $(\vDash_1^+)$ to Celarent, $(\vDash_2^+)$ to Ferio, $(\vDash_3^+)$ to Baroco, and $(\vDash_4^+)$ to subalt2. So each is justified as shown in Theorem 3.17. ■

In an analogous way to Lemma 3.9, we have the following.

Lemma 3.19 Let $\mathcal{P}_1, \dots, \mathcal{P}_n$ be E-syllogistic formulas in GS such that singular terms appearing in $\mathcal{P}_i$ $(1 \le i \le n)$ are different from each other. Let π be a normal proof in GS of an E-syllogistic formula $\mathcal{P}$ from $\mathcal{P}_1, \dots, \mathcal{P}_n$.

1. If $\mathcal{P}$ is an A-formula, then all the assumptions are A-formulas.
2. If $\mathcal{P}$ is an N-formula, then one of the assumptions is an N-formula and the other assumptions are A-formulas.
3. If $\mathcal{P}$ is an existential formula, then one of the assumptions is an existential formula and the other assumptions are A-formulas or N-formulas.

Proof. Similarly to the proof of Lemma 3.9. ■

Lemma 3.11 can be extended to E-syllogistic formulas.

Lemma 3.20 *Let $\mathcal{P}, \mathcal{P}_1, \ldots, \mathcal{P}_n$ be E-syllogistic formula in* GS. *Let π be a proof in* GS *of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$ such that (i) the sequence $\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{P}$ is a cycle and (ii) any singular term appearing in $\mathcal{P}_i$ $(1 \leq i \leq n)$ is different from each other. If π' is normal proof of π, then π' has the same assumptions as π.*

Proof. The proof proceeds in a similar way to that of Lemma 3.11. Suppose, for contradiction, that some formula $\mathcal{P}_i$ $(1 \leq i \leq n)$ is not among the assumptions of π'. Let $gtm(\mathcal{P}_i) = \{A_i, A_{i+1}\}$. We observe that neither A_i nor A_{i+1} appears in the conclusion of π'. Thus we may assume, without loss of generality, that there are assumptions, say $\mathcal{P}_j$ and $\mathcal{P}_k$, which contain A_i and A_{i+1}, respectively, and no other assumptions contain A_i nor A_{i+1}. By Lemma 3.19 at least one of $\mathcal{P}_j$ and $\mathcal{P}_k$ must be an A-formula or N-formula. Suppose then, without loss of generality, that $\mathcal{P}_j$, which contains A_i, is an A-formula. (The same argument holds for the case when $\mathcal{P}_j$ is an N-formula.) Since A_i does not appear in the conclusion of π', A_i must be eliminated in π' either (i) by an application of $(\sqsubset)$, in which case A_i appears as a middle term, or (ii) by an application of $(-)$. The former case leads to a contradiction, as we saw in the proof of Lemma 3.11. In the latter case, $\mathcal{P}_j$ must be of the form $a \sqsubset B_1, B_1 \sqsubset A_i$. Since the singular term a cannot be eliminated by any rule, a must appear in the conclusion $\mathcal{P}$ of π'. We divide the argument into two cases, depending on whether (1) the conclusion $\mathcal{P}$ is an A- or N-formula, or (2) $\mathcal{P}$ is an existential formula.

(1) Suppose, for concreteness, that the conclusion is an A-formula of the form $a \sqsubset B_m, B_m \sqsubset C$, where $m \geq 1$. Then π', which is a normal proof by our assumption, would look like:

$$\cfrac{\cfrac{\begin{array}{c}\cfrac{a \sqsubset B_1, B_1 \sqsubset A_i}{a \sqsubset B_1}\,(-)\\ \ddots \\ a \sqsubset B_{m-1}\end{array} \qquad \begin{array}{c}\Gamma_1\\ \vdots\\ B_{m-1} \sqsubset B_m\end{array}}{a \sqsubset B_m}\,(\sqsubset) \qquad \begin{array}{c}\Gamma_2\\ \vdots\\ B_m \sqsubset C\end{array}}{a \sqsubset B_m, B_m \sqsubset C}\,(+)$$

Here we see that the general term B_m, which appears in the conclusion $\mathcal{P}$, must also appear in two assumptions $\mathcal{P}_{i'} \in \Gamma_1$ and $\mathcal{P}_{j'} \in \Gamma_2$ with $i' \neq j'$. This is a contradiction to our assumption that $\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{P}$ is a cycle. Note that the same argument applies to the case where the conclusion is an N-formula of the form $a \sqsubset B_m, b \sqsubset C, B_m \mathrel{\vdash\!\!\dashv} C$.

(2) When the conclusion is an existential formula, it should be of the form $a \sqsubset B, a \Box C$, where $\Box \in \{\sqsubset, \mathrel{\vdash\!\!\dashv}\}$ and $B \not\equiv C$. Then in order to derive both $a \sqsubset B$ and $a \Box C$, there must be a general term B_i $(1 \leq i)$ that appears in three different assumptions. That is, π' must be of the form

$$\dfrac{\begin{matrix} \dfrac{\dfrac{\dfrac{\ddots \atop a \sqsubset B_{i-1} \quad \overset{\Gamma_1}{\vdots} \atop B_{i-1} \sqsubset B_i}{a \sqsubset B_i}(\sqsubset) \quad \overset{\Gamma_2}{\vdots} \atop B_i \sqsubset B'}{a \sqsubset B'}(\sqsubset) \\ \ddots \\ a \sqsubset B \end{matrix} \qquad \begin{matrix} \dfrac{\dfrac{\ddots \atop a \sqsubset B_{i-1} \quad \overset{\Gamma_1}{\vdots} \atop B_{i-1} \sqsubset B_i}{a \sqsubset B_i}(\sqsubset) \quad \overset{\Gamma_3}{\vdots} \atop B_i \sqsubset C'}{a \sqsubset C'}(\sqsubset) \\ \ddots \\ a \Box C \end{matrix}}{a \sqsubset B, a \Box C}(+)$$

where $B' \not\equiv C'$ and $\Gamma_1, \Gamma_2, \Gamma_3$ must contain different assumptions in which B_i appears. Again, this is a contradiction to our assumption. ■

We say that a GS-proof beginning with E-syllogistic formulas as assumptions and proceeding by the rules introduced in Lemma 3.18, i.e., $(\sqsubset_1^+)$, $(\sqsubset_2^+)$, $(\sqsubset_3^+)$, $(\sqsubset_4^+)$, $(\mathrel{\vdash\!\!\dashv}_1^+)$, $(\mathrel{\vdash\!\!\dashv}_2^+)$, $(\mathrel{\vdash\!\!\dashv}_3^+)$, and $(\mathrel{\vdash\!\!\dashv}_4^+)$, is an *E-syllogistic* proof in GS. Note that by this definition E-syllogistic proofs do not involve applications of $(\sqsubset)$, $(\mathrel{\vdash\!\!\dashv})$, $(+)$, nor $(-)$ rules.

Lemma 3.21 Let $\mathcal{P}, \mathcal{P}_1, \ldots, \mathcal{P}_n$ be E-syllogistic formulas in GS and, π be a GS-proof of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$ such that (i) the sequence $\sigma = \mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{P}$ is a cycle and (ii) all singular terms appearing in $\mathcal{P}_i$ $(1 \leq i \leq n)$ are different from each other. Then π can be transformed into an E-syllogistic proof with the same assumptions and conclusion.

Proof. The proof proceeds in a similar way to that of Lemma 3.14. By Theorem 2.8, π can be transformed into a normal proof π'. It is seen that by cyclicity condition, π' has the same assumption, $\mathcal{P}_1, \ldots, \mathcal{P}_n$, as π.

We show that π' can be transformed into an E-syllogistic proof of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$. We distinguish the cases according to the form of $\mathcal{P}$.

(Case 1) $\mathcal{P}$ is an A-formula, i.e., a formula of the form $\{a \sqsubset A, A \sqsubset B\}$.

By Lemma 2.10, using the same reasoning as in the case of Lemma 3.14, we have a path $\alpha = \mathcal{P}_i, \mathcal{Q}_1, \ldots, \mathcal{Q}_k, \mathcal{P}$ and $\beta = \mathcal{P}_j, \mathcal{R}_1, \ldots, \mathcal{R}_m, \mathcal{P}$ such that $\mathcal{P}_i$ contains $s \sqsubset A$ (resp. $\mathcal{P}_j$ contains $A \sqsubset B_1$) and $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ (resp. $\mathcal{R}_1, \ldots, \mathcal{R}_m$) is the transitive part, in which each formula $\mathcal{Q}_l$ is of the form $s_l \sqsubset A$ with $1 \leq l \leq k$, $s_k \equiv a$ (resp. $\mathcal{R}_l$ is of the form $A \sqsubset B_l$ with $1 \leq l \leq m$, $B_m \equiv B$). By cyclicity of σ, we have $\mathcal{P}_i \equiv \mathcal{P}_j$.

By Lemma 3.19(1), $\mathcal{P}_i$ ($\equiv \mathcal{P}_j$) is an A-formula, so it has the form $\{c \sqsubset A, A \sqsubset B_1\}$. Moreover, we have $c \equiv a$. Otherwise the transitive part of α contains both $c \sqsubset A$ and $a \sqsubset A$ with $c \not\equiv a$. But this is clearly impossible. (Note that no formula of the form $u \sqsubset c$ appears in the syllogistic fragment of GS.) So the transitive part of α consists only of the formula $a \sqsubset A$. Thus π' looks like:

$$\dfrac{\dfrac{a \sqsubset A, A \sqsubset B_1}{a \sqsubset A}\,(-) \qquad \dfrac{\begin{array}{c}\dfrac{\dfrac{a \sqsubset A, A \sqsubset B_1}{A \sqsubset B_1}\,(-) \quad \begin{array}{c}\vdots\ \pi_1\\ B_1 \sqsubset B_2\end{array}}{A \sqsubset B_2}\,(\sqsubset)\\ \ddots\\ A \sqsubset B_{m-1}\end{array} \qquad \begin{array}{c}\vdots\ \pi_{m-1}\\ B_{m-1} \sqsubset B\end{array}}{A \sqsubset B}\,(\sqsubset)}{a \sqsubset A, A \sqsubset B}\,(+)$$

The assumptions on which each π_i ($1 \leq i \leq m$–1) depends are all A-formulas, and in π_i there can only be applications of the ($\sqsubset$) rule, except the ($-$) rule applied to these assumptions.

We introduce a transformation procedure to reduce applications of ($\sqsubset$) by permuting down applications of ($-$). Take a topmost application of ($\sqsubset$) in π': it has the form on the left, which can be transformed into the form on the right.

$$\dfrac{\dfrac{a \sqsubset A, A \sqsubset B}{A \sqsubset B}\,(-) \quad \dfrac{b \sqsubset B, B \sqsubset C}{B \sqsubset C}\,(-)}{A \sqsubset C}\,(\sqsubset) \quad \rhd_1 \quad \dfrac{\dfrac{a \sqsubset A, A \sqsubset B \quad b \sqsubset C, B \sqsubset C}{a \sqsubset A, A \sqsubset C}\,(\sqsubset_1^+)}{A \sqsubset C}\,(-)$$

Note that this transformation preserves the assumptions and conclusion of the original proof. By repeated applications of $\rhd_1$, we obtain:

$$\dfrac{\dfrac{a \sqsubset A, A \sqsubset B_1}{a \sqsubset A}\,(-) \qquad \dfrac{\dfrac{\dfrac{\dfrac{a \sqsubset A, A \sqsubset B_1 \quad \overset{\vdots\, \pi_1^*}{b \sqsubset B_1, B_1 \sqsubset B_2}}{a \sqsubset A, A \sqsubset B_2}\,(\sqsubset_1^+)}{\ddots\; a \sqsubset A, A \sqsubset B_{m-1}} \quad \overset{\vdots\, \pi_{m-1}^*}{b_{m-1} \sqsubset B_{m-1}, B_{m-1} \sqsubset B}}{\dfrac{a \sqsubset A, A \sqsubset B}{A \sqsubset B}\,(-)}\,(\sqsubset_1^+)}{a \sqsubset A, A \sqsubset B}\,(+)$$

where $\pi_1^*, \ldots, \pi_{m-1}^*$ are E-syllogistic proofs consisting only of applications of $(\sqsubset_1^+)$.

We take a subproof ending with $a \sqsubset A, A \sqsubset B$, which yields an E-syllogistic proof of $a \sqsubset A, A \sqsubset B$ with the same assumptions and conclusion as π'.

(Case 2) $\mathcal{P}$ is an N-formula: it has the form $\{a \sqsubset A, b \sqsubset B, A \vdash\!\dashv B\}$. By Lemma 2.10, we can assume (without loss of generality) that π' has the form:

$$\dfrac{\dfrac{\overset{\Gamma_1}{\overset{\vdots\, \pi_1}{a \sqsubset A}} \quad \overset{\Gamma_2}{\overset{\vdots\, \pi_2}{b \sqsubset B}}}{a \sqsubset A, b \sqsubset B}\,(+) \qquad \overset{\Gamma_3}{\overset{\vdots\, \pi_3}{A \vdash\!\dashv B}}}{a \sqsubset A, b \sqsubset B, A \vdash\!\dashv B}\,(+)$$

where Γ_1 (Γ_2) contains an assumption $\mathcal{P}_i$ ($\mathcal{P}_j$) in which a formula of the form $s \sqsubset A$ ($t \sqsubset B$) appears, and Γ_3 contains assumptions $\mathcal{P}_{i'}$ and $\mathcal{P}_{j'}$ in which a formula of the form $A \sqsubset C$ or $A \vdash\!\dashv C$ (resp. $B \sqsubset C'$ or $B \vdash\!\dashv C'$) appears. By cyclicity of σ, we have $\mathcal{P}_i \equiv \mathcal{P}_{i'}$ and $\mathcal{P}_j \equiv \mathcal{P}_{j'}$. Since $\mathcal{P}_i$ and $\mathcal{P}_j$ are syllogistic formulas, s and t must be singular terms.

We can see that Γ_1 (resp. Γ_2) contains no assumptions other than $\mathcal{P}_i$ (resp. $\mathcal{P}_j$): if there is such an assumption in Γ_1, then in π_1 we have a path whose transitive part begins with the formula $s \sqsubset A$ which is a premise of an application of $(\sqsubset)$. But then s must be a general term, which is impossible. The same reasoning can be applied to the case of Γ_2.

So in order to transform π' into an E-syllogistic proof with the same assumptions and conclusion as π', we only need to consider the subproof π_3

ending with $A \mathrel{\vdash\!\dashv} B$. The general form of π_3 is as follows (each application of ($\vdash\!\dashv$) may appear in a different order).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a_1 \sqsubset A_1, b_1 \sqsubset B_1, A_1 \mathrel{\vdash\!\dashv} B_1}{A_1 \mathrel{\vdash\!\dashv} B_1}\,(-) \qquad \begin{matrix}\Gamma_1^1\\ \vdots\ \pi_1^1\\ A_2 \sqsubset A_1\end{matrix}}{A_2 \mathrel{\vdash\!\dashv} B_1}\,(\vdash\!\dashv)}{\begin{matrix}\ddots\\ A_{k-1} \mathrel{\vdash\!\dashv} B_1\end{matrix}} \qquad \begin{matrix}\Gamma_k^1\\ \vdots\ \pi_k^1\\ A \sqsubset A_{k-1}\end{matrix}}{A \mathrel{\vdash\!\dashv} B_1}\,(\vdash\!\dashv) \qquad \begin{matrix}\Gamma_1^2\\ \vdots\ \pi_1^2\\ B_2 \sqsubset B_1\end{matrix}}{\begin{matrix}A \mathrel{\vdash\!\dashv} B_2\\ \ddots\\ A \mathrel{\vdash\!\dashv} B_{m-1}\end{matrix}}\,(\vdash\!\dashv) \qquad \begin{matrix}\Gamma_m^2\\ \vdots\ \pi_m^2\\ B \sqsubset B_{m-1}\end{matrix}}{A \mathrel{\vdash\!\dashv} B}\,(\vdash\!\dashv)
$$

Since the premises $\Gamma_1^1, \ldots, \Gamma_k^1, \Gamma_1^2, \ldots, \Gamma_m^2$ only contain A-formulas, we can apply the rewriting procedure $\rhd_1$ introduced above from a topmost application of ($\sqsubset$) in each π_i^1 $(1 \leq i \leq k)$ and π_j^2 $(1 \leq j \leq m)$, and obtain proofs π_i^{1*} and π_j^{2*}, which do not contain applications of ($\sqsubset$). Next we apply the transformation procedure $\rhd_2$ to the topmost application of ($\vdash\!\dashv$) in π_3.

$$
\dfrac{\dfrac{a \sqsubset A, b \sqsubset B, A \mathrel{\vdash\!\dashv} B}{A \mathrel{\vdash\!\dashv} B}\,(-) \quad \dfrac{c \sqsubset C, C \sqsubset B}{C \sqsubset B}\,(-)}{A \mathrel{\vdash\!\dashv} C}\,(\vdash\!\dashv) \quad \rhd_2 \quad \dfrac{\dfrac{a \sqsubset A, b \sqsubset B, A \mathrel{\vdash\!\dashv} B \quad c \sqsubset C, C \sqsubset B}{a \sqsubset A, c \sqsubset C, A \mathrel{\vdash\!\dashv} C}\,(\vdash\!\dashv_1^+)}{A \mathrel{\vdash\!\dashv} C}\,(-)
$$

Then we obtain:

$$
\dfrac{\dfrac{\dfrac{\dfrac{a_1 \sqsubset A_1, b_1 \sqsubset B_1, A_1 \mathrel{\vdash\!\dashv} B_1 \quad \begin{matrix}\vdots\ \pi_1^{1*}\\ a_2 \sqsubset A_2, A_2 \sqsubset A_1\end{matrix}}{a_2, \sqsubset A_2, b_1 \sqsubset B_1, A_2 \sqsubset B_1}\,(\vdash\!\dashv_1^+)}{\begin{matrix}\ddots\\ a \sqsubset A, b_{m-1} \sqsubset B_{m-1}, A \mathrel{\vdash\!\dashv} B_{m-1}\end{matrix}} \qquad \begin{matrix}\vdots\ \pi_m^{2*}\\ b \sqsubset B, B \sqsubset B_{m-1}\end{matrix}}{a \sqsubset A, b \sqsubset B, A \mathrel{\vdash\!\dashv} B}\,(\vdash\!\dashv_1^+)}{A \mathrel{\vdash\!\dashv} B}\,(-)
$$

Hence the subtree ending with $a \sqsubset A, b \sqsubset B, A \mathrel{\vdash\!\dashv} B$ yields a syllogistic proof with the same assumptions and conclusion as π'.

(Case 3) $\mathcal{P}$ is an existential formula. π' has the same form as the one described in the proof of Lemma 3.14, except that each assumption can be an A-formula or an N-formula. We need to check all the possible proofs beginning with A-formulas, N-formulas, and existential formulas as assumptions. Accordingly, we need the additional transformation rules, shown below, which permute down applications of $(-)$ and replace an application of $(\sqsubset)$ and (H) with corresponding derived rules.

$$\dfrac{\dfrac{a \sqsubset A, a \sqsubset B}{a \sqsubset B}\,(-) \qquad \dfrac{b \sqsubset B, B \sqsubset C}{B \sqsubset C}\,(-)}{a \sqsubset C}\,(\sqsubset) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, a \sqsubset B \quad b \sqsubset B, B \sqsubset C}{a \sqsubset A, a \sqsubset C}\,(\sqsubset_2^+)}{a \sqsubset C}\,(-)$$

$$\dfrac{\dfrac{a \mathsf{H} A, a \sqsubset B}{a \sqsubset B}\,(-) \qquad \dfrac{b \sqsubset B, B \sqsubset C}{B \sqsubset C}\,(-)}{a \sqsubset C}\,(\sqsubset) \quad \rhd \quad \dfrac{\dfrac{a \mathsf{H} A, a \sqsubset B \quad b \sqsubset B, B \sqsubset C}{a \mathsf{H} A, a \sqsubset C}\,(\sqsubset_3^+)}{a \sqsubset C}\,(-)$$

$$\dfrac{\dfrac{a \sqsubset A, a \sqsubset B}{a \sqsubset B}\,(-) \qquad \dfrac{b \sqsubset B, c \sqsubset C, B \mathsf{H} C}{B \mathsf{H} C}\,(-)}{a \mathsf{H} C}\,(\mathsf{H}) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, a \sqsubset B \quad b \sqsubset B, c \sqsubset C, B \mathsf{H} C}{a \sqsubset A, a \mathsf{H} C}\,(\mathsf{H}_2^+)}{a \mathsf{H} C}\,(-)$$

$$\dfrac{\dfrac{a \sqsubset A, a \mathsf{H} B}{a \mathsf{H} B}\,(-) \qquad \dfrac{c \sqsubset C, C \sqsubset B}{C \sqsubset B}\,(-)}{a \mathsf{H} C}\,(\mathsf{H}) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, a \mathsf{H} B \quad c \sqsubset C, C \sqsubset B}{a \sqsubset A, a \mathsf{H} C}\,(\mathsf{H}_3^+)}{a \mathsf{H} C}\,(-)$$

$$\dfrac{\dfrac{a \sqsubset A, A \sqsubset B}{a \sqsubset A}\,(-) \qquad \dfrac{a \sqsubset A, A \sqsubset B}{A \sqsubset B}\,(-)}{a \sqsubset B}\,(\sqsubset) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, A \sqsubset B}{a \sqsubset A, a \sqsubset B}\,(\sqsubset_4^+)}{a \sqsubset B}\,(-)$$

$$\dfrac{\dfrac{a \sqsubset A, b \sqsubset B, A \mathsf{H} B}{a \sqsubset A}\,(-) \qquad \dfrac{a \sqsubset A, b \sqsubset B, A \mathsf{H} B}{A \mathsf{H} B}\,(-)}{a \mathsf{H} B}\,(\mathsf{H}) \quad \rhd \quad \dfrac{\dfrac{a \sqsubset A, b \sqsubset B, A \mathsf{H} B}{a \sqsubset A, a \mathsf{H} B}\,(\mathsf{H}_4^+)}{a \mathsf{H} B}\,(-)$$

By successively applying these transformation rules as well as $\rhd_1$ and $\rhd_2$ to topmost applications of $(\sqsubset)$ and (H) in the same way as described in Lemma 3.14, we can obtain a syllogistic proof whose assumptions and conclusion are the same as π'.

This completes the proof of Lemma 3.21. ■

Finally, we show the faithfulness of the translation $(\cdot)^{\bullet}$.

Theorem 3.22 (Faithfulness) Let π be a GS-proof of $S^{\bullet}$ from $S_1^{\bullet}, \ldots, S_n^{\bullet}$ such that (i) the sequence $S_1^{\bullet}, \ldots, S_n^{\bullet}, S^{\bullet}$ is a cycle and (ii) any singular term appearing in $S_i^{\bullet}$ $(1 \leq i \leq n)$ is different from the others. Then π can be translated into a proof in CS of S from $S_1, \ldots, S_n$.

Proof. The proof is similar to that for CS. By Lemma 3.21, π can be transformed into a proof π' in which only E-syllogistic formulas appear. We show that π' can be translated into a proof in CS^{+} by induction on the length of π'. The base case is immediate: if π consists only of a formula $S^{\bullet}$, S is clearly a proof in CS^{+}.

For the induction step, we divide the cases according to the last inference rule applied in π'. If the last inference is $(\sqsubset_1^{+})$, $(\sqsubset_2^{+})$, $(\sqsubset_3^{+})$, $(\vdash\!\dashv_1^{+})$, $(\vdash\!\dashv_2^{+})$, or $(\vdash\!\dashv_3^{+})$, the proof is essentially the same as that in Theorem 3.15: each rule is translated into Barbara, Celarent, Darii, Ferio, Baroco and Bocardo, respectively, together with conv1 and conv2 if needed.

We consider the remaining two cases.

1. The last inference is $(\sqsubset_4^{+})$. We have the following proof in GS:

$$\dfrac{\begin{array}{c}\vdots\\ a \sqsubset A, A \sqsubset B\end{array}}{a \sqsubset A, a \sqsubset B}\ (\sqsubset_4^{+})$$

By the induction hypothesis, we have a proof of $\mathsf{All}\ A\ \mathsf{are}\ B$ in CS^{+}. Hence we can obtain a proof of $\mathsf{Some}\ A\ \mathsf{are}\ B$ in the following way.

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \mathsf{All}\ A\ \mathsf{are}\ B\end{array}}{\mathsf{Some}\ B\ \mathsf{are}\ A}\ \mathsf{subalt1}}{\mathsf{Some}\ A\ \mathsf{are}\ B}\ \mathsf{conv1}$$

2. The last inference is $(\vdash\!\dashv_4^{+})$. We have the following proof in GS:

$$\dfrac{\begin{array}{c}\vdots\\ a \sqsubset A, b \sqsubset B, A \vdash\!\dashv B\end{array}}{a \sqsubset A, a \vdash\!\dashv B}\ (\vdash\!\dashv_4^{+})$$

By the induction hypothesis, we have a proof of No A are B or No B are A in CS^+. In the former case, by applying subalt2 and conv2, we obtain a proof of Some A are not B. In the latter case, an application of subalt2 yields the desired proof. ■

4. GS and a natural deduction system of minimal logic ML

In this section, we show that the syllogistic fragment of GS is naturally embedded into an implicational fragment of propositional minimal logic. By minimal logic we mean intuitionistic logic minus the absurdity rule (the $\perp$ rule), i.e., the rule licensing to infer an arbitrary formula from a contradiction.[1] We provide a transformation procedure to convert proofs in the syllogistic fragments of GS into proofs in a natural deduction system of propositional minimal logic, and vice versa. We thereby make explicit the relationship between the syllogistic fragments of GS and a well-established system in proof theory.

4.1 The proof theory of ML

We first present the language of an implicational fragment of propositional minimal logic, which we henceforth call ML. For the easiness of comparison with GS, we generalize a conjunction $\wedge$ to be applied to a finite set of formulas.

Definition 4.1 The language of ML contains propositional variables corresponding to singular and general terms in GS, denoted by $a, b, c, \dots$ and $A, B, C, \dots$, respectively, a propositional constant $\perp$, and generalized conjunction $\bigwedge$. The formulas of ML are inductively defined as follows:

1. $\perp$ and all propositional variables are formulas.

[1]See Prawitz (1965) for the standard definition of minimal logic. See also Troelstra and Schwichtenberg (2000) for a textbook treatment of minimal logic.

2. If ϕ and ψ are formulas, then $(\phi \to \psi)$ is a formula.
3. If Φ is a finite non-empty set of formulas, then $(\bigwedge \Phi)$ is a formula.

Notation. We use variables (possibly with subscripts) $s, t, u, \ldots$ to denote propositional variables in lower and upper case, $\phi, \psi, \ldots$ to denote a formula, and $\Phi, \Psi, \ldots$ to denote a finite set of formulas. We usually omit external brackets of formulas. We write $\neg s$ for $s \to \bot$ and $s \wedge t$ for $\bigwedge \{s, t\}$.

Definition 4.2 (Inference rules of ML) Inference rules of ML are introduction and elimination rules for $\to$ and $\wedge$:

$$\dfrac{\begin{matrix}[\phi]\\ \vdots \\ \psi\end{matrix}}{\phi \to \psi} \to I \qquad \dfrac{\phi \quad \phi \to \psi}{\psi} \to E \qquad \dfrac{\bigwedge \Phi \quad \bigwedge \Psi}{\bigwedge (\Phi \cup \Psi)} \wedge I \qquad \dfrac{\bigwedge \Phi}{\bigwedge \Phi'} \wedge E$$

where we assume the following restrictions.

- In $\to I$, exactly one assumption ϕ is discharged.
- In $\wedge I$, Φ and Ψ are non-empty and $\Phi \neq \Psi$.
- In $\wedge E$, Φ and Φ' are non-empty and $\Phi' \subset \Phi$.

By the restriction on $\to I$, our system ML can be said to be *contraction-free*.

As usual, the introduction and elimination rules for negation are spacial cases of $\to I$ and $\to E$. We denote them by $\neg I$ and $\neg E$.

$$\dfrac{\begin{matrix}[\phi]\\ \vdots \\ \bot\end{matrix}}{\neg\phi} \neg I \qquad \dfrac{\phi \quad \neg\phi}{\bot} \neg E$$

The notion of proof in ML is defined in the same way as that of GS.

The notion of *normal* proof in ML is defined in a usual way, except that the normalization procedure for conjunction is modified in a similar way as that of GS.

Definition 4.3 (Normal proof in ML) A formula ϕ is said to be a *cut-formula* in a proof π when ϕ is the conclusion of an application of introduction rule and a premise of an application of elimination rule. When π contains a cut formula of the forms on the left below, it can be transformed into the forms on the right:

$$\dfrac{\begin{matrix}\vdots\ \pi_1\\ \phi\end{matrix} \qquad \dfrac{\begin{matrix}[\phi]\\ \vdots\ \pi_2\\ \psi\end{matrix}}{\phi\to\psi}\,{\to}I}{\psi}\,{\to}E \qquad \rhd \qquad \begin{matrix}\vdots\ \pi_1\\ \phi\\ \vdots\ \pi_2\\ \psi\end{matrix}$$

$$\dfrac{\dfrac{\begin{matrix}\vdots\ \pi_1\\ \bigwedge\Phi_1\end{matrix} \qquad \begin{matrix}\vdots\ \pi_2\\ \bigwedge\Phi_2\end{matrix}}{\bigwedge(\Phi_1\cup\Phi_2)}\,{\wedge}I}{\bigwedge\Psi}\,{\wedge}E \quad \rhd \quad \begin{cases} \dfrac{\begin{matrix}\vdots\ \pi_i\\ \bigwedge\Phi_i\end{matrix}}{\bigwedge\Psi}\,{\wedge}E & \text{when } \Psi\subset\Phi_i \text{ for } i=1 \text{ or } 2 \\[3em] \dfrac{\dfrac{\begin{matrix}\vdots\ \pi_1\\ \bigwedge\Phi_1\end{matrix}}{\bigwedge\Psi_1}\,{\wedge}E \qquad \dfrac{\begin{matrix}\vdots\ \pi_2\\ \bigwedge\Phi_2\end{matrix}}{\bigwedge\Psi_2}\,{\wedge}E}{\bigwedge\Psi}\,{\wedge}I & \begin{matrix}\text{when } \Psi=\Psi_1\cup\Psi_2 \text{ such that}\\ \Psi_1\subset\Phi_1 \text{ and } \Psi_2\subset\Phi_2\end{matrix} \end{cases}$$

In the resulting proof for conjunction, when $\Phi_i = \Psi$ (resp. $\Phi_i = \Psi_i$) for $i = 1$ or 2, it should be understood that the $(\wedge E)$ rule is not applied and thus $\dfrac{\Phi_i}{\Psi}$ (resp. $\dfrac{\Phi_i}{\Psi_i}$) is replaced by $\mathcal{P}_i$.

We say that a proof in ML is in *normal form* when it contains no cut formula.

It should be noted that the normalization procedure for conjunction is the same as the standard one (cf. Prawitz 1965) when Φ_1 and Φ_2 consist of a single formula.

We have the normalization theorem for ML. The proof is essentially the same as the one for the standard natural deduction system for minimal logic (see Prawitz 1965; Troelstra and Schwichtenberg 2000).

Theorem 4.4 (Normalization) *Every proof π in* ML *can be transformed into a normal proof with the same conclusion as π.*

4.2 The relation between GS and ML

Now we give a translation of a formula in GS into a formula in ML. Here atomic formulas of GS are translated into *implicational* formulas in ML.

Definition 4.5 (Translation) Each singular or general term of GS is translated into the corresponding propositional variable of ML. Then the translation $(\cdot)^+$ of a formula in GS into a formula in ML is defined as follows.

$$\begin{aligned} (s \sqsubset t)^+ &:= s \to t \\ (s \boxminus t)^+ &:= s \to \neg t \\ \{P_1, \ldots, P_n\}^+ &:= \bigwedge \{P_1^+, \ldots, P_n^+\} \end{aligned}$$

Given this definition, it is straightforward to prove the soundness of the translation $(\cdot)^+$.

Theorem 4.6 (Soundness) *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{P}$ be syllogistic (E-syllogistic) formulas in* GS*, and let π be a proof in* GS *of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$. Then π can be translated into a proof in* ML *of $\mathcal{P}^+$ from $\mathcal{P}_1^+, \ldots, \mathcal{P}_n^+$.*

Proof. By induction of the length of π. Note that there is no formula of the forms $s \sqsubset s$ and $s \sqsubset a$ in the syllogistic and E-syllogistic fragments of GS. Thus it is sufficient to check ($\sqsubset$), ($\boxminus$), ($+$), and ($-$). Each rule is simulated in the following way.

1. $\dfrac{s \sqsubset t \quad t \sqsubset u}{s \sqsubset u}$ ($\sqsubset$) is translated as $\dfrac{\dfrac{\dfrac{[s] \quad s \to t}{t}\ {\to_E} \quad t \to u}{u}\ {\to_E}}{s \to u}\ {\to_I}$

2. $\dfrac{s \sqsubset t \quad t \boxminus u}{s \boxminus u}$ ($\boxminus$) is translated as $\dfrac{\dfrac{\dfrac{[s] \quad s \to t}{t}\ {\to_E} \quad t \to \neg u}{\neg u}\ {\to_E}}{s \to \neg u}\ {\to_I}$

3. $\dfrac{\mathcal{P} \quad \mathcal{Q}}{\mathcal{P} \cup \mathcal{Q}}\ (+)$ is translated as $\dfrac{\bigwedge \mathcal{P}^+ \quad \bigwedge \mathcal{P}^+}{\bigwedge (\mathcal{P}^+ \cup \mathcal{Q}^+)} \wedge I$

4. $\dfrac{\mathcal{P}}{\mathcal{Q}}\ (-)$ is translated as $\dfrac{\bigwedge \mathcal{P}^+}{\bigwedge \mathcal{Q}^+} \wedge E$

■

Example 4.7 Here is an example of a translation. A proof in GS

$$\dfrac{\dfrac{c \sqsubset A_1, c \sqsubset A_2}{c \sqsubset A_1}\,(-) \qquad \dfrac{\dfrac{c \sqsubset A_1, c \sqsubset A_2}{c \sqsubset A_2}\,(-) \quad A_2 \vDash A_3}{c \vDash A_3}\,(\vDash)}{c \sqsubset A_1, c \vDash A_3}\,(+)$$

is translated into the following proof in ML.

$$\dfrac{\dfrac{(c \to A_1) \wedge (c \to A_2)}{c \to A_1}\wedge E \qquad \dfrac{\dfrac{\dfrac{[c]^1 \quad \dfrac{(c \to A_1) \wedge (c \to A_2)}{c \to A_2}\wedge E}{A_2}\to E \quad A_2 \to \neg A_3}{\neg A_3}\to E}{c \to \neg A_3}\to I,\ 1}{(c \to A_1) \wedge (c \to \neg A_3)}\wedge I$$

Next we show the converse of Theorem 4.6, i.e., the faithfulness of the translation $(\cdot)^+$. The proof proceeds in a similar way to the proof of the faithfulness of the translations from CS and CS$^+$ to GS. As noted above, the $\wedge I$ and $\wedge E$ rules in ML directly correspond to the $(+)$ and $(-)$ in GS. Hence, in transforming a proof in ML whose assumptions and conclusion are restricted to the translations of syllogistic (E-syllogistic) formulas, the crucial task is the one to rewrite the steps involving applications of the $\to I$ and $\to E$ rules in terms of the $(\sqsubset)$ and $(\vDash)$ rules in GS.

Theorem 4.8 (Faithfulness) *Let $\mathcal{P}$ be a syllogistic (E-syllogistic) formula, and let Γ be a set of syllogistic (E-syllogistic) formulas in* GS. *Every proof of P^+ from Γ^+ in* ML *can be transformed into a proof of $\mathcal{P}$ from Γ or a subset of Γ in* GS, *where $\Gamma^+ := \{\mathcal{P}^+ \mid \mathcal{P} \in \Gamma\}$.*

Proof. By Theorem 4.4, every proof of $\mathcal{P}^+$ from Γ^+ in ML is transformed into a normal proof of $\mathcal{P}^+$ from Γ^+ or a subset of Γ^+. Let π be such a normal proof. We show that π can be translated into a proof of $\mathcal{P}$ in GS. Note that $\mathcal{P}^+$ has the form $\bigwedge\{P_1^+, \ldots, P_n^+\}$ $(n \geq 1)$ where each P_i^+ is an implicational formula of the form $s \to t$ or $s \to \neg t$. When $n \geq 2$, the last inference rule applied in π must be $\wedge I$, since π is normal and all assumptions are translations of syllogistic (E-syllogistic) formulas. As noted before, each application of $\wedge I$ can be translated as an application of $(+)$ in GS. Thus it is sufficient to show the translation of a proof whose conclusion is an implicational formula. We divide cases depending on whether the conclusion is of the form $s_1 \to t_1$ or $s_1 \to \neg t_1$.

(1) When the conclusion is of the form $s_1 \to t_1$, π must have the following form:

$$\dfrac{\dfrac{\dfrac{\dfrac{[s_1]^1 \quad \begin{array}{c}\vdots\ \pi_1\\ s_1 \to s_2\end{array}}{s_2}\,{\scriptstyle \to E} \quad \begin{array}{c}\vdots\ \pi_2\\ s_2 \to s_3\end{array}}{\begin{array}{c}s_3\\ \ddots\ \pi'\\ s_n\end{array}}\,{\scriptstyle \to E} \quad \begin{array}{c}\vdots\ \pi_n\\ s_n \to t_1\end{array}}{t_1}\,{\scriptstyle \to E}}{s_1 \to t_1}\,{\scriptstyle \to I,\ 1}$$

where π' consists of repeated applications of $\to E$ $(n \geq 1)$. Note that atomic formulas $s_1, \ldots, s_n, t_1$ can only be obtained by an application of $\to E$, because π is normal and all the assumptions in π are implicational formulas or a conjunction of implicational formulas. By the restriction on $\to I$, s_1 is discharged exactly once, so it cannot be discharged in $\pi_1, \ldots, \pi_n$. By the normality of π, the only possibility is that each π_i $(1 \leq i \leq n)$ consists of applications of $\wedge E$, and hence, each can be translated as a proof π_i^* consisting of applications of $(-)$. Hence we can translate π into the proof in GS

$$\dfrac{\dfrac{\begin{array}{c}\vdots\ \pi_1^*\\ s_1 \sqsubset s_2\end{array} \quad \begin{array}{c}\vdots\ \pi_2^*\\ s_2 \sqsubset s_3\end{array}}{\begin{array}{c}s_1 \sqsubset s_3\\ \ddots\ \pi'^*\\ s_1 \sqsubset s_n\end{array}}\,(\sqsubset) \quad \begin{array}{c}\vdots\ \pi_n^*\\ s_n \sqsubset t_1\end{array}}{s_1 \sqsubset t_1}\,(\sqsubset)$$

where π'^* consists of repeated applications of ($\sqsubset$).

(2) The conclusion is of the form $s_1 \to \neg t_1$. When $\neg t_1$ is obtained by an application of $\to E$, π has essentially the same structure as in the case of (1), hence a desired proof can be obtained in the same way as above, except that the last inference rule applied is ($\vDash$), rather than ($\sqsubset$).

When $\neg t_1$ is obtained by an application of $\neg I$, we may assume, without loss of generality, that π has the following form:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[s_1]^1 \quad \overset{\vdots\,\pi_1^1}{s_1 \to s_2}}{s_2}\,{\to}E \quad \overset{\vdots\,\pi_2^1}{s_2 \to s_3}}{s_3}\,{\to}E}{\overset{\ddots\,\pi'}{s_{n-1}}} \quad \overset{\vdots\,\pi_{n-1}^1}{s_{n-1} \to s_n}}{s_n} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[t_1]^2 \quad \overset{\vdots\,\pi_1^2}{t_1 \to t_2}}{t_2}\,{\to}E \quad \overset{\vdots\,\pi_2^2}{t_2 \to t_3}}{t_3}\,{\to}E}{\overset{\ddots\,\pi''}{t_m}} \quad \overset{\vdots\,\pi_m^2}{t_m \to \neg s_n}}{\neg s_n}}{\dfrac{\dfrac{\bot}{\neg t_1}\,\neg I,\,1}{s_1 \to \neg t_1}\,{\to}I,\,2}\,\neg E$$

where $n, m \geq 1$, π' and π'' consist of applications of $\to E$, and π_i^1 $(1 \leq i \leq n-1)$ and π_j^2 $(1 \leq j \leq m)$ consist of applications of $\wedge E$. Note that by the restriction on $\to I$, s_1 and t_1 are different terms and each must be discharged exactly once.

Now each π_i^1 and π_j^2 can be translated as a proof π_i^{1*} and π_j^{2*} consisting of applications of ($-$), and then we can translate π into the following proof in GS.

$$\dfrac{\dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1^{1*}}{s_1 \sqsubset s_2} \quad \overset{\vdots\,\pi_2^{1*}}{s_2 \sqsubset s_3}}{s_1 \sqsubset s_3}\,(\sqsubset)}{\overset{\ddots\,\pi'^*}{s_1 \sqsubset s_{n-1}}} \quad \overset{\vdots\,\pi_{n-1}^{1*}}{s_{n-1} \sqsubset s_n}}{s_1 \sqsubset s_n}\,(\sqsubset) \qquad \dfrac{\dfrac{\dfrac{\overset{\vdots\,\pi_1^{2*}}{t_1 \sqsubset t_2} \quad \overset{\vdots\,\pi_2^{2*}}{t_2 \sqsubset t_3}}{t_1 \sqsubset t_3}\,(\sqsubset)}{\overset{\ddots\,\pi''^*}{t_1 \sqsubset t_m}} \quad \overset{\vdots\,\pi_m^{2*}}{t_m \vDash s_n}}{t_1 \vDash s_n}\,(\vDash)}{s_1 \vDash t_1}\,(\vDash)$$

where π'^* and π''^* are solely composed of applications of ($\sqsubset$). (When $n=1$, the proof wholly consists of the subproof on the right side branch.)

This completes the proof of Theorem 4.8. ■

We have shown a sound and faithful translation of the syllogistic and E-syllogistic fragments of GS into ML by giving an explicit proof-transformation procedure. What we have established so far is summarized in the following figure.

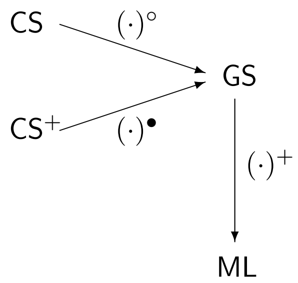


Fig. 4.1 The relationship between the inference systems.

Here each mapping $(\cdot)^\circ$, $(\cdot)^\bullet$, $(\cdot)^+$ stands for a sound and faithful translation from one system to another, with suitable restrictions on the treatment of singular terms as well as the cyclicity condition for the faithfulness of $(\cdot)^\circ$ and $(\cdot)^\bullet$.

As is suggested in Figure 4.1, it can also be shown that compositional mappings $((\cdot)^\circ)^+$ and $((\cdot)^\bullet)^+$ provide a sound and faithful translation of CS and CS^+ into ML, with suitable provisions for the faithfulness as before. More specifically, the soundness is a straightforward consequence of the soundness of the translations $(\cdot)^\circ$, $(\cdot)^\bullet$, and $(\cdot)^+$, which are established in Theorem 3.6, Theorem 3.17, and Theorem 4.6, respectively. The faithfulness can be shown as a consequence of the faithfulness of $(\cdot)^\circ$ and $(\cdot)^\bullet$ in Theorem 3.15 and Theorem 3.22, respectively, and the faithfulness of $(\cdot)^+$ in Theorem 4.8. For the faithfulness of the translation from GS to ML, we need in ML suitable restrictions on the translations of propositional variables in lower case (corresponding to singular terms in GS), as well as the cyclicity condition formulated for propositional variables in upper case (corresponding to general terms in GS). Since they are essentially similar to the ones for

GS, we do not enter into a detailed formulation and a proof here.

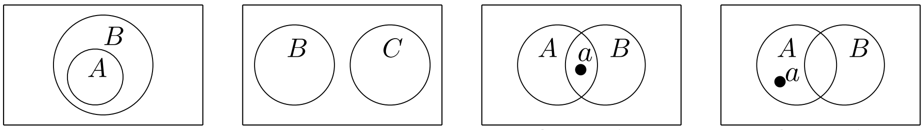
5. GS and an inference system for Euler diagrams

In this section, we present an inference system with Euler diagrams, called the Generalized Diagrammatic Syllogistic inference system (GDS), and establish a relationship between GS and GDS. The diagrammatic inference system GDS is first introduced in Mineshima, Okada and Takemura (2012a), where a completeness theorem and a normalization theorem are established using some specific notion of normal proofs of diagrams.

After introducing some background for formalizations of diagrammatic inferences (Section 5.1), we present the syntax of GDS (Section 5.2). The exposition in these two subsections is self-contained, but a more detailed discussion on the motivation behind our approach to formalizing Euler diagrams as well as more examples and other technical materials, can be found in Mineshima, Okada and Takemura (2012a). In Section 5.3, we show a faithful embeddability of GDS into GS, which is the main goal of this section.

5.1 Background: formalization of inferences with diagrams

Euler diagrams were originally introduced in Euler (1768) to illustrate syllogistic reasoning. As is already noted in Section 1, Euler diagrams represent logical relations between the terms of a categorical sentence in terms of the topological relations holding between circles. For example, universal sentences of the form All A are B and No A are B are represented by the inclusion and the exclusion relations between circles, respectively, as seen in D_1 and D_2 in Figure 5.1.



D_1: All A are B D_2: No B are C D_3: Some A are B D_3: Some A are not B

Fig. 5.1 Correspondence between categorical sentences and Euler diagrams.

Regarding existential sentences, there have been proposed several versions of Euler diagrams in the literature.[1] In Euler's original version of diagrams (Euler 1768; Kneale and Kneale 1962, 349–352), which was further developed by Gergonne (1817), it is assumed that (i) a diagram consists only of circles and that (ii) every minimal region in a diagram denotes a non-empty set. As a consequence, in order to represent an existential sentence in syllogisms, one has to use more than one diagram. Thus, the sentence Some A are B is represented by the *disjunction* of four diagrams, namely, diagrams of the forms: (1) circle A is inside circle B, (2) circle B is inside circle A, (3) circle A and circle B coincide, and (4) circle A partially overlaps circle B (see the diagram in Figure 5.2 below). Similarly, Some B are not C requires three diagrams. This means that when we consider a syllogism with these two premises, we have to take into account twelve ways of combining diagrams.

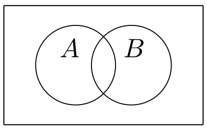


Fig. 5.2 Partially overlapping circles

To avoid such a complexity, we adopt the following conventions (these conventions will be formalized in the next section): (i) each minimal region

[1]See Hammer and Shin 1998; Stapleton 2005 for a survey on various systems of Euler diagrams.

in a diagram does not have existential import, and (ii) the existence of an object in a region is indicated by a *name point* such as $\overset{a}{\bullet}$. As a consequence, partially overlapping circles as in Figure 5.2 are semantically vacuous, i.e., they deliver no information about the relation between the sets denoted by the circles. Thus, they can be used to express the fact that the semantic relationship between the circles is *indeterminate*. Given these conventions, we represent categorical sentences Some A are B and Some A are not B by diagrams D_3 and D_4, respectively, in Figure 5.1 above.

Using the correspondences between Euler diagrams and categorical sentences, syllogistic inferences can be naturally replaced by a task of manipulating Euler diagrams, in particular of unifying premise diagrams and extracting information from them. For example, the syllogism Barbara

$$\textsf{All } A \textsf{ are } B,\ \textsf{All } B \textsf{ are } C \vdash \textsf{All } A \textsf{ are } C$$

is represented diagrammatically as in Figure 5.3.

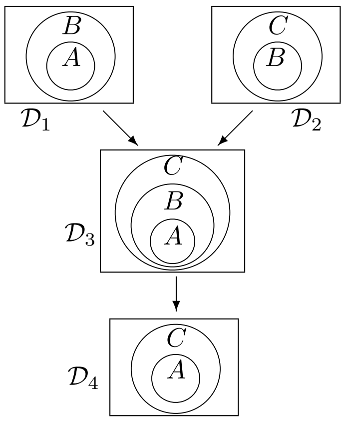


Fig. 5.3 Barbara with Euler diagrams

The operation of combining two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ is an instance of an application of *unification* rule. It consists in identifying the circle B and keeping all the relations holding on the premise diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$. Here we can see that the transitivity of inclusion relations between circles plays a crucial role in this simple inference process. As we will see below, the rule of unification plays a central role in formalizing the reasoning with Euler diagrams.

The logical properties of such diagrammatic inferences have been studied in the field of *diagrammatic logic*, which was initiated by philosophers and logicians in the 1990's.[2] A chief goal is to formalize the semantics and proof theory underlying such diagrammatic representations and inferences, and then to prove logical properties, in an analogous way to symbolic logic.

In these studies, however, the focus has been on formalizations of *Venn diagrams*, originally proposed in Venn (1881), where the set-theoretical relations are represented in terms of *shading*, as exemplified in Figure 5.4. Here the shaded region denotes the empty set; this means that in Venn diagrams, the categorical sentence All A are B is interpreted as its equivalent form: There is nothing which is A but not B.

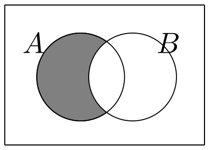


Fig. 5.4 Representation of All A are B in Venn diagrams

Venn diagrams can be abstractly defined as a set of *region*, i.e., by specifying which regions are shaded and which regions are not (see Shin 1994; Howse, Molina, and Taylor 2000, among others). Such a framework of defining logic diagrams may be called a *region-based* framework. This framework has been extended to formalizations of Euler diagrams since Hammer's (1995) pioneering work. (For recent developments, see Euler/Venn diagrams of Swoboda and Allwein 2005; Spider diagrams ESD2 and SD3 of Molina 2001 and Howse, Stapleton, and Taylor 2005.) In these studies, Euler diagrams are formalized indirectly, in terms of the methods developed in the study on Venn diagrams. In particular, few attentions have been paid to formalization of the inference rule of unification, such as the one illustrated in Figure 5.3. Some systems (e.g. Hammer 1995) lack the rule of unification, hence are not able to handle syllogistic inferences at all. Other systems (e.g. Swoboda

[2]See, in particular, Barwise and Etchemendy (1996), Shin (1994), and Hammer (1995).

and Allwein 2005; Molina 2001; Howse, Stapleton, and Taylor 2005) have the rule of unification but it is defined indirectly, in terms of operations on Venn diagrams, i.e., what we call *superposition* in Mineshima, Okada and Takemura (2012a); hence they fail to capture simple and intuitive inference processes of unifying Euler diagrams as exemplified in Figure 5.3.[3] Thus it is fair to say that the exact formulation of inference rules directly operating on Euler diagrams was not clear until recently.[4]

In contrast to the region-based approach, we introduced in Mineshima, Okada and Takemura (2012a) a novel approach to formalizing the reasoning with Euler diagrams, where diagrams are directly defined in terms of topological relations (inclusion and exclusion relations) between circles and points. We call our framework a *relation-based* framework. We introduced an Euler diagrammatic representation system EUL, and an inference system called the Generalized Diagrammatic Syllogistic inference system GDS. Our framework avoids some complications involved in the region-based framework. In particular, categorical sentences are naturally translated without making use of negation, and unification of Euler diagrams is directly formalized without making a detour to Venn diagrams. See Mineshima, Okada and Takemura (2012a) for a more detailed discussion on advantages and disadvantages of the region-based and relation-based frameworks.[5]

[3] In Sato, Mineshima and Takemura (2010a), we discuss some cognitive differences between reasoning with Euler diagrams and reasoning with Venn diagrams. We test subjects' performances in syllogism solving in case where these two types of diagrams are used. The results indicate that Euler diagrams are more effective in actual syllogistic reasoning than the corresponding Venn diagrams. See also Sato, Mineshima and Takemura (2010b) for a more general discussion on the efficacy of logic diagrams in deductive reasoning.

[4] For a discussion on the difficulty of formalizing reasoning with Euler diagrams, see Hammer and Shin (1995).

[5] A detailed comparison between these two frameworks from a proof-theoretical viewpoint is also found in Mineshima, Okada and Takemura (2010), where the region-based inference system is formalized as resolution calculus, in contrast to the relation-based system formalized as natural deduction system.

5.2 A representation system EUL for Euler diagrams

We present a representation system for Euler diagrams EUL. In our approach, diagrams (called EUL-diagrams) are primarily defined as geometric objects on the real plane. In order to capture the role of Euler diagrams in deductionve inferences, we introduce a certain equivalence relation between diagrams, abstracting away the information irrelevant to the purpose of deduction. Such "equivalent diagrams" are defined in terms of topological relations. Thus, we distinguish two levels of "diagrams," which have usually been referred to as the distinction between *concrete* diagrams and *abstract* diagrams (cf. Howse, Molina, Shin, and Taylor 2002). When disambiguation is necessary, we will call EUL-diagrams defined as geometric objects *concrete* EUL-diagrams, and those defined as a set of topological relations *abstract* EUL-diagrams. The formal treatment of diagrams in our semantics and proof theory in later sections is based on the level of abstract EUL-diagrams.

We will consider a simple diagrammatic representation system in which the only diagrammatic objects are simple closed curves, called *contours*, and points. A (concrete) EUL-diagrams is a set of contours and points enclosed by a rectangle in the real plane. Each contour and point has a unique and distinct name chosen from some fixed set of names, $\mathcal{L}$. See, e.g., Seligman (1995) and Flower and Howse (2002) for similar approach to definitions of concrete diagrams.

Definition 5.1 (EUL-diagram) An EUL-*diagram* $\mathcal{D} = \langle \mathbb{C}, \mathbb{P}, \mathbb{L} \rangle$ consists of a finite set $\mathbb{C}$ of contours on the real place ($\mathbb{R}^2$), a finite set $\mathbb{P}$ of points, and $\mathbb{L} : \mathbb{C} \cup \mathbb{P} \to \mathcal{L}$ is an injective function that returns the name of each contour and point, such that the following conditions hold.

1. $|\mathbb{C} \cup \mathbb{P}| \geq 2$.
2. No two contours intersect without crossing.
3. No two contours intersect at more than finitely many points.

Each member of $\mathbb{C}$ is called a (*named*) *coutour* of $\mathcal{D}$.

Notation. We use $A, B, C, \ldots, X, Y, \ldots$ (possibly with subscripts) as variables ranging over names of contours and $a, b, c, \ldots, x, y, \ldots$ (possibly with subscripts) as variables ranging over names of points. Contours and points are collectively called (*diagrammatic*) *objects*, and denoted by s, t, u (possibly with subscripts). We use a rectangle, $\square$, to represent the plane for an EUL-diagram. We use $\mathcal{D}, \mathcal{E}, \mathcal{F}, \ldots$ (possibly with subscripts) to denote EUL-diagrams.

Some examples of non-well-formed diagrams are given in Figure 5.5. In (i), names A and B are assigned to the same contour, hence violating the condition that $\mathbb{L}$ is a function. In (ii) and (iii), two objects have the same name, hence violating the condition that $\mathbb{L}$ is injective. Various possible restrictions on the concrete syntax of Euler diagrams are discussed in e.g., Stapleton, Zhang, Howse, and Rodgers (2010). Here we adopt a liberal approach to restrictions on the concrete syntax of diagrams, as compared to the one in e.g., Flower and Howse (2002). See Stapleton, Rodgers, Howse, and Taylor (2007) for definitions of various properties of concrete Euler diagrams.

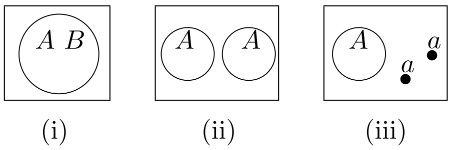


Fig. 5.5 Examples of non-well-formed diagrams

For an EUL-diagram $\mathcal{D}$, we denote by $pt(\mathcal{D})$ the set of named points of $\mathcal{D}$, by $cr(\mathcal{D})$ the set of (named) contours of $\mathcal{D}$, and by $ob(\mathcal{D})$ the set of objects of $\mathcal{D}$, i.e., $ob(\mathcal{D}) = pt(\mathcal{D}) \cup cr(\mathcal{D})$.

Definition 5.2 (Minimal diagram) An EUL-diagram that contains only two objects is called a *minimal diagram*. Minimal diagrams are denoted by α, β, γ (possibly with subscripts).

Now we define spatial relations that hold between two diagrammatic objects in the place. We call such relations EUL-*relations*, or sometimes simply *relations*. We use the same symbols as used in GS, i.e., $\sqsubset$ and $\vdash\!\dashv$, for the inclusion and exclusion relations between objects, for it would be clear from context within which system we are using these symbols.

Definition 5.3 (EUL-relation) We denote the interior of a contour A in $\mathbb{R}^2$ by $i(A)$. EUL-relations are the following binary relations between diagrammatic objects:

$$
\begin{array}{lcl}
A \sqsubset B & \Longleftrightarrow & i(A) \subseteq i(B) \\
A \mapsfromchar\!\!\!\dashv B & \Longleftrightarrow & i(A) \cap i(B) = \emptyset \\
A \bowtie B & \Longleftrightarrow & i(A) \cap i(B) \neq \emptyset \text{ and } i(A) \not\subseteq i(B) \text{ and } i(B) \not\subseteq i(A) \\
a \sqsubset A & \Longleftrightarrow & a \in i(A) \\
a \vdash\!\dashv A & \Longleftrightarrow & a \notin i(A) \\
a \sqsubset b & \Longleftrightarrow & a = b \\
a \vdash\!\dashv b & \Longleftrightarrow & a \neq b
\end{array}
$$

The relation $\sqsubset$ is a reflexive asymmetric relation, and both $\vdash\!\dashv$ and $\bowtie$ are irreflexive symmetric relations.

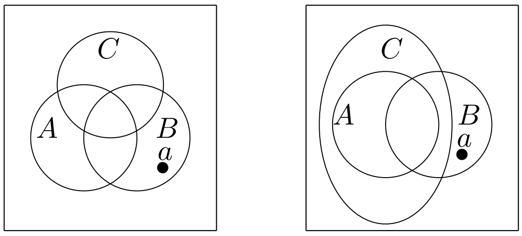
Proposition 5.4 *Let $\mathcal{D}$ be an* EUL*-diagram. For any distinct objects s and t of $\mathcal{D}$, exactly one of the* EUL*-relations $s \sqsubset t, t \sqsubset s, s \vdash\!\dashv t$ or $s \bowtie t$ holds.*

Observe that, by Proposition 5.4, for a given EUL-diagram $\mathcal{D}$, the set of EUL-relations holding on $\mathcal{D}$ is uniquely determined. We denote such a set by $\mathsf{rel}(\mathcal{D})$.

The following properties, as well as Proposition 5.4, characterize EUL-diagrams.

Lemma 5.5 Let $\mathcal{D}$ be an EUL-diagram. For any objects $s, t, u \in ob(\mathcal{D})$, we have the following:

1. *Reflexivity.* $s \sqsubset s \in \mathsf{rel}(\mathcal{D})$.
2. *Transitivity.* If $s \sqsubset t, t \sqsubset u \in \mathsf{rel}(\mathcal{D})$, then $s \sqsubset u \in \mathsf{rel}(\mathcal{D})$.
3. $\vdash\!\dashv$*-downward closedness.* If $s \vdash\!\dashv t, u \sqsubset t \in \mathsf{rel}(\mathcal{D})$, then $s \vdash\!\dashv u \in \mathsf{rel}(\mathcal{D})$.
4. *Point determinacy.* For any point a in $\mathcal{D}$, either $a \sqsubset s$ or $a \vdash\!\dashv s$ is in $\mathsf{rel}(\mathcal{D})$.
5. *Point minimality.* For any point a in $\mathcal{D}$ other than s, $s \sqsubset a \notin \mathsf{rel}(\mathcal{D})$.



$\mathcal{D}_1$ $\mathcal{D}_2$

Fig. 5.6 Examples of EUL-diagrams.

Example 5.6 Consider the EUL-diagram $\mathcal{D}_1$ and $\mathcal{D}_2$ in Figure 5.6.

We have:

$cr(\mathcal{D}_1) = cr(\mathcal{D}_2) = \{A, B, C\}$,
$pt(\mathcal{D}_1) = pt(\mathcal{D}_2) = \{a\}$,
$\mathsf{rel}(\mathcal{D}_1) = \{A \bowtie B, A \bowtie C, B \bowtie C, a \vdash\!\dashv A, a \sqsubset B, a \vdash\!\dashv C\}$,
$\mathsf{rel}(\mathcal{D}_2) = \{A \bowtie B, A \sqsubset C, B \bowtie C, a \vdash\!\dashv A, a \sqsubset B, a \vdash\!\dashv C\}$.

For notational convenience, in describing a set of relations $\mathsf{rel}(\mathcal{D})$, we usually omit the reflexive relation $s \sqsubset s$ for each object s. It should be understood that the complete description of, say, $\mathsf{rel}(\mathcal{D}_1)$ above is:

$$\{A \bowtie B, A \bowtie C, B \bowtie C, a \vdash\!\dashv A, a \sqsubset B, a \vdash\!\dashv C, A \sqsubset A, B \sqsubset B, C \sqsubset C, a \sqsubset a\}.$$

Next we present a natural, set-theoretical semantics for EUL.[6] In this semantics, each contour is interpreted as denoting a set of individuals, and each point is interpreted as denoting a singleton set of an individual, as in the interpretations of singular terms in GS. This makes it possible to interpret the EUL-relations $\sqsubset$ and $\vdash\!\dashv$ uniformly as the subset relation and the disjointness relation, respectively.

Definition 5.7 A *model* M is a pair (U, I) where U is a non-empty set (the domain of M), and I is an interpretation function which assigns to each object (contour and point) a non-empty subset of U; in particular $I(a)$ is a singleton for all point a.

[6] See also Mineshima, Okada and Takemura (2009, 2012a) for a detailed exposition of the semantic of EUL.

It should be emphasized that we assign a non-empty set to each contour.

Definition 5.8 Let $\mathcal{D}$ be an EUL-diagram. $M = (U, I)$ is a *model of* $\mathcal{D}$, written as $M \models \mathcal{D}$, if the following (1) and (2) hold: For all objects s, t of $\mathcal{D}$,

(1) $I(s) \subseteq I(t)$ if $s \sqsubset t$ holds on $\mathcal{D}$,

(2) $I(s) \cap I(t) = \emptyset$ if $s \vdash\!\dashv t$ holds on $\mathcal{D}$.

The well-definedness of the truth-conditions follows from Proposition 5.4. Note that when s is a named point, $I(s)$ is a singleton, hence $I(s) \subseteq I(t)$ is equivalent to $I(s) \in I(t)$ and $I(s) \cap I(t) = \emptyset$ is equivalent to $I(s) \notin I(t)$.

Remark 5.9 In EUL, we cannot form a diagram in which two objects coincide (cf. Definition 5.1). In particular, in contrast to GS, there is no relation of the form $a \sqsubset b$. This means that in EUL we cannot express the identity relation between points, $I(a) = I(b)$, as well as the identity relation between contours, $I(A) = I(B)$. Note that we can express the *non*-identity relation between points by $a \vdash\!\dashv b$.

Remark 5.10 By Definition 5.8, the EUL-relation $\bowtie$ does not contribute to the truth-condition of EUL-diagrams. Informally speaking, $A \bowtie B$ may be understood as expressing

$$(i) \quad I(A) \cap I(B) = \emptyset \ \text{ or } \ I(A) \cap I(B) \neq \emptyset,$$

which is true in any model. Thus, as we mentioned in Section 5.1, partially overlapping contours of the form $A \bowtie B$ can be used to express the indeterminacy with respect to the interpretations of contours. Note that in contrast to some other systems with Euler and Venn diagrams (cf. Stapleton 2005), the EUL representation system does not have any syntactic convention to express the indeterminacy with respect to the interpretations of a point and a contour, as well as the interpretations of two points. That is, there are no diagrammatic devices in EUL to express

$$(ii) \quad I(a) \in I(A) \text{ or } I(a) \notin I(A)$$

and

$$(iii) \quad I(a) = I(b) \text{ or } I(a) \neq I(b)$$

for any points a, b and any contour A.

The notion of validity is defined in a usual way.

Definition 5.11 (Validity) An EUL-diagram $\mathcal{E}$ is a *semantically valid consequence* of EUL-diagrams $\mathcal{D}_1, \ldots, \mathcal{D}_n$, written as $\mathcal{D}_1, \ldots, \mathcal{D}_n \models \mathcal{E}$, when the following holds: For any model M, if $M \models \mathcal{D}_1$ and ... and $M \models \mathcal{D}_n$, then $M \models \mathcal{E}$.

5.3 An inference system GDS for Euler diagrams

Intuitive manipulation of Euler diagrams is formalized as applications of inference rules in GDS. There are two kinds of inference rules in GDS: *unification* and *deletion*. For instance, consider the example of Barbara with Euler diagrams as illustrated in Figure 5.3 of Section 5.1. Here the first step to combine the diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ is an application of unification rule, while the second step to eliminate the contour B to obtain the conclusion is an application of deletion rule. We will call such a diagram tree a *diagrammatic proof* in GDS.

Given that an Euler diagram is abstractly identified as a set of EUL-relations, it is convenient to introduce abstract representations of diagrammatic proofs as well, where the premises and conclusion of each step are described in terms of the EUL-relations. For example, the following is an abstract representation of the diagrammatic proof of Barbara in Figure 5.3.

$$\dfrac{\dfrac{\mathcal{D}_1 : \{A \sqsubset B\} \qquad \mathcal{D}_2 : \{B \sqsubset C\}}{\mathcal{D}_3 : \{A \sqsubset B, A \sqsubset C, B \sqsubset C\}}\,\text{unification}}{\mathcal{D}_4 : \{A \sqsubset C\}}\,\text{deletion}$$

Here by $D_1 : \{A \sqsubset B\}$ we denote the diagram D_1 whose set of relations $\mathsf{rel}(\mathcal{D}_1)$ is $\{A \sqsubset B\}$, and so on. We do not omit brackets $\{\cdot\}$ in the abstract representation of a proof in GDS, so as to distinguish it from a proof in GS.

(Note that strictly speaking, the set of relations for each diagram contains a relation of the form $s \sqsubset s$ for each object s in that diagram; but as noted in Section 5.2, we omit it here.)

There are two preliminary remarks on formalization of unification rule. Firstly, in GDS the unification of two diagrams is formalized by restricting one of the premises to be a *minimal diagram*, namely, a diagram composed of two objects. Although a general formulation of unification rule that dispenses with such a restriction is technically possible at an abstract level, this restriction makes it much easier to formulate the operational meaning of unification processes. More specifically, a minimal diagram α can be regarded as an *instruction* on how to modify another premise, say $\mathcal{D}$, by introducing a new object or by arranging the configuration of objects that are already in $\mathcal{D}$. Such an operational aspect of unification will become clear in the definition of inference rules below.[7] The completeness theorem of GDS ensures that by the restricted form of unification, any diagrams $\mathcal{D}_1, \dots, \mathcal{D}_n$ may be unified into one diagram whose semantic information is equivalent to the conjunction of those of $\mathcal{D}_1, \dots, \mathcal{D}_n$.

Secondly, two kinds of constraint are imposed on unification. One is what we call the *constraint for determinacy*, which blocks the ambiguity with respect to the location of a point in a unified diagram. For example, two diagrams $\mathcal{D}_1$ and $\mathcal{D}_2$ in Figure 5.7 are not permitted to be unified into one diagram, since the location of the point c is not determined: it may be inside A or outside A. Similarly, two diagrams $\mathcal{D}_3$ and $\mathcal{D}_4$ in Figure 5.7 are not permitted to be unified into one diagram, since the semantic relationship between the points c and d is not determined: c may or may not be equal to d in a unified diagram. Since there is no syntactical devices in EUL to express this kind of indeterminacy, we prohibit the unification of these two diagrams.[8]

[7]For a more discussion on the operational interpretation of unification rules, see Section 3.2 of Mineshima, Okada and Takemura (2012a).

[8]With regard to this respect, our formulation of the constraint for determinacy differs from the one given in Mineshima, Okada and Takemura (2009, 2012a), where $\mathcal{D}_3$ and $\mathcal{D}_4$ in Figure 5.7 is permitted to be unified and it is stipulated that in the unified diagram $\mathcal{D}_3 + \mathcal{D}_4$ we always have $c \vdash\!\dashv d$. Correspondingly, in the semantics it is stipulated that

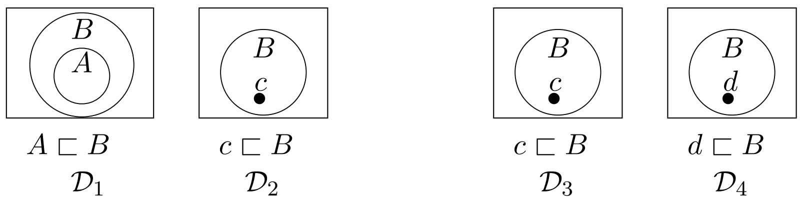


Fig. 5.7 Diagrams that cannot be unified due to the constraint for indeterminacy.

The other is the *constraint for consistency*, which is imposed to avoid representing inconsistent graphical information in a single diagram. For example, two diagrams $\mathcal{D}_3$ and $\mathcal{D}_4$ (resp. $\mathcal{D}_5$ and $\mathcal{D}_6$) in Figure 5.8 below are not permitted to be unified, because they semantically contradict each other. Note that in our semantics each contour is interpreted as denoting a non-empty set; thus the pair of $\mathcal{D}_5$ and $\mathcal{D}_6$ is semantically inconsistent as well as the pair of D_3 and D_4.

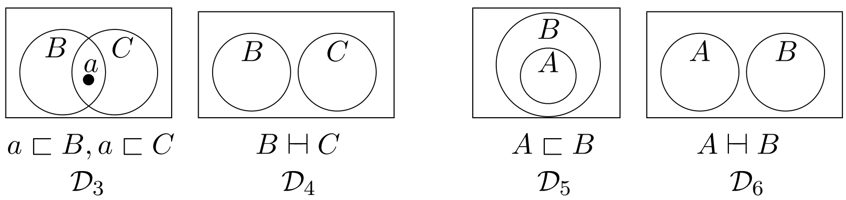


Fig. 5.8 Diagrams that cannot be unified due to the constraint for inconsistency.

Now we introduce unification rules of GDS. For each rule, we specify the following four components: (i) Premise, (ii) Precondition, (iii) Operation, and (iv) Conclusion.

(i) Each unification rule has two premises: a diagram $\mathcal{D}$ and a minimal diagram α. Premise describes the configurations of objects in these two diagrams.

$I(a) \neq I(b)$ holds for all points a and b, and in the proof system we have an axiom of the form $a \vdash\!\dashv b$. Given the syntax of EUL it is less stipulative to impose a restriction on unification rules. So we prefer the current approach.

(ii) Precondition specifies the constraints for determinacy and consistency imposed on each rule. Each rule is applicable only when the premise diagram $\mathcal{D}$ satisfies this condition.

(iii) In Operation, we describe a diagrammatic operation that allows us to introduce a new object into the premise $\mathcal{D}$ or to rearrange the configuration of objects in $\mathcal{D}$.

(iv) In Conclusion, we describe the configuration of the unified diagram $\mathcal{D} + \alpha$ that is obtained by the operation in (iii). The description is given in terms of the set of relations holding on $\mathcal{D} + \alpha$, i.e., $\mathsf{rel}(\mathcal{D} + \alpha)$. As before, for simplicity we omit a reflexive relation of the form $s \sqsubset s$ in the description of Conclusion.

Thus the definition of each unification rule can be read as follows:

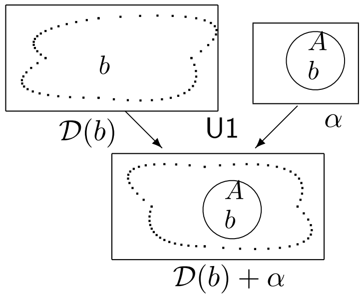
> Given a diagram $\mathcal{D}$ and a minimal diagram α with such and such configurations of objects (Premise), which satisfy such and such conditions (Precondition), one may modify the diagram $\mathcal{D}$ in such and such a way (Operation), so that the resulting diagram $\mathcal{D} + \alpha$ has such and such relations (Conclusion).

The unification rules of GDS are subdivided into ten rules (U1–U10 rules), depending on the number and type of objects shared by a diagram $\mathcal{D}$ and a minimal diagram α. We also have what we call *point insertion* rule, which allows us to combine two diagrams with the same configurations of contours. Here, instead of providing a complete description of unification rules, we give some illustrative examples. The full listing of inference rules in GDS is found in Section 5.4. See also Mineshima, Okada and Takemura (2012a) for a slightly different presentation of unification rules in GDS.

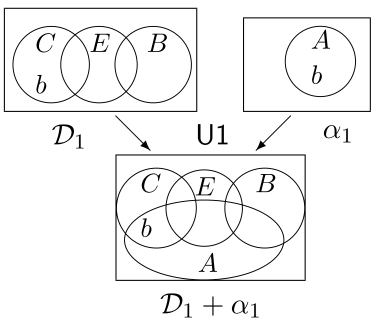
We start with a description of the U1 rule, which allows us to unify any diagram $\mathcal{D}$ containing a point b with any minimal diagram α in which $A \sqsubset B$ obtains. In the following descriptions, we sometimes denote a diagram $\mathcal{D}$ containing an object s by $\mathcal{D}(s)$, and a diagram $\mathcal{D}$ in which the relation $s \,\square\, t$ holds by $\mathcal{D}(s \,\square\, t)$, where $\square \in \{\sqsubset, \vdash\!\dashv, \bowtie\}$.

U1
Premise: $\mathcal{D}(b)$ and $\alpha : \{b \sqsubset A\}$
Precondition: In $\mathcal{D}(b)$, there is no point other than b, i.e., $pt(\mathcal{D}) = \{b\}$.
Operation on $\mathcal{D}(b)$: Let A be such that $b \sqsubset A$ and $A \bowtie X$ for all $X \in cr(\mathcal{D})$.
Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$

Schematically, U1 is applied as follows.



Note that when $\mathcal{D}(b)$ contains a contour, say B, the semantic relationship between B and A is underdetermined by the premies $\mathcal{D}(b)$ and α, hence we get $A \bowtie B$ in the unified diagram $\mathcal{D}(b) + \alpha$. The following is an example of an application of U1.



Here, following the Operation given in U1, we introduce the contour A into $\mathcal{D}_1$ so that $b \sqsubset A$, $A \bowtie B$ $A \bowtie C$, and $A \bowtie E$ obtain in $\mathcal{D}_1 + \alpha_1$. For the full understanding of the rule, let us provide the abstract description of this example, although it is somewhat complicated.

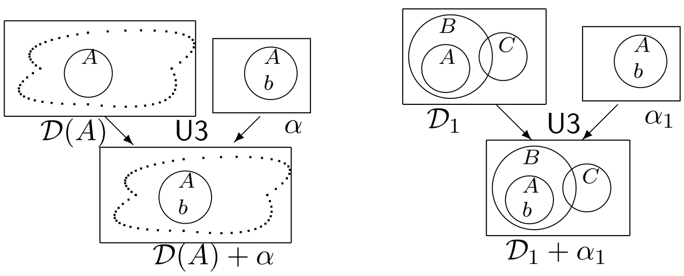
$$\frac{\mathcal{D}_1 : \{b \sqsubset C, b \vdash\!\dashv E, b \vdash\!\dashv B, C \bowtie E, C \vdash\!\dashv B, E \bowtie B\} \qquad \alpha_1 : \{b \sqsubset A\}}{\mathcal{D}_1 + \alpha_1 : \{b \sqsubset A, b \sqsubset C, b \vdash\!\dashv E, b \vdash\!\dashv B, A \bowtie C, A \bowtie E, A \bowtie B, C \bowtie E, C \vdash\!\dashv B, E \bowtie B\}}\ \mathsf{U1}$$

Here the premises and the conclusion are described in terms of the set of relations holding on them.

The U1 rule is a relatively trivial rule in that no new relations other than $\bowtie$-relations are introduced in the conclusion. The following is a more interesting example of unification rules:

> U3
> Premise: $\mathcal{D}(A)$ and $\alpha : \{b \sqsubset A\}$
> Precondition: In $\mathcal{D}(A)$, for all contour X, either $A \sqsubset X$ or $A \vdash\!\dashv X$, and for all point x, $x \vdash\!\dashv A$.
> Operation on $\mathcal{D}(A)$: Let b be such that $b \sqsubset A$.
> Conclusion:
> $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{b \sqsubset X \mid A \sqsubset X \text{ in } \mathcal{D}\} \cup \{b \vdash\!\dashv s \mid A \vdash\!\dashv s \text{ in } \mathcal{D}\}$

In this rule, the minimal diagram α, where $b \sqsubset A$ obtains, provides an instruction to introduce a point b in $\mathcal{D}(A)$. The schematic representation of this rule is shown on the left below, and a concrete example of its application on the right.



This example is abstractly represented as follows:

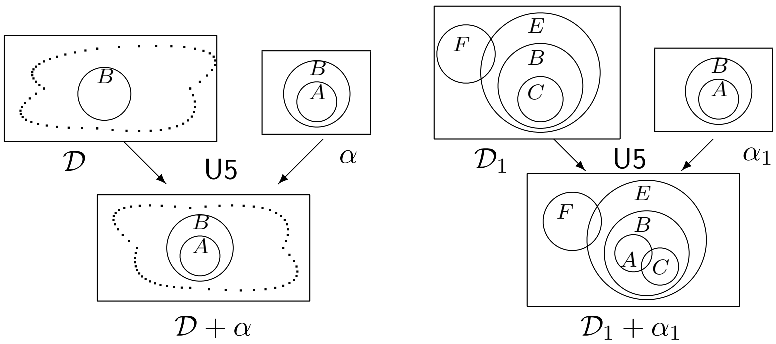
$$\frac{\mathcal{D}_1 : \{A \sqsubset B, A \vdash\!\dashv C, B \bowtie C\} \qquad \alpha_1 : \{b \sqsubset A\}}{\mathcal{D}_1 + \alpha_1 : \{b \sqsubset A, b \sqsubset B, b \vdash\!\dashv C, A \sqsubset B, A \vdash\!\dashv C, B \bowtie C\}}\ \mathsf{U3}$$

It should be remarked here that by the application of U3, the point b is introduced and thereby new relations $b \sqsubset B$ and $b \vdash\!\dashv C$ are *automatically* inferred in the conclusion.[9]

Next, the following is a definition of the U5 rule, where any diagram $\mathcal{D}(B)$ and any minimal diagram α with the relation $A \sqsubset B$ are unified.

U5
Premise: $\mathcal{D}(B)$ and $\alpha : \{A \sqsubset B\}$
Precondition: In $\mathcal{D}(B)$, for all point x, $x \vdash\!\dashv B$.
Operation on $\mathcal{D}(B)$: Let A be such that $A \sqsubset B$ and $A \bowtie X$ for all X with $X \sqsubset B$ or $B \bowtie X$.
Conclusion:
$\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{A \sqsubset X \mid B \sqsubset X \text{ in } \mathcal{D}\} \cup \{A \vdash\!\dashv s \mid B \vdash\!\dashv s \text{ in } \mathcal{D}\}$
$\cup \{A \bowtie X \mid X \sqsubset B \text{ or } B \bowtie X \text{ in } \mathcal{D}\}$

The schematic illustration of the U5 rule and a concrete example of its application are as follows:



The example on the right is abstractly represented in the following way:

[9] As is well known, such information that can be automatically read off from the conclusion of a diagrammatic operation is called *free ride* in Shimojima (1996), and given a detailed analysis from both logical and cognitive viewpoints there. See also Sato, Mineshima and Takemura (2010a, 2010b) for some discussion on a cognitive aspect of the free ride property of inferences with Euler and Venn diagrams.

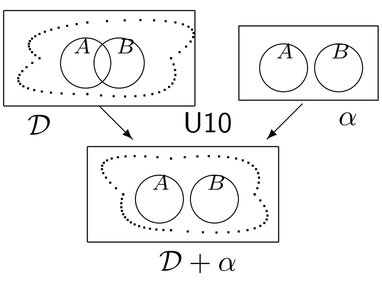
$$\frac{\mathcal{D}_1 : \{C \sqsubset B, B \sqsubset E, B \mathrel{\vdash\!\dashv} F, C \sqsubset E, C \mathrel{\vdash\!\dashv} F, E \bowtie F\} \qquad \alpha_1 : \{A \sqsubset B\}}{\mathcal{D}_1 + \alpha_1 : \{A \sqsubset B, A \bowtie C, A \sqsubset E, A \mathrel{\vdash\!\dashv} F, C \sqsubset B, B \sqsubset E, B \mathrel{\vdash\!\dashv} F, C \sqsubset E, C \mathrel{\vdash\!\dashv} F, E \bowtie F\}}\ \mathsf{U5}$$

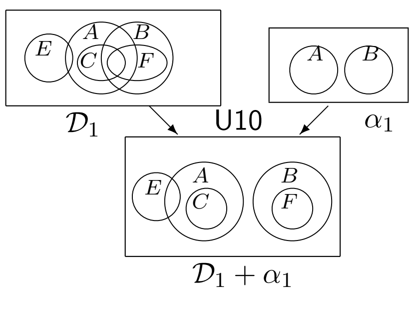
Finally, let us provide a case in which premise diagrams share *two* objects. The following is a definition of the U10 rule, where a diagram on which $A \bowtie B$ holds and a minimal diagram on which $A \mathrel{\vdash\!\dashv} B$ holds are unified.

U10
Premise: $\mathcal{D}(A \bowtie B)$ and $\alpha : \{A \mathrel{\vdash\!\dashv} B\}$
Precondition: In $\mathcal{D}(A \bowtie B)$, there is no object s such that $s \sqsubset A$ and $s \sqsubset B$.
Operation on $\mathcal{D}(A \bowtie B)$: Move A and B so that $A \mathrel{\vdash\!\dashv} B$.
Conclusion:

$$\Big(\mathsf{rel}(D) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \text{ in } \mathcal{D}\}\Big) \cup \{X \mathrel{\vdash\!\dashv} Y \mid X \sqsubset A \text{ and } Y \sqsubset B \text{ in } \mathcal{D}\}$$

Recall that $A \bowtie B$ is a semantically vacuous relation according to the semantics of EUL. In such a case, one must keep the stronger relation $A \mathrel{\vdash\!\dashv} B$ in the conclusion; hence the minimal diagram here can be regarded as an instruction to *separate* the contours A and B in $\mathcal{D}$. Note that by this separation, some objects lying in A and B are separated as well. The following are the the schematic representation of the rule and a concrete example of its application.





The abstract description of the example on the right is:

$$\frac{\mathcal{D}_1 : \{\mathbf{A} \bowtie \mathbf{B}, C \sqsubset A, A \bowtie E, \mathbf{A} \bowtie \mathbf{F}, \mathbf{B} \bowtie \mathbf{C}, B \mathrel{\vdash\!\dashv} E, F \sqsubset B, C \mathrel{\vdash\!\dashv} E, \mathbf{C} \bowtie \mathbf{F}, E \mathrel{\vdash\!\dashv} F\} \qquad \alpha_1 : \{A \mathrel{\vdash\!\dashv} B\}}{\mathcal{D}_1 + \alpha_1 : \{\mathbf{A} \mathrel{\vdash\!\dashv} \mathbf{B}, C \sqsubset A, A \bowtie E, \mathbf{A} \mathrel{\vdash\!\dashv} \mathbf{F}, \mathbf{B} \mathrel{\vdash\!\dashv} \mathbf{C}, B \mathrel{\vdash\!\dashv} E, F \sqsubset B, C \mathrel{\vdash\!\dashv} E, \mathbf{C} \mathrel{\vdash\!\dashv} \mathbf{F}, E \mathrel{\vdash\!\dashv} F\}}\ \mathsf{U10}$$

Here, for the easiness of understanding, the relations that change from $\bowtie$ to $\vDash$ by this inference are marked in boldface type. The other relations are unchanged. Note that not only the relation $A \bowtie B$ but also all the other crossing relations holding inside the intersection of A and B are set apart by this application of the U10 rule.

We also have a *deletion* rule, which allows us to remove an object from a given diagram. The precondition of an application of the deletion rule is that the resulting diagram contains at least two objects, thus it satisfies the well-formedness condition of EUL-diagrams. Following the pattern of the definition of unification rules, we provide the definition of the deletion rule in the following way.

Deletion
Premise: $\mathcal{D}(s)$
Precondition: $\mathcal{D}(s)$ has at least three objects.
Operation: Delete s from $\mathcal{D}(s)$.
Conclusion: $\mathsf{rel}(\mathcal{D}) \setminus \{\varphi \in \mathsf{rel}(\mathcal{D}) \mid s \text{ occurs in } \varphi\}$

There is an alternative way of formulating diagrammatic inferences rules: Following Avigad, Dean and Mumma (2009), which aims to provide a faithful formalization of geometric proofs in Euclid's *Elements*, we could distinguish two kinds of inference rules with diagrammatic objects, namely, what they call *construction rule* and *deduction rule*. The former allows us to introduce a new object into a given diagram, whereas the latter allows us to infer the facts about objects that have already been introduced. From our viewpoint, these two kinds of rules correspond to what we specify for each rule in Operation and Conclusion, respectively. Such a separation is attractive because with two kinds of rule, the *operational* and *declarative* aspects of diagrammatic proofs could be distinguished and analyzed explicitly. One consequence of this approach is that an application of a rule does not necessarily correspond to a rewriting step of given diagrams as is assumed in our current framework. For this reason, we stick to the current approach in this thesis.

Now it is easily verified that for each rule, the set $\mathsf{rel}(\mathcal{D}+\alpha)$ in Conclusion satisfies the properties of Lemma 5.5, i.e., the well-formed condition of EUL-diagrams. This shows that each rule preserves the well-formedness condition of EUL-diagrams.[10]

The notion of *diagrammatic proof* (*d-proof*, for short) in GDS is inductively defined as follows.

Definition 5.12 (Diagrammatic proofs in GDS) A *diagrammatic proof* (or *d-proof*, for short) of an EUL-diagram $\mathcal{D}$ from a set of EUL-diagrams $\mathcal{D}_1, \dots, \mathcal{D}_n$ is defined inductively as follows:

1. A diagram $\mathcal{D}$ is a d-proof of $\mathcal{D}$ from $\mathcal{D}$.
2. A minimal diagram in which $A \bowtie B$ holds is an axiom, and hence it is a d-proof of itself from the empty set.
3. Let π_1 be a d-proof of $\mathcal{F}$ from $\mathcal{D}_1, \dots, \mathcal{D}_n$ and π_2 be a d-proof of $\mathcal{E}$ from $\mathcal{E}_1, \dots, \mathcal{E}_m$, respectively. If $\mathcal{D}$ is obtained by an application of unification to $\mathcal{F}$ and $\mathcal{E}$, then the following figure is a d-proof of $\mathcal{D}$ from $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}_1, \dots, \mathcal{E}_m$.

$$\begin{array}{ccc} \pi_1 & & \pi_2 \\ \mathcal{F} & & \mathcal{E} \\ & \searrow \swarrow & \\ & \mathcal{D} & \end{array}$$

4. Let π_1 be a d-proof of $\mathcal{E}$ from $\mathcal{D}_1, \dots, \mathcal{D}_n$. If $\mathcal{D}$ is obtained by an application of deletion to $\mathcal{E}$, then the following figure is a d-proof of $\mathcal{D}$ from $\mathcal{D}_1, \dots, \mathcal{D}_n$.

$$\begin{array}{c} \pi_1 \\ \mathcal{E} \\ \downarrow \\ \mathcal{D} \end{array}$$

Here $\begin{array}{c}\pi\\ \mathcal{D}\end{array}$ means a d-proof π whose conclusion is $\mathcal{D}$. The *length* of a d-proof is defined as the number of applications of inference rules.

[10] There remains the question of how to find a concrete diagram satisfying a given abstract description. Such an algorithm to generate (draw) a diagram from an abstract description is widely discussed in the literature; see, e.g., Flower and Howse (2002), Stapleton, Howse, Rodgers and Zhang (2008), and references therein. See also Mineshima, Okada and Takemura (2012a) for some discussion on a possible implementation of a generation algorithm in the framework of GDS.

Definition 5.13 (Provability) Let $\overrightarrow{\mathcal{D}}$ be a set of EUL-diagrams. An EUL-diagram $\mathcal{E}$ is *provable* from $\overrightarrow{\mathcal{D}}$, written as $\overrightarrow{\mathcal{D}} \vdash \mathcal{E}$, if there is a d-proof of $\mathcal{E}$ in GDS from a sequence $\mathcal{D}_1, \dots, \mathcal{D}_m$ such that $\mathcal{D}_i \in \overrightarrow{\mathcal{D}}$. We call $\overrightarrow{\mathcal{D}}$ (resp. $\mathcal{E}$) premise (resp. conclusion) diagrams.

The diagrammatic inference system GDS is proved to be sound and complete with respect to the set-theoretical semantics in Section 5.2. For the completeness, we impose the model existence condition in the same way as GS. The proof is presented in Mineshima, Okada and Takemura (2012a).

Theorem 5.14 (Soundness of GDS) *Let $\overrightarrow{\mathcal{D}}$ be a set of* EUL-*diagrams, and let $\mathcal{E}$ be an* EUL-*diagram. If $\overrightarrow{\mathcal{D}} \vdash \mathcal{E}$ in* GDS, *then $\overrightarrow{\mathcal{D}} \models \mathcal{E}$.*

Theorem 5.15 (Completeness of GDS) *Let $\overrightarrow{\mathcal{D}}$ be a semantically consistent set of* EUL-*diagrams, and let $\mathcal{E}$ be an* EUL-*diagram. If $\overrightarrow{\mathcal{D}} \models \mathcal{E}$, then $\overrightarrow{\mathcal{D}} \vdash \mathcal{E}$ in* GDS.

5.4 Full list of inference rules of GDS

In this section, we present the full list of an axiom and inference rules in GDS. There are two types of rule, unification and deletion. The unification rules are divided into three groups, Group (I), (II), and (III). The rules in Group (I) and (II) are classified according to the number and type of objects shared by a diagram $\mathcal{D}$ and a minimal diagram α.

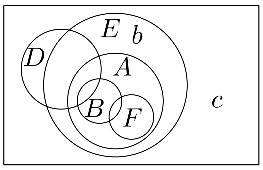
- In Group (I), $\mathcal{D}$ and α share one object. The rules in this group are further divided into two types: those in which one point is shared (U1-U2 rules) and those in which one contour is shared (U3-U8 rules). Each rule is specified by the relation holding on α, and has a constraint for determinacy.
- In Group (II), $\mathcal{D}$ and α share two contours (hence α consists of two contours). We distinguish two rules in this group (U9 and U10 rules), depending on whether $A \sqsubset B$ or $A \vdash\dashv B$ holds on α. Both rules have a constraint for consistency.

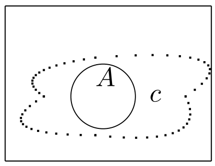
- The rule in Group (III) is Point Insertion, where neither of two premise diagrams is restricted to be minimal.

As noted in Section 5.2, for each unification rule, we specify the following four components: (i) Premise, (ii) Precondition, (iii) Operation, and (iv) Conclusion. In particular, the definition of unification rules in Group (I) and (II) can be read as follows:

> Given a diagram $\mathcal{D}$ and a minimal diagram α with such and such configurations of objects (Premise), which satisfy such and such conditions (Precondition), one may modify the diagram $\mathcal{D}$ in such and such a way (Operation), so that the resulting diagram $\mathcal{D}+\alpha$ has such and such relations (Conclusion).

For a better understanding of unification rule, we also give a schematic diagrammatic representation and a concrete example of each rule. The schematic representations of diagrams indicate the occurrence of objects in a context on a diagram. We write the indicated objects explicitly and indicate the context by "dots" as in the diagram shown to the right below. For example, when we need to indicate only A and c on the left hand diagram, we could write it as shown on the right.





In the following descriptions, we denote a diagram $\mathcal{D}$ containing an object s by $\mathcal{D}(s)$, and a diagram $\mathcal{D}$ in which $s \mathbin{\square} t$ obtains by $\mathcal{D}(s \mathbin{\square} t)$, where $\square \in \{\sqsubset, \vdash\!\dashv, \bowtie\}$.

Definition (**Inference rules of** GDS) *Axiom*, *unification*, and *deletion* of GDS are defined as follows.

Axiom:

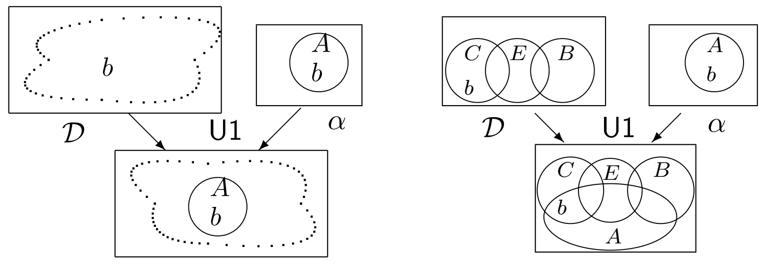
A1: For any contours A and B, any minimal diagram where $A \bowtie B$ holds is an axiom.

Unification: We denote by $\mathcal{D} + \alpha$ the diagram obtained by unifying a diagram $\mathcal{D}$ with a minimal diagram α. $\mathcal{D}+\alpha$ is defined when $\mathcal{D}$ and α share one or two objects.

(I) When $\mathcal{D}$ and α share one object:

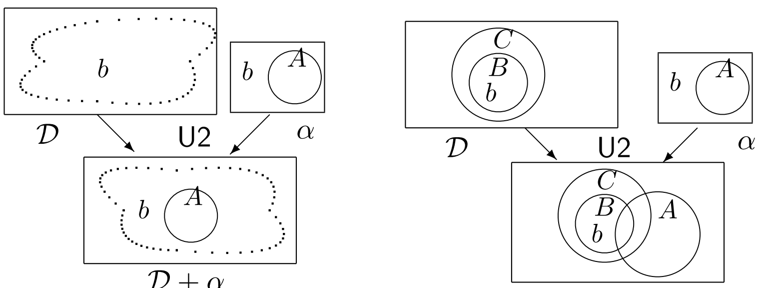
> U1
> Premise: $\mathcal{D}(b)$ and $\alpha : \{b \sqsubset A\}$
> Precondition: In $\mathcal{D}(b)$, there is no point other than b, i.e., $pt(\mathcal{D}) = \{b\}$.
> Operation on $\mathcal{D}(b)$: Let A be such that $b \sqsubset A$ and $A \bowtie X$ for all $X \in cr(\mathcal{D})$.
> Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$

Schema and Example (U1):



> U2
> Premise: $\mathcal{D}(b)$ and $\alpha : \{b \vDash A\}$
> Precondition: In $\mathcal{D}(b)$, there is no point other than b, i.e., $pt(\mathcal{D}) = \{b\}$.
> Operation on $\mathcal{D}(b)$: Let A be such that $b \vDash A$ and $A \bowtie X$ for all $X \in cr(\mathcal{D})$.
> Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{A \bowtie X \mid X \in cr(\mathcal{D})\}$

Schema and Example (U2):



U3

Premise: $\mathcal{D}(A)$ and $\alpha : \{b \sqsubset A\}$

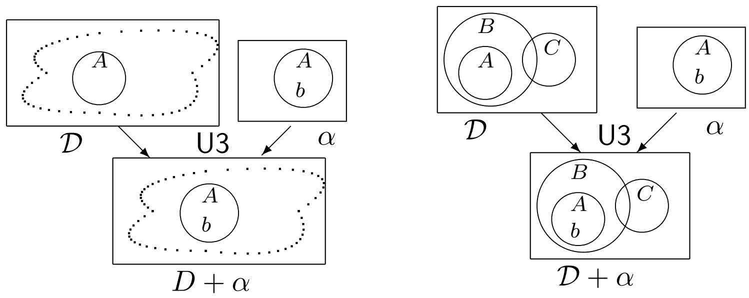
Precondition: In $\mathcal{D}(A)$, for all contours X, either $A \sqsubset X$ or $A \vdash\!\dashv X$, and for all point x, $x \vdash\!\dashv A$.

Operation on $\mathcal{D}(A)$: Let b be such that $b \sqsubset A$.

Conclusion:

$\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{b \sqsubset X \mid A \sqsubset X \text{ in } \mathcal{D}\} \cup \{b \vdash\!\dashv s \mid A \vdash\!\dashv s \text{ in } \mathcal{D}\}$

Schema and Example (U3):



U4

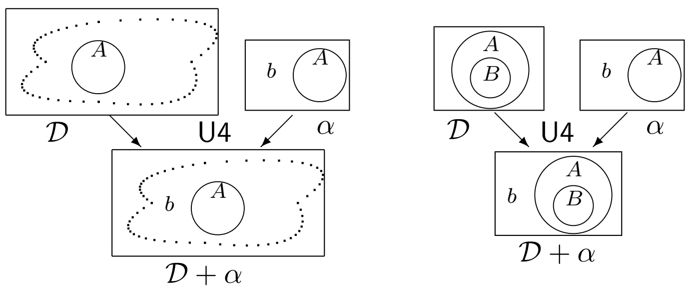
Premise: $\mathcal{D}(A)$ and $\alpha : \{b \vdash\!\dashv A\}$

Precondition: In $\mathcal{D}(A)$, for all object s, $s \sqsubset A$.

Operation on $\mathcal{D}(A)$: Let b be such that $b \vdash\!\dashv A$.

Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{b \vdash\!\dashv s \mid s \sqsubset A \text{ in } \mathcal{D}\}$

Schema and Example (U4):



U5

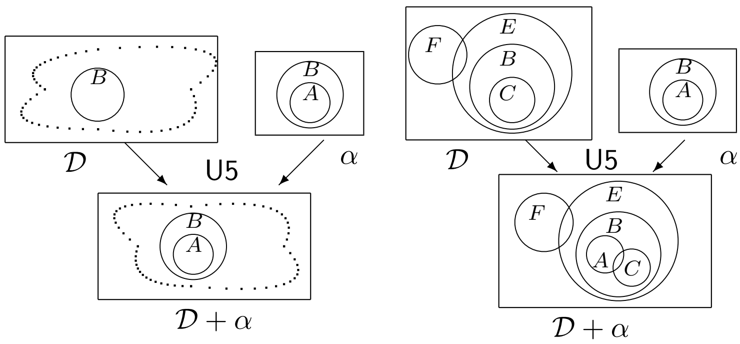
Premise: $\mathcal{D}(B)$ and $\alpha : \{A \sqsubset B\}$

Precondition: In $\mathcal{D}(B)$, for all point x, $x \vdash\dashv B$.

Operation on $\mathcal{D}(B)$: Let A be such that $A \sqsubset B$ and $A \bowtie X$ for all X with $X \sqsubset B$ or $B \bowtie X$.

Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{A \sqsubset X \mid B \sqsubset X \text{ in } \mathcal{D}\}$
$\cup \{A \vdash\dashv s \mid B \vdash\dashv s \text{ in } \mathcal{D}\} \cup \{A \bowtie X \mid X \sqsubset B \text{ or } B \bowtie X \text{ in } \mathcal{D}\}$

Schema and Example (U5):



U6

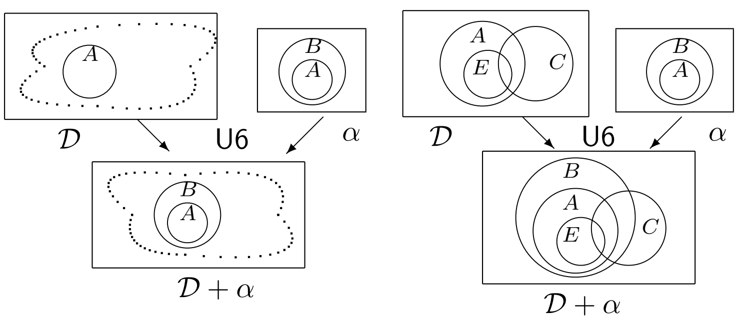
Premise: $\mathcal{D}(A)$ and $\alpha : \{A \sqsubset B\}$

Precondition: In $\mathcal{D}(A)$, for all point x, $x \sqsubset A$.

Operation on $\mathcal{D}(A)$: Let B be such that $A \sqsubset B$ and $B \bowtie X$ for all X with $A \sqsubset X$ or $A \vdash\dashv X$ or $A \bowtie X$.

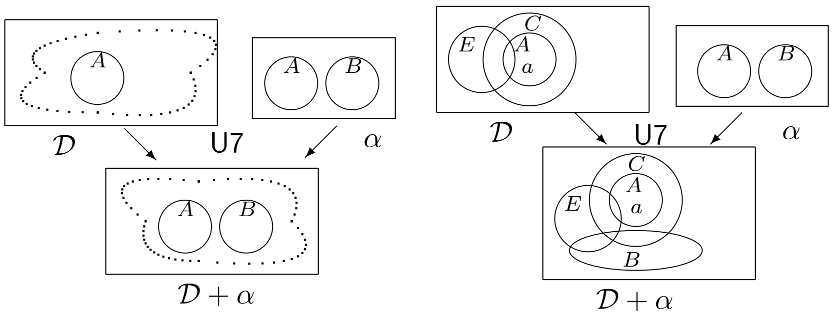
Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{s \sqsubset B \mid s \sqsubset A \text{ in } \mathcal{D}\}$
$\cup \{B \bowtie X \mid A \sqsubset X \text{ or } A \vdash\dashv X \text{ or } A \bowtie X \text{ in } \mathcal{D}\}$

Schema and Example (U6):



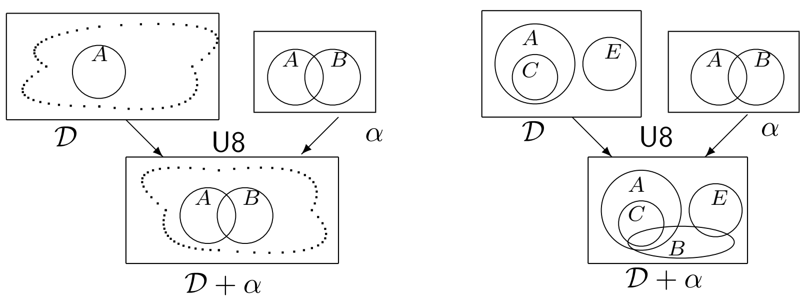
U7
Premise: $\mathcal{D}(A)$ and $\alpha : \{A \vDash B\}$
Precondition: In $\mathcal{D}(A)$, for all point x, $x \sqsubset A$.
Operation on $\mathcal{D}(A)$: Let B be such that $A \vDash B$ and $B \bowtie X$ for all X with $A \sqsubset X$ or $A \vDash X$ or $A \bowtie X$.
Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{B \vDash s \mid s \sqsubset A \text{ in } \mathcal{D}\}$
$\cup \{B \bowtie X \mid A \sqsubset X \text{ or } A \vDash X \text{ or } A \bowtie X \text{ in } \mathcal{D}\}$

Schema and Example (U7):



U8
Premise: $\mathcal{D}(A)$ and $\alpha : \{A \bowtie B\}$
Precondition: In $\mathcal{D}(A)$, there is no point, i.e., $pt(\mathcal{D}) = \emptyset$.
Operation on $\mathcal{D}(A)$: Let B be such that $A \bowtie B$.
Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{B \bowtie X \mid X \in cr(\mathcal{D})\}$

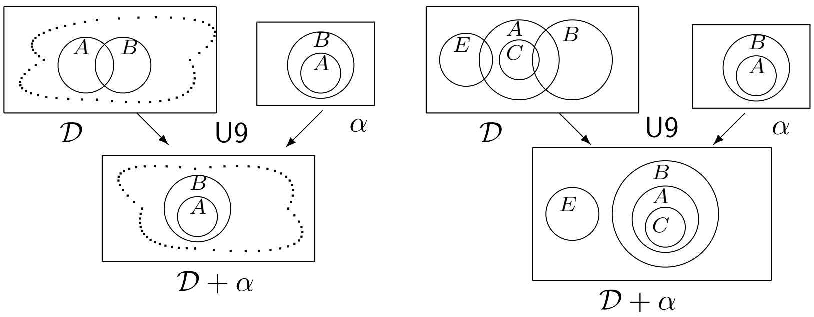
Schema and Example (U8):



(II) When $\mathcal{D}$ and α share two contours:

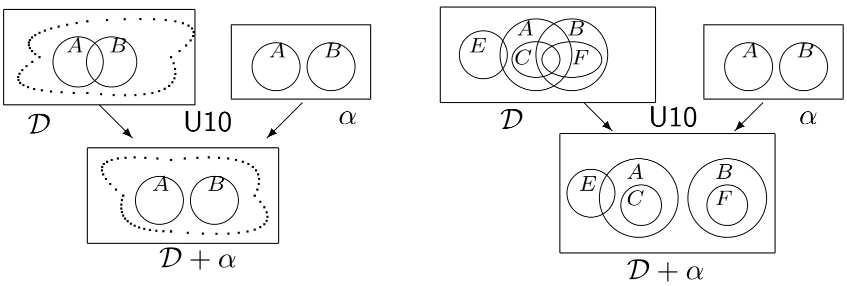
> U9
>
> Premise: $\mathcal{D}(A \bowtie B)$ and $\alpha : \{A \sqsubset B\}$
>
> Precondition: In $\mathcal{D}(A \bowtie B)$, there is no object s such that $s \sqsubset A$ and $s \vdash\!\dashv B$.
>
> Operation on $\mathcal{D}(A \bowtie B)$: Move A and B so that $A \sqsubset B$.
>
> Conclusion:
>
> $\big(\mathsf{rel}(D) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } (B \sqsubset Y \text{ or } B \vdash\!\dashv Y) \text{ in } \mathcal{D}\}\big)$
> $\cup \{X \sqsubset Y \mid X \sqsubset A \text{ and } B \sqsubset Y \text{ in } \mathcal{D}\}$
> $\cup \{X \vdash\!\dashv Y \mid X \sqsubset A \text{ and } B \vdash\!\dashv Y \text{ in } \mathcal{D}\}$

Schema and Example (U9):



> U10
>
> Premise: $\mathcal{D}(A \bowtie B)$ and $\alpha : \{A \vdash\!\dashv B\}$
>
> Precondition: In $\mathcal{D}(A \bowtie B)$, there is no s such that $s \sqsubset A$ and $s \sqsubset B$.
>
> Operation on $\mathcal{D}(A \bowtie B)$: Move A and B so that $A \vdash\!\dashv B$.
>
> Conclusion:
>
> $\big(\mathsf{rel}(D) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \text{ in } \mathcal{D}\}\big) \cup \{X \vdash\!\dashv Y \mid X \sqsubset A \text{ and } Y \sqsubset B \text{ in } \mathcal{D}\}$

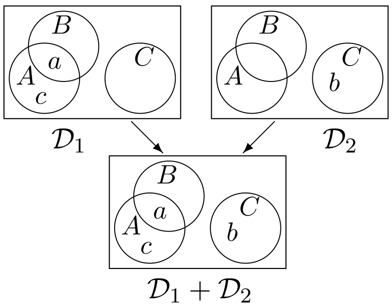
Schema and Example (U10):



(III)

Point Insertion
Premise: $\mathcal{D}_1$ and $\mathcal{D}_2(a)$
Precondition: (i) For any contour A, B and for any $\Box \in \{\sqsubset, \vdash\!\dashv, \bowtie\}$, $A \Box B$ in $\mathcal{D}_1$ iff $A \Box B$ in $\mathcal{D}_2$; (ii) $pt(\mathcal{D}_2) = \{a\}$ and $a \notin pt(\mathcal{D}_1)$; (iii) there is no point b in D_1 such that for any contour X, $b \sqsubset X$ in $\mathcal{D}_1$ iff $a \sqsubset X$ in $\mathcal{D}_2$.
Operation on $\mathcal{D}_1$: Let a be such that for all contour X, $a \sqsubset X$ in $\mathcal{D}_1$ iff $a \sqsubset X$ in $\mathcal{D}_2$.
Conclusion: $\mathsf{rel}(\mathcal{D}_1) \cup \mathsf{rel}(\mathcal{D}_2) \cup \{a \vdash\!\dashv x \mid x \in pt(\mathcal{D}_1)\}$

Example (Point Insertion):



Deletion:

Deletion
Premise: $\mathcal{D}(s)$
Precondition: $\mathcal{D}(s)$ has at least three objects.
Operation: Delete s from $\mathcal{D}(s)$.
Conclusion: $\mathsf{rel}(\mathcal{D}) \setminus \{\varphi \mid s \text{ occurs in } \varphi\}$

5.5 The relation between GS and GDS

We are now in a position to investigate a relationship between GS and GDS. We show that GDS can be faithfully embedded into GS by providing a proof-transformation procedure. This means that we interpret the diagrammatic proofs in GDS in terms of the proofs in GS, and in this way we view the symbolic and diagrammatic proofs from a unified perspective.

We start with providing a translation of EUL-diagrams into formulas in GS. We denote the translation of a diagram $\mathcal{D}$ by $\mathcal{D}^\sharp$. Throughout this section, we adopt the following notational convention. If $\mathcal{P}$ and $\mathcal{Q}$ are sets of atomic formulas in GS, we write

$$\mathcal{P}, \mathcal{Q}$$

to abbreviate $\mathcal{P} \cup \mathcal{Q}$. Recall that an atomic formula P and its singleton $\{P\}$ are syntactically identified in GS. Thus if $\mathcal{P}$ is a set of atomic formulas and P, Q are atomic formula, we write

$$\mathcal{P}, P$$

to mean $\mathcal{P} \cup \{P\}$ and

$$P, Q$$

to mean $\{P, Q\}$.

The EUL-relation $\sqsubset$ and $\vdash\!\dashv$ are translated into the corresponding relation symbols $\sqsubset$ and $\vdash\!\dashv$ in GS. The translation of an EUL-diagram into a GS-formula simply consists of neglecting an EUL-relation of the form $A \bowtie B$.

Definition 5.16 (Translation of EUL-diagrams) Each named contour A is translated into a corresponding general term A in GS, and each named point a is translated into a corresponding singular term a in GS. Then each EUL-relation φ is translated into a GS-formula $\varphi^\sharp$ by:

$$(s \sqsubset t)^\sharp := s \sqsubset t$$
$$(s \vdash\!\dashv t)^\sharp := s \vdash\!\dashv t$$
$$(s \bowtie t)^\sharp := \emptyset$$

Let $\mathcal{D}$ be an EUL-diagram whose set of relation $\mathsf{rel}(\mathcal{D})$ is $\{\varphi_1, \ldots, \varphi_n\}$. Then the diagram $\mathcal{D}$ is translated into a GS-formula $\mathcal{D}^\sharp$ by:

$$\mathcal{D}^\sharp := \varphi_1^\sharp, \ldots, \varphi_n^\sharp.$$

We next provide a translation of inference rules of GDS. An important characteristic of a diagrammatic proof in GDS, as compared to a proof in GS, is that all the information contained in premises is preserved in the conclusion. To illustrate the point, let us compare a simple GS-proof on the left below, which is an application of ($\sqsubset$), and the abstract representation of the corresponding one-step inference (an application of U5) in GDS with the same premises, shown on the right.

$$\frac{A \sqsubset B \quad B \sqsubset C}{A \sqsubset C}\ (\sqsubset) \qquad\qquad \frac{\mathcal{D}_1 : \{A \sqsubset B\} \qquad \mathcal{D}_2 : \{B \sqsubset C\}}{\mathcal{D}_1 + \mathcal{D}_2 : \{A \sqsubset B, B \sqsubset C, A \sqsubset C\}}\ \mathsf{U6}$$

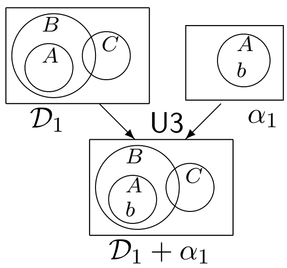
In the case of the GS-proof, only the relation that is newly introduced by a ($\sqsubset$) inference, namely, $A \sqsubset C$, appears in the conclusion. By contrast, in the case of the GDS-proof, all the relations contained in the premises, namely, $A \sqsubset B$ and $B \sqsubset C$, reappear in the conclusion as well. As a result, in order to simulate this single application of U5 within GS, we need successive applications of the (+) rule in the following way:

$$\frac{\dfrac{A \sqsubset B \quad B \sqsubset C}{A \sqsubset B, B \sqsubset C}\ (+) \qquad \dfrac{A \sqsubset B \quad B \sqsubset C}{A \sqsubset C}\ (\sqsubset)}{A \sqsubset B, B \sqsubset C, A \sqsubset C}\ (+)$$

Here each premise and conclusion of the diagrammatic proof is translated into a GS-formulas of the same form as its EUL-relation, that is:

$(\mathcal{D}_1)^\sharp = A \sqsubset B$;
$(\mathcal{D}_2)^\sharp = B \sqsubset C$;
$(\mathcal{D}_1 + \mathcal{D}_2)^\sharp = A \sqsubset B, B \sqsubset C, A \sqsubset C$.

As a more complex example, consider the following diagrammatic proof, which we mention in Section 5.3 as an example of application of U3.



We repeat the abstract representation of this proof.

$$\frac{\mathcal{D}_1 : \{A \sqsubset B, A \mathrel{\vdash\!\dashv} C, B \bowtie C\} \qquad\qquad \alpha_1 : \{b \sqsubset A\}}{\mathcal{D}_1 + \alpha_1 : \{b \sqsubset A, b \sqsubset B, b \mathrel{\vdash\!\dashv} C, A \sqsubset B, A \mathrel{\vdash\!\dashv} C, B \bowtie C\}}\ \mathsf{U3}$$

The premises and the conclusion are translated as follows:

$$(\mathcal{D}_1)^\sharp = A \sqsubset B, A \mathrel{\vdash\!\dashv} C$$
$$(\alpha_1)^\sharp = b \sqsubset A$$
$$(\mathcal{D}_1 + \alpha_1)^\sharp = b \sqsubset A, b \sqsubset B, b \mathrel{\vdash\!\dashv} C, A \sqsubset B, A \mathrel{\vdash\!\dashv} C$$

Following the same strategy as above, the proof is simulated in GS as follows.

$$\dfrac{\dfrac{\dfrac{(\mathcal{D}_1)^\sharp \quad b \sqsubset A}{(\mathcal{D}_1)^\sharp, b \sqsubset A}\,(+) \qquad \dfrac{b \sqsubset A \quad \dfrac{(\mathcal{D}_1)^\sharp}{A \sqsubset B}\,(-)}{b \sqsubset B}\,(\sqsubset)}{(\mathcal{D}_1)^\sharp, b \sqsubset A, b \sqsubset B}\,(+) \qquad \dfrac{b \sqsubset A \quad \dfrac{(\mathcal{D}_1)^\sharp}{A \mathrel{\vdash\!\dashv} C}\,(-)}{b \mathrel{\vdash\!\dashv} C}\,(\mathrel{\vdash\!\dashv})}{(\mathcal{D}_1)^\sharp, b \sqsubset A, b \sqsubset B, b \mathrel{\vdash\!\dashv} C}\,(+)$$

An application of unification in GDS with premise diagrams $\mathcal{D}$ and α gives rise to a diagram $\mathcal{D} + \alpha$ whose set of relations is decomposed into the relations already contained in the premises, i.e.,

$$\mathsf{rel}(\mathcal{D}) \cup \mathsf{rel}(\alpha),$$

and the newly introduced ones,

$$\{\varphi_1, \ldots, \varphi_n\}.$$

Figure 5.9 shows the general translation schema of an application of unification rule, whose premises are diagrams $\mathcal{D}$ and α and whose newly introduced relations are $\varphi_1, \ldots, \varphi_n$. Here π consists of successive applications of $(+)$, and each of $\pi_1, \ldots, \pi_n$ consists of applications of $(\sqsubset)$ and $(\mathrel{\vdash\!\dashv})$, following a (possibly empty) application of $(-)$ to $\mathcal{D}^\sharp$.

Using this general translation schema, we show the following:

Theorem 5.17 (Soundness) *Let $\mathcal{D}_1, \ldots, \mathcal{D}_n, \mathcal{E}$ be* EUL*-diagrams. Every diagrammatic proof in* GDS *of $\mathcal{E}$ from $\mathcal{D}_1, \ldots, \mathcal{D}_n$ can be translated into a proof in* GS *of $\mathcal{E}^\sharp$ from $\mathcal{D}_1^\sharp, \ldots, \mathcal{D}_n^\sharp$.*

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\mathcal{D}^\sharp \quad \alpha^\sharp}{\mathcal{D}^\sharp, \alpha^\sharp}\,(+) \qquad \begin{array}{c}\mathcal{D}^\sharp \ \ \alpha^\sharp \\ \vdots\ \pi_1 \\ \varphi_1^\sharp\end{array}}{\mathcal{D}^\sharp, \alpha^\sharp, \varphi_1^\sharp}\,(+) \\ \ddots\ \pi \\ \mathcal{D}^\sharp, \alpha^\sharp, \varphi_1^\sharp, \dots, \varphi_{n-1}^\sharp\end{array} \qquad \begin{array}{c}\mathcal{D}^\sharp \ \ \alpha^\sharp \\ \vdots\ \pi_n \\ \varphi_n^\sharp\end{array}}{\mathcal{D}^\sharp, \alpha^\sharp, \varphi_1^\sharp, \dots, \varphi_n^\sharp}\,(+)$$

Fig. 5.9 The general translation schema of unification rule.

Proof. By induction on the length of proofs in GDS. We only show two interesting cases: U5 and U10. It is straightforward to see that the other cases are treated in a similar way. A full list of inference rules in GDS can be found in the next section.

1. The Premise and Conclusion of U5 are:

 Premise: $\mathcal{D}(B)$ and $\alpha : \{A \sqsubset B\}$
 Conclusion: $\mathsf{rel}(D) \cup \mathsf{rel}(\alpha) \cup \{A \sqsubset X \mid B \sqsubset X \text{ in } \mathcal{D}\}$
 $\cup \{A \vdash\!\dashv s \mid B \vdash\!\dashv s \text{ in } \mathcal{D}\} \cup \{A \bowtie X \mid X \sqsubset B \text{ or } B \bowtie X \text{ in } \mathcal{D}\}$.

Here the "newly introduced" relations are ones of the forms $A \sqsubset X$, $A \vdash\!\dashv s$, and $A \bowtie X$. The last one is simply neglected in the translation. So it suffices to show how to derive $A \sqsubset X$ and $A \vdash\!\dashv s$ from the translations of the premises, i.e., $\mathcal{D}^\sharp$ and $\alpha^\sharp = A \sqsubset B$. They are derived as follows:

$$\dfrac{A \sqsubset B \quad \dfrac{\mathcal{D}^\sharp}{B \sqsubset X}\,(-)}{A \sqsubset X}\,(\sqsubset) \qquad\qquad \dfrac{A \sqsubset B \quad \dfrac{\mathcal{D}^\sharp}{B \vdash\!\dashv s}\,(-)}{A \vdash\!\dashv s}\,(\vdash\!\dashv)$$

Then by combining the derived formulas along the general schema in Figure 5.9, we obtain the desired proof in GS.

2. The Premise and Conclusion of U10 are:

 Premise: $\mathcal{D}(A \bowtie B)$ and $\alpha : \{A \vdash\!\dashv B\}$
 Conclusion: $\big(\mathsf{rel}(D) \setminus \{X \bowtie Y \mid X \sqsubset A \text{ and } Y \sqsubset B \text{ in } \mathcal{D}\}\big)$
 $\cup \{X \vdash\!\dashv Y \mid X \sqsubset A \text{ and } Y \sqsubset B \text{ in } \mathcal{D}\}$

Again, it suffices to show how to derive a newly introduced relation of the form $X \vdash\!\dashv Y$ from given premises $\mathcal{D}^\sharp$ and $\alpha^\sharp = A \vdash\!\dashv B$. The following proof in GS yields the desired result.

$$\dfrac{\dfrac{A \vdash\!\dashv B \quad \dfrac{\mathcal{D}^\sharp}{X \sqsubset A}\,(-)}{X \vdash\!\dashv B}\,(\vdash\!\dashv) \qquad \dfrac{\mathcal{D}^\sharp}{Y \sqsubset B}\,(-)}{X \vdash\!\dashv Y}\,(\vdash\!\dashv)$$

■

The converse of Theorem 5.17 is proved using the normalization theorem of GS. Recall that a normal proof in GS can be divided into three parts, the deletion part, the transitive part, and the addition part (Corollary 2.10). While it is immediate to translate the deletion parts and the transitive parts into diagrammatic proofs, it is not a trivial task to translate the addition parts, because (i) there are constraints imposed on unification that forbid combining certain forms of diagrams, and (ii) by an application of the (+) rule, there may arise, in the middle of a proof, a formula that does not correspond to any GDS-diagram.

Example 5.18 Consider three diagrams:

$\mathcal{D}_1 : \{A \sqsubset B, A \vdash\!\dashv C, B \bowtie C\}$,
$\mathcal{D}_2 : \{d \vdash\!\dashv C, d \sqsubset C, B \vdash\!\dashv C\}$,
$\mathcal{E} : \{d \vdash\!\dashv A, d \vdash\!\dashv B, d \sqsubset C, A \sqsubset B, A \vdash\!\dashv C, B \vdash\!\dashv C\}$.

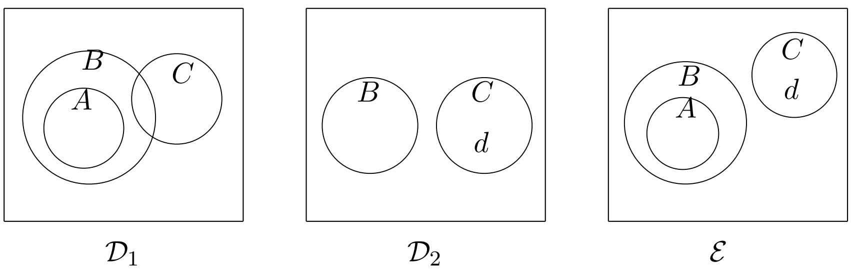


Fig. 5.10 Premise diagrams $\mathcal{D}_1$ and D_2 and conclusion diagram $\mathcal{E}$.

One way to prove $\mathcal{E}^\sharp$ from $\mathcal{D}_1^\sharp$ and $D_2^\sharp$ in GS is as follows:

$$\dfrac{\mathcal{D}_2^\sharp \qquad \dfrac{D_1^\sharp \qquad \dfrac{\dfrac{\mathcal{D}_1^\sharp}{A \sqsubset B}\,(-) \qquad \dfrac{\mathcal{D}_2^\sharp}{B \mathrel{\vdash\!\dashv} d}\,(-)}{A \mathrel{\vdash\!\dashv} d}\,(\vdash\!\dashv)}{\mathcal{D}_1^\sharp, A \mathrel{\vdash\!\dashv} d}\,(+)\clubsuit}{\mathcal{E}^\sharp}\,(+)$$

Here, omitting a formula of the form $s \sqsubset s$, we have:

$\mathcal{D}_1^\sharp = A \sqsubset B, A \mathrel{\vdash\!\dashv} C;$
$\mathcal{D}_2^\sharp = d \mathrel{\vdash\!\dashv} B, d \sqsubset C, B \mathrel{\vdash\!\dashv} C;$
$\mathcal{E}^\sharp = d \mathrel{\vdash\!\dashv} A, d \mathrel{\vdash\!\dashv} B, d \sqsubset C, A \sqsubset B, A \mathrel{\vdash\!\dashv} C, B \mathrel{\vdash\!\dashv} C.$

The subproof from $\mathcal{D}_1^\sharp$ and $\mathcal{D}_2^\sharp$ to $A \mathrel{\vdash\!\dashv} d$ can be translated into a diagrammatic proof in GDS as follows.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}_1}{\{A \sqsubset B\}}\,\textsf{deletion} \qquad \dfrac{\mathcal{D}_2}{\{d \mathrel{\vdash\!\dashv} B\}}\,\textsf{deletion}}{\{A \sqsubset B, d \mathrel{\vdash\!\dashv} A, d \mathrel{\vdash\!\dashv} B\}}\,\textsf{unification (U4)}}{\{d \mathrel{\vdash\!\dashv} A\}}\,\textsf{deletion}$$

Here the first two applications of $(-)$ in the GS-proof are translated as applications of deletion in GDS, and the application of $(\vdash\!\dashv)$ is translated as an application of unification (U4) followed by an application of deletion. Now the problem is how to translate the application of $(+)$ marked by $\clubsuit$. Although the GS-formula $A \mathrel{\vdash\!\dashv} d$ directly corresponds to a minimal diagram, say α, of the form $\{A \mathrel{\vdash\!\dashv} d\}$, and hence it may be a premise of unification, we cannot unify the diagrams $\mathcal{D}$ and α due to the constraint for determinacy: the location of the point d is indeterminate in a unified diagram. (Since $\mathcal{D}$ and α share the contour A, the type of rule that could be applied here is U4. For the precise description of the precondition of an application of U4, see Section 5.4.) Furthermore, even if such an application of $(+)$ observes the constraint, the resulting formula might not correspond to any EUL-diagram, as is the formula $\mathcal{D}_1^\sharp, A \mathrel{\vdash\!\dashv} d$ in our example.

To solve this problem, we essentially use a proof-construction of GDS given in Mineshima, Okada and Takemura (2012a), called a *canonical dia-*

grammatic proof. In the following proof, we will outline this construction; for more details, see the proof of Theorem 3.14 of that paper. See also Example 5.20 below.

Theorem 5.19 (Faithfulness) *Let $\overrightarrow{\mathcal{D}}$ be a set of* EUL*-diagrams, and let $\mathcal{E}$ be an* EUL*-diagram. Every proof in* GS *of $\mathcal{E}^\sharp$ from $\overrightarrow{\mathcal{D}}^\sharp$ can be translated into a diagrammatic proof in* GDS *of $\mathcal{E}$ from $\overrightarrow{\mathcal{D}}$ or a subset of $\overrightarrow{\mathcal{D}}$, where $\overrightarrow{\mathcal{D}}^\sharp := \{\mathcal{D}^\sharp \mid \mathcal{D} \in \overrightarrow{\mathcal{D}}\}$.*

Proof. By Theorem 2.8, any proof π in GS of $\mathcal{E}^\sharp$ from $\overrightarrow{\mathcal{D}}^\sharp$ can be transformed into a normal proof π' of $\mathcal{E}^\sharp$ from $\overrightarrow{\mathcal{D}}^\sharp$ or a subset of $\overrightarrow{\mathcal{D}}^\sharp$. By Corollary 2.10, π' can be divided into three parts, of which (i) the deletion part is translated as applications of deletion in GDS, and (ii) the transitive part, consisting of ($\sqsubset$) and ($\vdash\!\dashv$), is translated as a combination of unification (more specifically, U3–U7) and deletion in the following way:

$$\frac{s \sqsubset u \quad u \sqsubset t}{s \sqsubset t}\,(\sqsubset) \quad \rightsquigarrow \quad \cfrac{\cfrac{\{s \sqsubset u\} \qquad \{u \sqsubset t\}}{\{s \sqsubset u, u \sqsubset t, s \sqsubset t\}}\,\text{unification}}{\{s \sqsubset t\}}\,\text{deletion}$$

$$\frac{s \sqsubset u \quad u \vdash\!\dashv t}{s \vdash\!\dashv t}\,(\vdash\!\dashv) \quad \rightsquigarrow \quad \cfrac{\cfrac{\{s \sqsubset u\} \qquad \{u \vdash\!\dashv t\}}{\{s \sqsubset u, u \sqsubset t, s \vdash\!\dashv t\}}\,\text{unification}}{\{s \vdash\!\dashv t\}}\,\text{deletion}$$

Then by using these translations as well as applying deletion to the premise diagrams $\overrightarrow{\mathcal{D}}$ if necessary, we can construct proofs in GDS of all the minimal diagrams that correspond to the atomic formulas in GS derivable from $\overrightarrow{\mathcal{D}}^\sharp$. Let $\alpha_1, \ldots, \alpha_n$ be such minimal diagrams. For the translation of the addition part, the following proof-construction yields the desired proof in GDS.

1. Among the minimal diagrams $\alpha_1, \ldots, \alpha_n$, we pick up the ones containing a point corresponding to a singular term appearing in $\mathcal{E}^\sharp$. From them, we construct EUL-diagrams $\mathcal{D}'_1, \ldots, \mathcal{D}'_k$ such that each of them contains exactly one point as well as all contours that correspond to the general terms occurring in $\mathcal{E}^\sharp$, and in each of them $A \bowtie B$ holds for any pair of contours A and B. U1 and U2 rules are used for this proof-construction.

2. Then, by unifying $\mathcal{D}_1, \ldots, \mathcal{D}_k$ with the Point Insertion rule, we can construct a diagram $\mathcal{E}'$ consisting of all the points and contours which correspond to the terms occurring in $\mathcal{E}^\sharp$.

3. Finally, among $\alpha_1, \ldots, \alpha_n$, we pick up all the point-free minimal diagrams in each of which we have a relation of the form $A \sqsubset B$ or $A \vdash\!\dashv B$ that corresponds to each atomic formula in $\mathcal{E}^\sharp$. We then construct a diagram $\mathcal{F}$, by unifying the diagram $\mathcal{E}'$ with the point-free minimal diagrams one by one (using U9 and U10 rules). It is shown that the diagram $\mathcal{F}$ is equal to $\mathcal{E}$.

A detailed formalization of each step and the verification of the fact that the constructed diagram $\mathcal{F}$ coincides with the conclusion $\mathcal{E}$ are found in Mineshima, Okada and Takemura (2012a). (The proof essentially uses the completeness of GDS with respect to minimal diagrams.) ■

Example 5.20 Let us go back to the example in 5.18. In order to obtain a diagrammatic proof of $\mathcal{E}$ from $\mathcal{D}_1$ and $\mathcal{D}_2$, we first construct diagrammatic proofs of minimal diagrams:

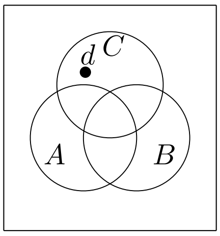
$$\alpha_1 : \{d \vdash\!\dashv A\},\ \alpha_2 : \{d \vdash\!\dashv B\},\ \alpha_3 : \{d \sqsubset C\},$$
$$\alpha_4 : \{A \sqsubset B\},\ \alpha_5 : \{A \vdash\!\dashv C\},\ \alpha_6 : \{B \vdash\!\dashv C\}.$$

α_1 is derived as shown in Example 5.18; $\alpha_2, \alpha_3, \alpha_6$ are derived by applying deletion to $\mathcal{D}_2$; α_4, α_5 are derived by applying deletion to $\mathcal{D}_1$. Now we construct a proof of $\mathcal{E}$ from $\alpha_1, \ldots, \alpha_6$, following the three steps indicated in the proof of Theorem 5.19.

1. First, we combine all the minimal diagrams containing the point d, i.e., α_1, α_2, and α_3 by using U1 and U2. Let $\mathcal{D}'$ be the resulting diagrams. The proof is as follows:

$$\dfrac{\dfrac{\alpha_1 : \{d \vdash\!\dashv A\} \quad \alpha_2 : \{b \vdash\!\dashv B\}}{\{d \vdash\!\dashv A, d \vdash\!\dashv B, A \bowtie B\}}\,\mathsf{U2} \qquad \alpha_3 : \{d \sqsubset C\}}{\mathcal{D}' : \{d \vdash\!\dashv A, d \vdash\!\dashv B, d \sqsubset C, A \bowtie B, B \bowtie C, A \bowtie C\}}\,\mathsf{U1}$$

The diagram $\mathcal{D}'$ looks like:



$\mathcal{D}'$

In Mineshima, Okada and Takemura (2012a), the diagrams of such a form are called *Venn-like diagrams* for the obvious reason.

2. Since d is the only point to be considered, we do not have to apply the Point Insertion rule in this example.

3. Then we unify $\mathcal{D}'$ with the point-free minimal diagrams $\alpha_4, \alpha_5, \alpha_6$ using U9 and U10 as follows.

$$\dfrac{\dfrac{\dfrac{\mathcal{D}' : \{d \dashv\vdash A, d \dashv\vdash B, d \sqsubset C, A \bowtie B, B \bowtie C, A \bowtie C\} \quad \alpha_4 : \{A \sqsubset B\}}{\{d \dashv\vdash A, d \dashv\vdash B, d \sqsubset C, A \sqsubset B, B \bowtie C, A \bowtie C\}}\,\mathsf{U9} \quad \alpha_6 : \{B \dashv\vdash C\}}{\{d \dashv\vdash A, d \dashv\vdash B, d \sqsubset C, A \sqsubset B, B \dashv\vdash C, A \bowtie C\}}\,\mathsf{U10} \quad \alpha_5 : \{A \dashv\vdash C\}}{\mathcal{E} : \{d \dashv\vdash A, d \dashv\vdash B, d \sqsubset C, A \sqsubset B, B \dashv\vdash C, A \dashv\vdash C\}}\,\mathsf{U10}$$

Here the applications of U9 and U10 change the $\bowtie$ relations in $\mathcal{D}'$ into $\sqsubset$ or $\dashv\vdash$ relations, and yields the required diagrammatic proof of $\mathcal{E}$. See again the diagram $\mathcal{E}$ in Figure 5.10.

To sum up, we have provided a sound and faithful translation of GDS into GS. Now together with the fact that we have a sound and faithful translation from CS (CS$^+$) into GS, and one from GS into ML, we can establish translations between CS (CS$^+$) and GDS, and between ML and GDS. The relationships between the inference systems we are concerned with so far are summarized in Figure 5.11 (cf. Figure 1.3 in Section 1).

In order to give translations between CS (CS$^+$), GDS, and ML, we first need to introduce a few notions. A categorical sentence can be translated into an EUL-diagram via its translation to GS-formulas. A problem here is that for each categorical sentence S, its translation S° (or $S^\bullet$) in GS is not

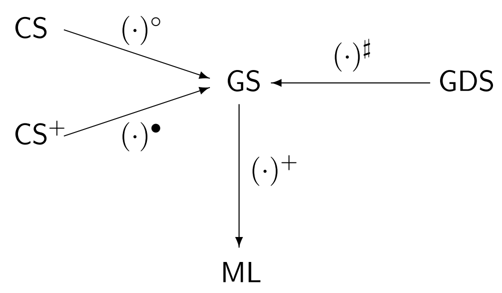


Fig. 5.11 The relationships between the inference systems.

in the image of the translation $(\cdot)^\sharp$ from GDS into GS. To avoid this problem, we define the *closure* $\mathcal{P}^{cl}$ of a GS-formula $\mathcal{P}$ to be the smallest set such that for all terms s, t, u,

1. if s appears in $\mathcal{P}$, then $s \sqsubset s$ is in $\mathcal{P}^{cl}$;
2. if $s \sqsubset t$ and $t \sqsubset u$ are in $\mathcal{P}^{cl}$, then $s \sqsubset u$ is in $\mathcal{P}^{cl}$;
3. if $s \sqsubset t$ and $t \vDash u$ are in $\mathcal{P}^{cl}$, then $s \vDash u$ is in $\mathcal{P}^{cl}$.

As an example, consider the case of a categorical sentence No A are B. We have:

$$((\textsf{No } A \textsf{ are } B)^\circ)^{cl} = \{A \vDash B\}^{cl} = \{A \vDash B, A \sqsubset A, B \sqsubset B\};$$

and

$$\begin{aligned}&((\textsf{No } A \textsf{ are } B)^\bullet)^{cl}\\ =&\{a \sqsubset A, b \sqsubset B, A \vDash B\}^{cl}\\ =&\{a \sqsubset A, b \sqsubset B, A \vDash B, a \vDash B, b \vDash A, a \vDash b, a \sqsubset a, b \sqsubset b, A \sqsubset A, B \sqsubset B\}.\end{aligned}$$

It is easily shown that in GS, every proof of $\mathcal{P}$ from $\mathcal{P}_1, \ldots, \mathcal{P}_n$ can be converted to a proof of $\mathcal{P}^{cl}$ from $(\mathcal{P}_1)^{cl}, \ldots, (\mathcal{P}_n)^{cl}$, and vice versa. It is also easily checked that for each categorical sentence S, $(S^\circ)^{cl}$ and $(S^\bullet)^{cl}$ are in the image of $(\cdot)^\sharp$. Now let $(\cdot)^\natural$ be the inverse translation of $(\cdot)^\sharp$ such that $\mathcal{P}^\natural = \mathcal{P} \cup \{s \bowtie t \mid s \sqsubset t \notin \mathcal{P}$ and $t \sqsubset s \notin \mathcal{P}$ and $s \vdash\dashv t \notin \mathcal{P}\}$. Clearly, $(\cdot)^\sharp$ and $(\cdot)^\natural$ are mutually inverse. We define the translation $(\cdot)^\triangleright$ from a categorical sentence in CS into an EUL-diagram by $S^\triangleright := ((S^\circ)^{cl})^\natural$, and the mapping $(\cdot)^\blacktriangleright$ from a categorical sentence in CS$^+$ into an EUL-diagram by $S^\blacktriangleright := ((S^\bullet)^{cl})^\natural$.

We also need the notion of cyclicity in GDS, which is defined as follows. Let $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}$ are EUL-diagrams. We say that a sequence $\mathcal{D}_1, \dots, \mathcal{D}_n, \mathcal{E}$ is a *cycle* if $cr(\mathcal{D}_1) = \{A_1, A_2\}, \dots, cr(\mathcal{D}_n) = \{A_n, A_{n+1}\}, cr(\mathcal{E}) = \{A_{n+1}, A_1\}$ for some contours $A_1, \dots, A_n$ such that $A_i \not\equiv A_j$ for all i, j $(1 \leq i, j \leq n+1)$.

Now we have:

Theorem 5.21 (Translation of CS and CS$^+$ into GDS) Let $S, S_1, \dots, S_n$ be categorical sentences in CS (resp. CS$^+$).
(1) *Soundness.* Every proof in CS (resp. CS$^+$) of S from $S_1, \dots, S_n$ can be translated into a diagrammatic proof in GDS of $S^\triangleright$ (resp. $S^\blacktriangleright$) from $S_1^\triangleright, \dots, S_n^\triangleright$ (resp. $S_1^\blacktriangleright, \dots, S_n^\blacktriangleright$), where (i) any named point appearing in $S_i^\triangleright$ $(1 \leq i \leq n)$ is different from each other and (ii) any named point appearing in $S^\triangleright$ also appears in some of the premises $S_1^\triangleright, \dots, S_n^\triangleright$.
(2) *Faithfulness.* Let π be a diagrammatic proof in GDS of $S^\triangleright$ (resp. $S^\blacktriangleright$) from $S_1^\triangleright, \dots, S_n^\triangleright$ (resp. $S_1^\blacktriangleright, \dots, S_n^\blacktriangleright$) such that (i) the sequence $S_1^\triangleright, \dots, S_n^\triangleright, S^\triangleright$ (resp. $S_1^\blacktriangleright, \dots, S_n^\blacktriangleright, S^\blacktriangleright$) is a cycle and (ii) any named point appearing in $S_i^\triangleright$ (resp. $S_i^\blacktriangleright$) for $1 \leq i \leq n$ is different from each other. Then π can be translated into a proof in CS (resp. CS$^+$) of S from $S_1, \dots, S_n$.

Proof. We only show the faithfulness. The soundness is shown in a similar way using Theorem 3.6 and Theorem 5.19. Let π be a diagrammatic proof as described in (2). By Theorem 5.17, π can be translated into a proof in GS of $(S^\circ)^{cl}$ [resp. $(S^\bullet)^{cl}$] from $(S_1^\circ)^{cl}, \dots, (S_n^\circ)^{cl}$ [resp. $(S_1^\bullet)^{cl}, \dots, (S_n^\bullet)^{cl}$], which in turn is converted to a proof π' in GS of S° (resp. $S^\bullet$) from $S_1^\circ, \dots, S_n^\circ$ (resp. $S_1^\bullet, \dots, S_n^\bullet$) by simply neglecting some parts of it. By Theorem 2.8 π can be converted into a normal proof π'' in GS, and by Lemma 3.11 [resp. Lemma 3.20] π'' has the same assumptions as π', thus its assumptions and conclusion form a cycle. Hence by Theorem 3.15, π'' can be translated into a proof in CS [resp. CS$^+$] of S from $S_1, \dots, S_n$. ■

Finally, we have the following results.

Theorem 5.22 (Translation of GDS into ML) Let $\overrightarrow{\mathcal{D}}$ be a set of EUL-diagrams and $\mathcal{E}$ be an EUL-diagram. Let $(\overrightarrow{\mathcal{D}}^{\sharp})^{+} := \{(\mathcal{D}^{\sharp})^{+} \mid \mathcal{D} \in \overrightarrow{\mathcal{D}}\}$.
(1) *Soundness.* Every proof in GDS of $\mathcal{E}$ from $\overrightarrow{\mathcal{D}}$ can be translated into a proof in ML of $(\mathcal{E}^{\sharp})^{+}$ from $(\overrightarrow{\mathcal{D}}^{\sharp})^{+}$ or a subset of $(\overrightarrow{\mathcal{D}}^{\sharp})^{+}$.

(2) *Faithfulness.* Every proof in ML of $(\mathcal{E}^{\sharp})^{+}$ from $(\overrightarrow{\mathcal{D}}^{\sharp})^{+}$ can be translated into a proof in GDS of $\mathcal{E}$ from $(\overrightarrow{\mathcal{D}}^{\sharp})^{+}$ or a subset of $(\overrightarrow{\mathcal{D}}^{\sharp})^{+}$.

Proof. The soundness follows from Theorem 3.6 and Theorem 5.19, using the normalization theorem of GS (Theorem 2.8). The faithfulness follows from by Theorem 3.15 and Theorem 5.17, using the normalization theorem of ML (Theorem 4.4). ∎

6. An extended system with conjunctive terms

The syntax of GS is quite simple but it can be naturally extended in various ways. In this section, as a first step, we consider an extension of GS with *intersection*, where a complex term of the form $A \sqcap B$ is introduced. We call the extended system "Generalized Syllogistic inference system with intersection", termed as $\mathsf{GS}^{\sqcap}$. The main result of this section is a proof of a completeness theorem of $\mathsf{GS}^{\sqcap}$.

6.1 An extended inference system $\mathsf{GS}^{\sqcap}$

In $\mathsf{GS}^{\sqcap}$, we can form a complex term of the form $A \sqcap B$, which denotes the intersection of the sets denoted by A and B. This extension enables us to deal with inferences involving categorical sentences with modifying phrases, such as intersective adjectives and relative clauses.

As an illustration, consider a monotonicity inference of the following form (see, e.g., Sánchez Valencia 1991 for a discussion on monotonicity inferences).

$$(1) \quad \frac{\textsf{All } A \textsf{ are } C}{\textsf{All } A \textsf{ who are } B \textsf{ are } C}$$

$$(2) \quad \frac{\textsf{No } A \textsf{ are } C}{\textsf{No } A \textsf{ who are } B \textsf{ are } C}$$

Here the conclusions are sentences with a relative clause. We regard A who are B as a complex term, and represent it as $A \sqcap B$. Then (1) and (2) are formalized in $\mathsf{GS}^{\sqcap}$ as (1′) and (2′), respectively.

$$(1') \quad \frac{A \sqsubset C}{A \sqcap B \sqsubset C}$$

$$(2')\ \frac{A \vDash C}{A \sqcap B \vDash C}$$

For simplicity, we assume that the terms of the language of $\mathsf{GS}^{\sqcap}$ consist only of general terms, and set aside the treatment of singular terms. This restriction is only for expository convenience.

Definition 6.1 (Symbol) The language of $\mathsf{GS}^{\sqcap}$ consists of the following symbols.

1. General terms: $A_1, A_2, A_3, \ldots$
2. Relation symbols: $\sqsubset$ (inclusion), $\vDash$ (exclusion)
3. Term-forming operator: $\sqcap$ (intersection)
4. Auxiliary symbols: , (comma), { } (braces), and ( ) (brackets).

Definition 6.2 (Terms and formulas) The terms of $\mathsf{GS}^{\sqcap}$ are inductively defined as follows.

(i) Every general term is a term.

(ii) If s and t are terms, then $(s \sqcap t)$ is a term.

The formulas of $\mathsf{GS}^{\sqcap}$ are defined as follows.

(iii) If s and t are terms, then $s \sqsubset t$ and $s \vDash t$ are formulas. The formulas of this form are called *atomic* formulas.

(iv) If $P_1, \ldots, P_n$ are atomic formulas ($n \geq 1$), then $\{P_1, \ldots, P_n\}$ is a formula. A singleton $\{P\}$ is identified with P.

Notation. As in GS, we use syntactic variables $A, B, C, \ldots$ to denote general terms, $s, t, u, \ldots$ to denote terms, $P, Q, \ldots$ to denote atomic formulas, $\mathcal{P}, \mathcal{Q}, \ldots$ to denote formulas, and Γ, Δ to denote a set of formulas.

We assume the following syntactic identification:

1. $\vDash$ is symmetric, i.e., for all s, t, $s \vDash t \equiv t \vDash s$;

2. $\sqcap$ is idempotent, symmetric, and associative, i.e., for all s, t, u, $s \sqcap s \equiv s$, $s \sqcap t \equiv t \sqcap s$, and $s \sqcap (t \sqcap u) \equiv (s \sqcap t) \sqcap u$.

Given that associativity holds for $\sqcap$, we henceforth omit the brackets of complex terms. Accordingly, a term of $\mathsf{GS}^{\sqcap}$ is to be considered to have the form $A_1 \sqcap \cdots \sqcap A_n$ $(n \geq 1)$, which is abbreviated as $\sqcap \vec{A_n}$ (when $n = 1$, $\sqcap A_1 \equiv A_1$).

The semantics of GS is naturally extended for that of $\mathsf{GS}^{\sqcap}$.

Definition 6.3 A model M for the language of $\mathsf{GS}^{\sqcap}$ is a pair (U, I), where U is a non-empty set (the domain of M), and I is an interpretation function such that

1. $I(A)$ is a non-empty subset of U for all general term A, and
2. $I(s \sqcap t) = I(s) \cap I(t)$ for all terms s and t.

Remark. In this definition, $I(A)$ must be non-empty for any general term A, but $I(s) \sqcap I(t)$ may be empty if there is no intersection between $I(s)$ and $I(t)$.

The other semantic notions, i.e., the satisfaction relation $\models$, the semantic consequence relation, and the semantic consistency, are defined in the same say as those of GS. See Definition 2.13 of Section 2.

Now we present the proof theory of $\mathsf{GS}^{\sqcap}$, which is an extension of the axiom and inference rules of GS. As in GS, we omit the braces { and } of formulas when they appear in a proof.

Definition 6.4 (Axiom and inference rules of $\mathsf{GS}^{\sqcap}$)
Axiom (ax) : $s \sqsubset s$
Inference rules:

$$\frac{s \sqsubset t \quad t \sqsubset u}{s \sqsubset u}\,(\sqsubset) \qquad \frac{s \sqsubset t \quad t \mathrel{\vdash\!\dashv} u}{s \mathrel{\vdash\!\dashv} u}\,(\mathrel{\vdash\!\dashv})$$

$$\frac{s \sqsubset u}{s \sqcap t \sqsubset u}\,(\sqcap L) \qquad \frac{s \sqsubset t \quad s \sqsubset u}{s \sqsubset t \sqcap u}\,(\sqcap R) \qquad \frac{s \sqcap u \mathrel{\vdash\!\dashv} t}{s \mathrel{\vdash\!\dashv} t \sqcap u}\,(\sqcap_{\vdash\!\dashv}) \qquad \frac{s \mathrel{\vdash\!\dashv} t}{s \sqcap t \sqsubset u}\,(\mathrel{\vdash\!\dashv} E)$$

$$\frac{\mathcal{P} \quad \mathcal{Q}}{\mathcal{P} \cup \mathcal{Q}}\,(+) \qquad\qquad \frac{\mathcal{P}}{\mathcal{Q}}\,(-)$$

where in $(+)$, $\mathcal{P} \neq \mathcal{Q}$, and in $(-)$, $\mathcal{Q} \subset \mathcal{P}$.

Remark. Instead of $(\vdash\!\dashv E)$, we could introduce $\bot$ as a primitive term denoting the empty set, and adopt the axiom $\bot \sqsubset s$ and the inference rule:

$$\frac{s \vdash\!\dashv t}{s \sqcap t \sqsubset \bot}$$

$(\vdash\!\dashv E)$ is plainly a derived rule of this modified system.

The notions of proofs and the provability relation are define in the same way as GS (cf. Definition 2.4 and 2.5).

Lemma 6.5 The following are provable in $\mathsf{GS}^{\sqcap}$.

1. $s \sqsubset t \vdash s \sqcap u \sqsubset t \sqcap u$

$$\frac{\dfrac{\dfrac{\dfrac{}{s \sqsubset s}\,ax}{s \sqcap u \sqsubset s}\,(\sqcap L) \quad s \sqsubset t}{s \sqcap u \sqsubset t}\,(\sqsubset) \quad \dfrac{\dfrac{}{u \sqsubset u}\,ax}{s \sqcap u \sqsubset u}\,(\sqcap L)}{s \sqcap u \sqsubset t \sqcap u}\,(\sqcap R)$$

2. $s \vdash\!\dashv t \vdash s \sqcap u \vdash\!\dashv t \sqcap u$

$$\frac{\dfrac{\dfrac{}{t \sqsubset t}\,ax}{t \sqcap u \sqsubset t}\,(\sqcap L) \quad \dfrac{\dfrac{s \sqsubset s}{s \sqcap u \sqsubset s}\,(\sqcap L) \quad s \vdash\!\dashv t}{s \sqcap u \vdash\!\dashv t}\,(\vdash\!\dashv)}{s \sqcap u \vdash\!\dashv t \sqcap u}\,(\vdash\!\dashv)$$

3. $s \sqcap t \vdash\!\dashv s \sqcap t \ \vdash \ s \vdash\!\dashv t$

$$\frac{\dfrac{s \sqcap t \vdash\!\dashv s \sqcap t}{s \sqcap t \vdash\!\dashv t}\,(\sqcap_{\vdash\!\dashv})}{s \vdash\!\dashv t}\,(\sqcap_{\vdash\!\dashv})$$

4. $s \vdash\!\dashv t \ \vdash \ s \sqcap t \vdash\!\dashv u$

$$\frac{\dfrac{s \vdash\!\dashv t \quad \dfrac{\dfrac{}{t \sqsubset t}\,ax}{t \sqcap u \sqsubset t}\,(\sqcap L)}{s \vdash\!\dashv t \sqcap u}\,(\vdash\!\dashv)}{s \sqcap t \vdash\!\dashv u}\,(\sqcap_{\vdash\!\dashv})$$

The soundness theorem is proved by straightforward induction on the length of a proof in $\mathsf{GS}^{\sqcap}$.

Theorem 6.6 (Soundness of $\mathsf{GS}^{\sqcap}$) *Let $\mathcal{P}_1, \ldots, \mathcal{P}_n, \mathcal{Q}$ be formulas in $\mathsf{GS}^{\sqcap}$. If $\mathcal{P}_1, \ldots, \mathcal{P}_n \vdash \mathcal{Q}$ in $\mathsf{GS}^{\sqcap}$, then $\mathcal{P}_1, \ldots, \mathcal{P}_n \models \mathcal{Q}$.*

6.2 Completeness of $\mathsf{GS}^{\sqcap}$

In our formulation of a completeness theorem of $\mathsf{GS}^{\sqcap}$, we require the set of assumptions to be semantically consistent in the same way as in GS. The following is proved in a similar way as the proof of Lemma 2.16.

Lemma 6.7 *Let Γ be a semantically consistent set of formulas. Then none of the following holds in $\mathsf{GS}^{\sqcap}$ for any general term A and B:*

1. $\Gamma \vdash A \vDash\!\dashv A$.
2. $\Gamma \vdash A \sqsubset B$ *and* $\Gamma \vdash A \vDash\!\dashv B$.
3. *There is a general term C such that* $\Gamma \vdash A \vDash\!\dashv B$, $\Gamma \vdash C \sqsubset A$ *and* $\Gamma \vdash C \sqsubset B$.

Remark. Lemma 6.7 cannot be generalized to arbitrary complex terms. For example, consider $\Gamma = \{A \vDash\!\dashv B\}$. It is clear that Γ is semantically consistent, but we have $\Gamma \vdash A \sqcap B \vDash\!\dashv A \sqcap B$ by Lemma 6.5 (4).

To show the completeness of $\mathsf{GS}^{\sqcap}$, we start with defining a *canonical model*, whose domain consists of the terms of $\mathsf{GS}^{\sqcap}$.

Definition 6.8 (Canonical model of $\mathsf{GS}^{\sqcap}$) Let Γ be a semantically consistent set of of formulas in $\mathsf{GS}^{\sqcap}$. A canonical model $M_\Gamma = (U_\Gamma, I_\Gamma)$ for Γ is defined as follows:

1. The domain $U_\Gamma = \{s \mid s \text{ is a term of } \mathsf{GS}^{\sqcap}\}$
2. I_Γ is an interpretation function such that

- $I_\Gamma(A) = \{s \mid \Gamma \vdash s \sqsubset A \text{ in } \mathsf{GS}^{\sqcap}\} \setminus \{s \sqcap t \mid \Gamma \vdash s \vDash\!\dashv t \text{ in } \mathsf{GS}^{\sqcap}\}$ for all general term A, and
- $I_\Gamma(s \sqcap t) = I_\Gamma(s) \cap I_\Gamma(t)$ for all term s, t.

Note that since we have $\Gamma \vdash A \sqsubset A$ by Axiom (ax), $I_\Gamma(A)$ is not empty for any general term A.

Example 6.9 Consider

$$\Gamma = \{A \mathrel{\vdash\!\dashv} D, B \sqsubset A, C \sqsubset A, C \mathrel{\vdash\!\dashv} D, B \mathrel{\vdash\!\dashv} D\}.$$

We have:

$$\begin{aligned}
I_\Gamma(A) &= \{A, B, C, A \sqcap B, A \sqcap C, B \sqcap C, A \sqcap B \sqcap C\}, \\
I_\Gamma(B) &= \{B, A \sqcap B, B \sqcap C, A \sqcap B \sqcap C\}, \\
I_\Gamma(C) &= \{C, A \sqcap C, B \sqcap C, A \sqcap B \sqcap C\}, \\
I_\Gamma(D) &= \{D\}.
\end{aligned}$$

Lemma 6.10 *If* Γ *is a semantically consistent set of atomic formulas of* $\mathsf{GS}^\sqcap$*, then* M_Γ *is a model of* Γ*.*

Proof. Let Γ be a semantically consistent set of atomic formulas of $\mathsf{GS}^\sqcap$, and suppose that $\mathcal{P} \in \Gamma$ (where we may assume that $\mathcal{P}$ has the form $\{P_1, \ldots, P_n\}$). We claim that $M_\Gamma \models \mathcal{P}$. As in the case of GS (cf. the proof of Lemma 2.21), in order to show this claim, it suffices to show that $M_\Gamma \models P_i$ for each P_i $(1 \leq i \leq n)$. Note that given $\mathcal{P} \in \Gamma$, we may assume $\Gamma \vdash P_i$. We have two cases, depending on whether P_i is of the form $\sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m$ or $\sqcap \vec{A}_k \mathrel{\vdash\!\dashv} \sqcap \vec{B}_m$.

(Case 1) When P_i is of the form $\sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m$, We show that $M_\Gamma \models \sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m$, i.e., $I_\Gamma(\sqcap \vec{A}_k) \subseteq I_\Gamma(\sqcap \vec{B}_m)$. Let $u \in I_\Gamma(\sqcap \vec{A}_k)$, where we have for any u_1, u_2, if $u = u_1 \sqcap u_2$ then $\Gamma \nvdash u_1 \mathrel{\vdash\!\dashv} u_2$. By definition, $u \in I_\Gamma(A_1) \cap \cdots \cap I_\Gamma(A_k)$, so $u \in I_\Gamma(A_j)$ for all j $(1 \leq j \leq k)$. Hence $\Gamma \vdash u \sqsubset A_j$. Then, by repeated applications of the $(\sqcap R)$ rule, we have $\Gamma \vdash u \sqsubset \sqcap \vec{A}_k$. By the assumption that $\Gamma \vdash \sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m$, we obtain $\Gamma \vdash u \sqsubset \sqcap \vec{B}_m$ using the $(\sqsubset)$ rule. Since we have, by the $(\sqcap L)$ rule, $\Gamma \vdash \sqcap \vec{B}_m \sqsubset B_j$ for all j $(1 \leq j \leq m)$, it follows that $\Gamma \vdash u \sqsubset B_j$ by the $(\sqsubset)$ rule. Thus we have $u \in I_\Gamma(B_j)$ for all j $(1 \leq j \leq m)$, i.e., $u \in I_\Gamma(B_1) \cap \cdots \cap I_\Gamma(B_m)$. Hence $u \in I_\Gamma(\sqcap \vec{B}_m)$, as required.

(Case 2) P_i is of the form $\sqcap \vec{A}_k \vdash\!\dashv \sqcap \vec{B}_m$. We show $I_\Gamma(\sqcap \vec{A}_k) \cap I_\Gamma(\sqcap \vec{B}_m) = \emptyset$. Assume to the contrary that there is a term u such that $u \in I_\Gamma(\sqcap \vec{A}_k)$ and $u \in I_\Gamma(\sqcap \vec{B}_m)$. Then we have (i) $\Gamma \vdash u \sqsubset \sqcap \vec{A}_k$, (ii) $\Gamma \vdash u \sqsubset \sqcap \vec{B}_m$, and (iii) there is no u_1, u_2 such that $u = u_1 \sqcap u_2$ and $\Gamma \vdash u_1 \vdash\!\dashv u_2$. Since $\Gamma \vdash \sqcap \vec{A}_k \vdash\!\dashv \sqcap \vec{B}_m$ by our assumption, we have $\Gamma \vdash u \vdash\!\dashv u$ using the ($\vdash\!\dashv$) rule with (i) and (ii). When u is a general term, this is a straightforward contradiction by Lemma 6.7(1). Otherwise, we would have $u = u_1 \sqcap u_2$ for some u_1, u_2, i.e., $\Gamma \vdash u_1 \sqcap u_2 \vdash\!\dashv u_1 \sqcap u_2$. Then by Lemma 6.5(3), we have $\Gamma \vdash u_1 \vdash\!\dashv u_2$, which is a contradiction to (iii), as required. ∎

We proceed to prove a completeness theorem of $\mathsf{GS}^{\sqcap}$.

Theorem 6.11 (Completeness of $\mathsf{GS}^{\sqcap}$) *Let Γ be a semantically consistent set of formulas, and let $\mathcal{P}$ be a formula in $\mathsf{GS}^{\sqcap}$. If $\Gamma \models \mathcal{P}$, then $\Gamma \vdash \mathcal{P}$ in $\mathsf{GS}^{\sqcap}$.*

Proof. Suppose $\Gamma \models \mathcal{P}$. Then by Lemma 6.10 we have $M_\Gamma \models \Gamma$, hence $M_\Gamma \models \mathcal{P}$. We may assume that $\mathcal{P}$ has the form $\{P_1, \ldots, P_n\}$, where $P_1, \ldots, P_n$ are atomic formulas. So we have $M_\Gamma \models P_i$ for each P_i ($1 \leq i \leq n$). Now we show that $\Gamma \vdash P_i$ in $\mathsf{GS}^{\sqcap}$. It then follows that $\Gamma \vdash \{P_1, \ldots, P_n\}$ by repeated applications of the (+) rule, which completes the proof. We have two cases according to the form of P_i.

(Case 1) When P_i is of the form $\sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m$, we have $I_\Gamma(\sqcap \vec{A}_k) \subseteq I_\Gamma(\sqcap \vec{B}_m)$. We claim that $\Gamma \vdash \sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m$. The argument splits into two cases, depending on whether $\Gamma \vdash A_j \vdash\!\dashv A_l$ for some j, l ($1 \leq j, l \leq k$) or not.

1. When $\Gamma \vdash A_j \vdash\!\dashv A_l$ for some j, l ($1 \leq j, l \leq k$), the following derivation yields the desired result.

$$\dfrac{\dfrac{A_j \vdash\!\dashv A_l}{A_j \sqcap A_l \sqsubset \sqcap \vec{B}_m}\,(\vdash\!\dashv E)}{\sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m}\,(\sqcap L)$$

2. When $\Gamma \not\vdash A_j \vdash\!\dashv A_l$ for any j, l ($1 \leq j, l \leq k$), we have $\sqcap \vec{A}_k \in I_\Gamma(A_i)$ for each i ($1 \leq i \leq k$), since $\Gamma \vdash A_i \sqsubset A_i$ by (ax) and $\Gamma \vdash \sqcap \vec{A}_k \sqsubset A_i$

by the $(\sqcap L)$ rule. Thus $\sqcap \vec{A}_k \in I_\Gamma(\sqcap \vec{A}_k)$. Since $I_\Gamma(\sqcap \vec{A}_k) \subseteq I_\Gamma(\sqcap \vec{B}_m)$, we have $\sqcap \vec{A}_k \in I_\Gamma(\sqcap \vec{B}_m)$, hence $\Gamma \vdash \sqcap \vec{A}_k \sqsubset \sqcap \vec{B}_m$.

(Case 2) When P_i is of the form $\sqcap \vec{A}_k \mathrel{\vdash\!\dashv} \sqcap \vec{B}_m$, we have $I_\Gamma(\sqcap \vec{A}_k) \cap I_\Gamma(\sqcap \vec{B}_m) = \emptyset$. We claim that $\Gamma \vdash \sqcap \vec{A}_k \mathrel{\vdash\!\dashv} \sqcap \vec{B}_m$. The argument splits into the following cases:

1. When $\Gamma \vdash A_j \mathrel{\vdash\!\dashv} A_l$ for some j, l $(1 \leq j, l \leq k)$, the following derivation gives the desired result:

$$\dfrac{\dfrac{A_j \mathrel{\vdash\!\dashv} A_l}{A_j \sqcap A_l \mathrel{\vdash\!\dashv} \sqcap \vec{B}_m}\text{ By Lemma 6.5(4)}}{\sqcap \vec{A}_k \mathrel{\vdash\!\dashv} \sqcap \vec{B}_m}\ (\sqcap L)$$

   Similarly for the case where $\Gamma \vdash B_j \mathrel{\vdash\!\dashv} B_l$ for some j, l $(1 \leq j, l \leq m)$.

2. When $\Gamma \nvdash A_j \mathrel{\vdash\!\dashv} A_l$ for any j, l $(1 \leq j, l \leq k)$ and $\Gamma \nvdash B_{j'} \mathrel{\vdash\!\dashv} B_{l'}$ for any j', l' $(1 \leq j', l' \leq m)$, suppose for contradiction that $\Gamma \nvdash \sqcap \vec{s}_t \mathrel{\vdash\!\dashv} \sqcap \vec{B}_m$. By (ax) we have $\Gamma \vdash A_i \sqsubset A_i$ for all i $(1 \leq i \leq k)$. Using the $(\sqcap L)$ rule, then, we have $\Gamma \vdash (\sqcap \vec{A}_k) \sqcap (\sqcap \vec{B}_m) \sqsubset A_i$. Thus, given our assumption, we have $(\sqcap \vec{A}_k) \sqcap (\sqcap \vec{B}_m) \in I_\Gamma(A_i)$ for all $1 \leq i \leq k$, hence $(\sqcap \vec{A}_k) \sqcap (\sqcap \vec{B}_m) \in I_\Gamma(\sqcap \vec{A}_k)$. By the same reasoning, we have $(\sqcap \vec{A}_k) \sqcap (\sqcap \vec{B}_m) \in I_\Gamma(\sqcap \vec{B}_m)$. Hence $I_\Gamma(\sqcap \vec{A}_k) \cap I_\Gamma(\sqcap \vec{B}_m) \neq \emptyset$. This is a contradiction, as required. ■

Concluding Remarks

As for the diagrammatic representation systems corresponding to extended syllogistic logics, Mineshima, Okada and Takemura (2009) discuss various extensions of EUL representation system and GDS inference system, including extensions with intersection, union, and complement. See also Nishihara and Morita (1988) and Nishihara, Morita, and Iwata (1990) for syllogisms with conjunctive and disjunctive terms and Moss (2010c) for syllogisms with complements. There are various others ways of extending the basic fragment of categorical syllogisms; relational syllogisms (i.e., syllogisms that involve relations and hence allow a limited sort of multiple quantification; see Pratt-Hartmann and Moss 2009); syllogisms involving proportional quantifiers like *most* (Moss 2008), and numerical syllogisms (Pratt-Hartmann 2009). A comparison between inferences with various forms of sentences and diagrams would contribute to making progress in understanding the nature of both linguistic and diagrammatic inferential processes in human reasoning. It is left for future work to explore the relationship between extended diagrammatic systems and the corresponding linguistic systems including $\mathsf{GS}^{\sqcap}$.

Chapter 2

Presupposition and Descriptions: A Proof-Theoretical Approach

1. Introduction to Chapter 2

The aim of this chapter is to present a proof-theoretic analysis of presuppositions in natural language, focusing on the interpretation of definite descriptions. We introduce a natural deduction framework to deal with presuppositional phenomena: this new framework has mechanisms on a par with those of recent dynamic frameworks, such as dynamic semantics and discourse representation theory, but it avoids problems inherent to these standard approaches.

As is well known, the notion of presupposition as currently studied in philosophy, linguistics, and logic can be traced back to some of Frege's writings on the foundation of mathematics and logical analysis of a language (see, in particular, Frege 1892). It has received special attention since Strawson's (1950) seminal objection to Russell's treatment of definite description. However, it was relatively recently that systematic theories of presuppositions with wide empirical coverage were developed and studied in connection with the tradition of formal semantics of natural language, beginning with Montague (1973). Notably, a new formal approach to the problems of presupposition was launched in the 1980s by emphasizing that presuppositional phenomena in natural language motivate the so-called *dynamic* conception of meaning, whose central idea is to regard meanings as *context change potentials* or *context update conditions*, rather than truth-conditions.

There are two influential theories of presuppositions in this tradition: dynamic semantics and discourse representation theory. The former, sometimes called the *satisfaction theory of presuppositions*, was initiated by the seminal work of Heim (1983). The latter was originally developed by Kamp (1981) and augmented with a module to handle presupposition by the influential work of van der Sandt (1992). Both approaches make substantial revisions to the standard logical framework in the tradition of Frege and Montague, claiming that standard logical systems are insufficient to treat presuppositional phenomena. One central issue here is how to formalize the notion of contexts that interact with asserted contents in certain complex ways. In particular, there is the problem of how to handle *local* contexts, i.e., contexts that are updated in the middle of a sentence.

With this problem, we attempt to pursue an option to preserve standard logical systems. A guiding idea is that presuppositional elements require reasoning about contexts that are structured in a certain way. In our view, presuppositions are best understood in terms of a deductive perspective, in which reasoning about contents and contexts plays a central role. For this purpose, we use natural deduction systems developed in the tradition of Gentzen's proof theory and apply them to analyses of presuppositional phenomena in natural language. One main goal of this chapter is to show that such a logical framework augmented with proof theory (natural deduction system) can handle well-known data that have been used to argue for dynamic theories of presupposition. Our proposal is based on the natural deduction system of ε-calculus (Stenlund 1973, 1975; Carlström 2005) and on constructive type theory (Martin-Löf 1984; Nordström, Petersson, and Smith 1990).

Our study is also intended as an attempt to show that proof-theoretic methods can be applied to the domain of natural language semantics. As noted above, we will argue that proof-theoretic methods are particularly useful in analyzing presuppositions in natural language. We will show that our proof-theoretic framework opens up the possibility of providing a novel semantic analysis on natural language inferences, which are neglected in the model-theoretic tradition.

Among various expressions that generate presuppositions (i.e., what are called *presupposition triggers*), we will concentrate on the treatment of existential presuppositions of definite descriptions. One reason is that there are well-studied proof systems in mathematical logic, such as ε-calculus, that are suitable for our purpose. Although existing proof systems are concerned with descriptions in mathematical discourse, we show that they can also be fruitfully applied to descriptions in natural languages.

The structure of Chapter 2

The structure of this chapter is as follows. In Section 2, we first review two major theories of definite descriptions, namely, Russell's quantificational theory and Strawson's referential theory (Section 2.2). We then introduce the problem of presupposition projection and argue that if Strawson's theory can be incorporated into a formal theory of presupposition projection, then it has descriptive and explanatory advantages over Russell's theory (Section 2.3).

In Section 3, we first outline the basic conception of meaning shared by two major theories of presupposition projection, namely, dynamic semantics and discourse representation theory (Section 3.1). We then critically examine these two theories in Sections 3.2 and 3.3. We argue that there are empirical and conceptual problems with these two accounts of presupposition and that these problems call for an alternative approach, in which *reasoning* about presuppositions plays a central role in accounting for their projection behavior (Section 3.4).

In Section 4, we present a proof-theoretic framework for handling existential presuppositions associated with definite descriptions. We outline the basic idea behind the proof-theoretic analysis of descriptions (Section 4.1) and then introduce a natural deduction system with ε-operators, called $\mathsf{IL}\varepsilon$ (Section 4.2) — a proof system that will be used throughout the subsequent sections. We also describe a relation between $\mathsf{IL}\varepsilon$ and Constructive Type Theory of Martin-Löf, which has been recently developed as an alternative to dynamic theories such as discourse representation theory.

In Section 5, we apply the proof-theoretic framework introduced in Section 4 to the problems of presuppositions discussed in the earlier sections, restricting our attention to existential presuppositions triggered by definite descriptions. We show how reasoning about presuppositions can be formalized as processes of constructing and transforming formal derivations in the proof system and argue that the problems confronting dynamic semantics and discourse representation theory can be avoided within this framework.

2. Background on definite descriptions

2.1 Two approaches to the semantics of descriptions

We start with reviewing two major views on the interpretation of definite descriptions, namely, Russell's and Strawson's analyses.[1] Russell's analysis consists of the following ideas (Russell 1905; Neale 1990):

(R1) A definite description "the *F*" is a *quantificational* expression;

(R2) A sentence of the form "The *F* is *G*" *entails* that there is exactly one thing that is an F.

A syntactic observation motivating Russell's analysis, which is particularly emphasized by Neale (1990), is that a definite description is parallel in form to other quantificational expressions (what Russell calls "denoting phrases"), i.e., those expressions consisting of a determiner followed by a noun phrase, such as *every F*, *some F*, *no F*, and so on. It is then claimed that such expressions should be subject to parallel semantic treatment and thus analyzed uniformly as quantificational devices.

As stressed by Neale (1990), among others, Russell's analysis is consistent with the theory of *generalized quantifiers* (Barwise and Cooper 1981), whose central idea is that a determiner such as *all*, *some*, and *no* is best analyzed as a *binary* quantifier denoting a relation between sets. For example, a sentence of the form *Every F is G* can be analyzed as having the

[1]Throughout this section, we focus on *definite* descriptions, setting aside issues concerned with *indefinite* descriptions. A modern Russellian treatment of indefinite descriptions can be found in Ludlow and Neale (1991). An alternative analysis in which indefinite descriptions are treated as a certain kind of referring expression has been explored within dynamic frameworks, such as Kamp (1981) and Heim (1982).

logical form [every $x : Fx$] Gx, where [every $x : Fx$] is a restricted quantifier whose scope is the predicate Gx, rather than as having the first-order representation $\forall x(Ax \rightarrow Bx)$ with the unary universal quantifier.[2] Then, the truth condition of such quantificational structures can be specified along the following lines.[3]

(1) [every $x : Fx$] Gx is true if and only if $\mathbf{F} \subseteq \mathbf{G}$

(2) [no $x : Fx$] Gx is true if and only if $\mathbf{F} \cap \mathbf{G} = \emptyset$.

(3) [some $x : Fx$] Gx is true if and only if $\mathbf{F} \cap \mathbf{G} \neq \emptyset$.

(4) [most $x : Fx$] Gx is true if and only if $|\mathbf{F} \cap \mathbf{G}| \geq |\mathbf{F} \setminus \mathbf{G}|$.

Here, a bold $\mathbf{F}$ refers to the set denoted by a predicate F and $|\mathbf{F}|$ denotes the cardinality of the set $\mathbf{F}$. The truth condition for a sentence containing definite description can be given in a parallel way:

(5) [the $x : Fx$] Gx is true if and only if $\mathbf{F} \cap \mathbf{G} \neq \emptyset$ and $|\mathbf{F}| = 1$.

This means that according to Russellian analysis, a sentence of the form "The F is G" is analyzed as making an *existential* statement, "There is exactly one F and it is G." Given this truth-condition, Russell's analysis is committed to the claim that a sentence containing an empty description is plainly false.

It is now well known that Strawson opposed Russell's analysis, insisting that if nothing satisfies the description, then the question of whether the sentence containing it is true or false does not arise.[4] In such a case, the speaker would fail to say anything true or false. The Strawsonian analysis then claims the following:

[2]An alternative is a relational representation such as $All(A, B)$, which highlights the relational structure of a quantified sentence. In Chapter 1, we used a variant of such a relational representation of universal quantified sentences. The restricted quantifier notation has been widely adopted in the literature since it mirrors the surface structure of natural language.

[3]Special attention has been paid to a proportional quantifier such as *most*, which is a cornerstone of the generalized quantifier theory because it is not definable in first-order logic; see Barwise and Cooper (1981).

[4]See Strawson (1950, 1952). As noted in the Introduction, the presuppositional analysis of definite descriptions can also be found in Frege (1892).

(S1) A definite description "the F" is a *referring* expression;

(S2) A sentence containing "the *F*" *presupposes* that there exists one thing that is an F. [5]

As a syntactic motivation for the claim in (S1), it might be pointed out that definite descriptions are parallel in form not only to quantificational phrases, but also to complex demonstratives, such as *this table* and *that man*, which apparently act as referring expressions.[6] As for the claim in (S2), compare the following:

(6) a. The F is G.
  b. There is exactly one F and it is G.

According to the the Russellian analysis, there is no difference in semantic content between (6a) and (6b); both *assert* that there is exactly one F and that it is G. By contrast, the Strawsonian analysis holds that (6a) *presupposes* the existence of a unique individual satisfying F and *asserts* that it is G. Strong evidence for the Strawsonian analysis comes from constructions in which a definite description is embedded under an attitude verb, as in (7a), or in the antecedent of a conditional, as in (7b).[7]

(7) a. I wonder whether the ghost in my attic will be quiet tonight.
  b. If the ghost in my attic is quiet tonight, I will hold a party.

[5]Strawson (1950) distinguishes between a sentence (a sentence type) and the statement made by uttering a sentence. The former is the bearer of linguistic meaning, and the latter the bearer of truth-value. For Strawson, presupposition is a relation between statements, not between sentences; a statement P is said to presuppose a statement Q if the truth of P is a necessary condition of Q's having a truth value. In the main text, here and henceforth, we avoid complications caused by the issue of the kind of items between which the relation of presupposition obtains; following a standard practice, we will talk as if presuppositions hold between a sentence and a proposition.

[6]Note, however, that the semantic status of complex demonstratives is controversial, since some authors—most notably King (2001)—claim that complex demonstratives should be analyzed not as referring expressions as standardly assumed since Kaplan (1989), but as quantificational expressions.

[7]These examples are originally from Heim (1991) and examined in detail in Elbourne (2005: 109–112) and Elbourne (2010).

Under the Russellian analysis, "the ghost in my attic will be quiet tonight" is equivalent to "there is exactly one ghost in my attic and it will be quiet tonight." Hence the Russellian analysis seems to predict that (7a) and (7b), when uttered in appropriate contexts, would have the readings in (8a) and (8b), respectively.

(8) a. I wonder whether there is exactly one ghost in my attic and it will be quiet tonight.

b. If there is exactly one ghost in my attic and it is quiet tonight, I will hold a party.

However, neither of (7a) and (7b) has such readings. Intuitively, in uttering (7a), the speaker is *assuming* that there is exactly one ghost in her attic and wondering whether it will be quiet tonight. Similarly, in saying (7b), the speaker seems to be presupposing the (unique) existence of such a ghost, and asserting that if it is quiet, she will hold a party. As we will see below, such readings are correctly predictable from the Strawsonian claim in (S2) augmented with suitable assumptions on the behavior of descriptions embedded in complex constructions. Thus, it seems to be fair to say that the contrast between (7) and (8) constitutes a problem for Russellian analysis and provides initial support for the Strawsonian view that uniqueness and existence conditions are not entailments but presuppositions.[8]

[8] In defense of Russellian analysis, it might be argued that the correct reading of a complex sentence in which a definite description appears in an embedded environment can be accounted for as a scopal interaction between definite descriptions and other operators. Specifically, it might be argued that the definite description in question would take wide scope over the attitude verb in (7a) and over the conditional in (7b). However, it is still not clear how to account for the intuition that in these examples, uniqueness and existence conditions are not asserted but assumed or taken for granted by the speaker. Another problem with this proposal is that in the case of (7a), the wide scope interpretation forces a *de re* reading for the definite description *the ghost*. Accordingly, a *de dicto* reading, under which the speaker does not believe that there is any ghost in my attic, is ruled out, contrary to our intuition that it is the preferred reading of (7a). A more detailed argument against the Russellian response appealing to a scope distinction can be found in Elbourne (2005, 2010).

The classical Strawsonian analysis also, however, faces some difficulty in accounting for the behavior of definite descriptions appearing in certain complex sentences. We address two issues that have been discussed by both philosophers and linguists. These problems will be relevant to our later discussion on analyses of presuppositional phenomena in general.

The problem of "filtration"

Consider the following examples:

(9) John's children are wise.

(10) a. John has children and his children are wise.
b. If John has children, his children are wise.
c. Either John does not have any children or his children are wise.

The Strawsonian analysis in (S2) predicts that a sentence containing the definite description *John's children* presupposes that John has children. So if John has no children, it would be misleading to use such a sentence: it would not express any proposition and the question of whether it is true or false would not arise. Now this account seems to correctly predict the behavior of the simple sentence in (9). However, it fails to capture the correct readings of the examples in (10). For example, we have a strong intuition that if John has no children, (10a) should be judged false, since the first conjunct is false. In the case of (10b), it is clear that it can be used to express a proposition even if John has no children. The same remark can be made about (10c). Thus, we observe that these three constructions do not presuppose that John has children; using the terminology in the presupposition literature, the existence presuppositions are *filtered* out in these environments. Indeed, the same kind of examples were already observed by Russell himself:

(11) If France were a monarchy, the King of France would be of the House of Orleans. (Russell and Whitehead 1910: 69)

Obviously, this sentence might be held to be true even though the king of France does not exist. The classical Strawsonian view does not account for this phenomenon.

The problem of open descriptions

Another objection to Strawson's theory is based on the constructions in which descriptions contain variables that are bound by quantifiers. Consider the following examples, taken from Mates (1973): [9]

(12) a. Every boy kissed the girl who loved him.
b. Every positive integer is the positive square root of some positive integer.

In (12a), the description *the girl who loved him* contains the pronoun that is bound by the outside quantifier. In (12c), the description contains a quantificational expression *some positive integer*. Consequently, in both examples, there are no unique objects associated with the descriptions in question, and thus there seems to be no way to analyze these descriptions as referring expressions in Strawson's sense.

By contrast, Russell's theory has no problems with these examples and can systematically provide interpretations. Using the restricted quantifier notation, (12a) and (12b) can be analyzed as (13a) and (13b), respectively:

(13) a. [every x : boy x] [the y : girl y & y loved x] x kissed y
b. [every x : positive integer x] [some y : positive integer y] [the z : z is positive square root of y] x is z

The problem with Strawson's original account is that it confines its attention to simple constructions in which descriptions do not interact with binding operators such as quantifiers. All of this suggests that Strawson's analysis is not yet a systematic theory that can be regarded as a serious rival to Russell's theory of descriptions.

2.2 Presupposition projection

Let us go back to the examples in (10), which pose a difficulty to Strawsonian analysis. In the literature of semantics and pragmatics of natural

[9] See also van der Sandt (1992) and Kripke (2005) for a similar argument.

languages, these phenomena have been treated as instances of the "projection problem" of presuppositions. This is the problem of predicting the presupposition of complex sentences in a compositional fashion from the entailment and presupposition of their parts.[10] The notion of presuppositions has played a central role in the recent development of formal semantic and pragmatics, and several well-developed theories of the projection problem for presuppositions have mechanisms that deal with the presuppositional effect of definite descriptions. We will review two influential approaches below. Before moving on, we will review some of the relevant data and attempt to understand the behavior of the definite descriptions we saw above in a broader perspective.

Traditionally, philosophers have tended to confine themselves to a limited range of expressions that are considered to give rise to presuppositions, such as definite descriptions, universal quantifiers, and proper names. In the 1970s, linguists discovered that a great variety of expressions or syntactic constructions give rise to inferences that show a different behavior from ordinary entailments and thus should be accounted for by presupposition theory. Such expressions or constructions that give rise to presuppositions are standardly called *presupposition triggers*. In addition to those discussed by Frege and Strawson, the standard examples that have been widely agreed to be presupposition triggers include the following:[11]

[10] The problem was first posed by Langendoen and Savin (1971).

[11] For a more comprehensive list with references to relevant literature on each presupposition trigger, see Nishiyama (1983), Levinson (1983), Soames (1989), Geurts (1999), and Beaver (2001). It should be noted that although the list indicated here is fairly standard in the literature on presuppositions, whether it accords with our intuitions is controversial. Thus, Kripke (2009: 370) says, "[such a list] doesn't get our intuitions about the relevant presuppositions as we would naturally think of them, and is even highly counterintuitive in many cases." Specifically, concerning the additive particle *too* and cleft constructions, Kripke (2009) convincingly argues that the standard descriptions of their presuppositional content, as shown in (18b) and (20b), are wrong and provides alternative descriptions. For the presupposition of *too*, see also Soames (1989: 613–4, fn. 54) and Heim (1992). Moreover, it might be too simplistic to assume that there is no essential difference in the projection behavior between the various kinds of triggers (cf. Charlow 2009). For the current purposes, however, we are making the simplifying assumption that

(14) a. Bill regrets that he lied to Mary. Factive verb

b. Bill lied to May.

(15) a. John has stopped beating his wife. Aspectual verb

b. John has beaten his wife.

(16) a. Harry managed to find the book. Implicative verb

b. Finding the book required some effort.

(17) a. Sam broke the window again today. Iterative

b. Sam broke the window before.

(18) a. It was Sam who broke the window. Cleft

b. Someone broke the window.

(19) a. What John broke was his typewriter. Pseudo-cleft

b. John broke something.

(20) a. $[\text{Pat}]_F$ is leaving, too. (Focus on *Pat*) Additive

b. Someone other than Pat is leaving.

A characteristic property of presupposition is that when a speaker utters any of the (a) sentences, she can normally be assumed to take the truth of the corresponding (b) sentence for granted; intuitively, the presupposed information is conveyed not as something worthy of discussion but rather as something that is already agreed upon by the participants in the conversation.

Another characteristic property is that presuppositions differ from entailments in how they are *inherited* by complex sentences. Specifically, a sentence S is said to presuppose p when p is normally implied not only by S but also by the family of sentences in which S is embedded, i.e., those

the standard descriptions are correct and that there is no relevant difference between the various triggers.

constructions that are dubbed "hole" according to Karttunen's (1974) classification of presupposition triggers.[12]

(21) **Generalization 1.** The following constructions inherit the presuppositions of S:

a. It is not the case that S. NEGATION

b. Maybe it is the case that S. EPISTEMIC MODAL

c. If it is the case that S, then S'. THE ANTECEDENT OF A CONDITIONAL

d. Is it the case that S? QUESTION

e. Suppose that it is the case that S. HYPOTHETICAL ASSUMPTION

The basic observation here is that the presuppositions of sentence S normally project out of negations, epistemic modals, the antecedents of conditionals, questions, and hypothetical assumptions. This observation serves as an empirical test to identity presuppositional information.[13]

To see how the test works, consider the following example:

(22) It was Sam who broke the window.

(23) Someone broke the window.

(24) a. It wasn't Sam who broke the window. NEGATION

b. Maybe it was Sam who broke the window. EPISTEMIC MODAL

[12]Throughout this chapter, following the standard terminology, "imply" and "implication" are used as cover terms standing for inference relations in general, including entailment and presupposition.

[13]A more detailed procedure to test whether given information is a presupposition or not can be found in Geurts (1999). It may be worthwhile to point out that such a test, called the "families of sentences" test by Chierchia and McConnell-Ginet (2000), has some limitations: it cannot be applied to the types of sentences that are unable to appear in the environments indicated in (21). Thus, interrogative sentences cannot be embedded in any of the environments in (21); hence the "families of sentences" test says nothing about whether it has any presuppositions. For some discussion on this point, see Beaver (2001: 18–20).

c. If it was Sam who broke the window, then he will have to fix it. THE ANTECEDENT OF A CONDITIONAL

d. Was it Sam who broke the window? QUESTION

e. Suppose that it was Sam who broke the window. HYPOTHETICAL ASSUMPTION

(22) is embedded in various environments in (24). What is remarkable is that in a normal conversational situation, (23) is implied not only by (22) but also by each embedding construction in (24). In other words, the proposition in (23) is *projected* out of these environments. Based on this observation, we can conclude that (23) is a presupposition of (22).

To confirm that presuppositions differ from entailments in their inference pattern, let us look at a typical case of entailment:

(25) Sam broke the window.

(26) Sam broke something.

(27) a. Sam didn't break the window.

b. Maybe Sam broke the window.

c. If Sam broke the window, the glass fell down.

d. Did Sam break the window?

e. Suppose that Sam broke the window.

(25) is embedded in various environments in (27). But in this case, whereas (25) implies (26), all of the embedding constructions in (27) do not. Thus, the test confirms the standard assumption that the relation holding between (25) and (26) is not a presupposition but an entailment. We can see that entailments and presuppositions are systematically distinguished in terms of their projection behavior in embedded contexts.

A naive hypothesis about the projection problem is what is called the *cumulative hypothesis*, originally discussed by Morgan (1969) and Langendoen and Savin (1971):

> **The cumulative hypothesis** If an elementary clause S has a presupposition p, then p is also a presupposition of the whole complex sentence containing S.

As already observed by Morgan (1969), there are clear counter-examples to this hypothesis.[14] Consider the following examples:

(28) a. If the window was broken, then it was Sam who broke it.
b. Either the window wasn't broken, or it was Sam who broke it.

As we saw above, the sentence *It was Sam who broke it* triggers the presupposition in (23) when uttered in isolation.

(23) Someone broke the window.

The cumulative hypothesis predicts that this presupposition is inherited in complex constructions such as (28a) and (28b). However, it is obvious that neither implies (23). This point is further confirmed by the following example:

(29) The window was broken and it was Sam who broke it.

Obviously, (23) is not a presupposition but an entailment of (29). Such examples illustrate yet another important property of presuppositions; namely, the presuppositions associated with elementary clauses disappear in certain environments.

(30) **Generalization 2**. Let S and S' be sentences such that S implies the presuppositions of S'. Then, the following constructions do not inherit the presuppositions of S':
a. S and S' CONJUNCTION
b. If S then S' CONDITIONAL
c. Either it is not the case that S or S'. DISJUNCTION

[14] Langendoen and Savin (1971) also discussed conditional sentences like (28a).

With this background on presuppositions in mind, let us go back to the Strawsonian analysis of descriptions. Now, it is easy to see that the behavior of definite descriptions in (10), repeated below, which is problematic for Strawson, is subject to the same patterns in (30). That is, (10a), (10b), and (10c) are instances of the three patterns in (30), respectively:

(10) a. John has children and his children are wise.
 b. If John has children, his children are wise.
 c. Either John does not have any children or his children are wise.

The examples in (7a) and (7b), repeated below, which provide initial motivation for the Strawsonian analysis, are also consistent with the projection behavior standardly assumed in the literature on presupposition.

(7) a. I wonder whether the ghost in my attic will be quiet tonight.
 b. If the ghost in my attic is quiet tonight, I will hold a party.

The conditional sentence (7b) is an instance of the schema given in (21c). In (7a), the description *the ghost in my attic*, which acts as a presupposition trigger, appears in the complement of the propositional attitude construction. This example conforms to the standard account of the interaction between propositional attitude verbs and presuppositions, first proposed by Karttunen (1974) and further developed by Heim (1992).[15] Let S be an elementary sentence that triggers the presupposition that P. Then, according to Karttunen (1974), a sentence of the form (31a) gives rise to an inference shown in (31b).

(31) a. A wonders whether S
 b. A believes that P.

More generally, according to the Karttunen-Heim view, attitude verbs such as *doubt*, *hope*, and *want*, i.e., those that are called *filters* in Karttunen (1974), give rise to the same pattern of presuppositions as in (31). Thus we would normally infer from the utterance of (32a) that (32b) is true.

[15]For the presuppositions of attitude verbs, see also Geurts (1996).

(32) a. Patrick wants to sell his cello. (Heim 1992)
b. Patrick believes that he owns a cello.

All the evidence suggests is that the problem of “filtration” as posed for the Strawsonian analysis of definite descriptions is not special but common to various presupposition triggers; it is an instance of the general problem of presupposition projection. This view suggests the possibility of applying to descriptions whatever theories we already have for other presupposition triggers. Thus, we can say that if the Strawsonian analysis of definite descriptions was combined with some suitable theory of presupposition projection, it would have an explanatory advantage: such an analysis would enable us to avoid the problem of filtration and, furthermore, to understand the behavior of descriptions in a much broader perspective — in a parallel manner to the other presupposition triggers. This is why in recent work on descriptions, including Heim and Krartzer (1998), von Fintel (2004), Szabó (2005), Elbourne (2005), Glanzberg (2007), and Rothschild (2007), among others, Strawsonian analysis was widely preferred to Russellian analysis with respect to the presupposition data seen in this section. Furthermore, as we saw before, the problem of open descriptions shows that Strawsonian analysis needs to be enriched with some suitable account of the interaction of presupposition and quantification. Since the seminal work of Heim (1983), such an account of “presuppositions below clauses” has been elaborated in the tradition of *dynamic semantics*. In the next sections, we review two influential approaches to presupposition projection in a dynamic setting.

3. Two theories of presupposition projection

The two most influential theories of the projection problem are Heim's (1983) dynamic semantics and van der Sandt's (1992) discourse representation theory. Both theories can be subsumed under the "dynamic" account of meaning, which departs from the traditional truth-conditional conception in a crucial way. The basic idea behind this dynamic approach—which was most notably proposed by Stalnaker (1978)—is that the utterance of a sentence *changes*, in a certain way, the context in which it was made; when the sentence uttered has a complex structure the process by which the utterance changes the context is composed of several intermediate steps, and this process of context change plays a crucial role in accounting for the projection behavior of presuppositions. Heim's dynamic semantics and van der Sandt's discourse representation theory implement Stalnaker's original idea in different ways, making different predictions of presupposition projection patterns for some cases. In the following sections, we review the two theories and identify some problems that confront them.[1]

3.1 Basic assumptions of dynamic approach

A theory of presupposition based on the dynamic approach was initially proposed by Stalnaker (1973, 1974). As mentioned above, the key observation is that utterances are made in a context and that they bring about some change in the context. If we confine our attention to the type of conversation

[1]See Okada and Mineshima (2009) for a brief survey of dynamic semantics and other logical systems handling dynamic state transition that are currently studied in logic, linguistics, and computer science.

that aims to exchange information, the effect of an utterance is normally to add some information to the context shared by the participants in the conversation. More specifically, Stalnaker's proposal is based on the following assumptions:

(D1) A context is identified with a *common ground*—the body of information that is presumed to be common to the participants in the conversation.

(D2) An utterance is intended to affect the context: if the utterance of a declarative sentence is successful, then the (assertive) content of the utterance is added to the common ground, which provides the context for the subsequent discourse.

(D3) The presuppositions associated with a sentence are conditions it imposes on the context in which it is uttered—conditions that must be satisfied if the utterance is to affect the context in a successful way.

The crucial assumption that is used to account for the presupposition projection is (D2): as a conversation proceeds, the context does not remain fixed but is dynamically incremented according to what was asserted in the previous discourse. As we will see below, such a change is brought about not only by a whole sentence but also by subsentential components. To be more explicit, consider the case of the conjunctive sentence of the form "A and B". Following Karttunen (1974) and Schlenker (2009), among others, let us call the context in which the utterance of an expression E is evaluated the *local context* for E. The local context for "A and B" is calculated as follows:

(33) If "A and B" is uttered in an initial context σ, the local context for A is σ and the local context for B is $\sigma + A$, i.e., the context that is obtained by updating σ with the assertive content of A.

Stalnaker claims that the determination of a local context as in (33) derives from a general pragmatic consideration about a rational process of information exchange and, specifically, that the order in which the expressions are uttered plays a crucial role. Roughly speaking, the local context of a sentence S incorporates information delivered by a sentence that appears before S but is not normally affected by sentences that come after S. As emphasized by Schlenker (2008, 2009), however, a problem for Stalnaker's pragmatic approach is that it is not clear how to extend it to other connectives (e.g., disjunction) and quantifiers. Subsequent authors have instead developed semantic accounts that deal with a wide range of constructions, based on the basic assumptions in (D1), (D2), and (D3).

3.2 Dynamic semantics

Heim (1983) developed the dynamic approach to presuppositions by assuming that a sentence semantically encodes the information about how to change contexts, i.e., what she calls the *context change potential*. In Heim's theory, context change potentials are assigned to sentences, including quantified ones, in a compositional way. That is, given the syntactic structure of the sentence, the context change potential of a complex sentence is defined in terms of the context change potentials of its parts.

More specifically, Heim (1983) implements the basic assumptions in (D1), (D2), and (D3) along the following line.

(H1) The notion of a context is identified with a set of possible worlds (i.e., what Stalnaker (1974) called a *context set*).

(H2) The meaning of the (assertoric) utterance of a sentence S, i.e., the context change potential of S, is identified with a function that takes an input context and returns an updated context; typically, the effect of updating a context σ with a simple sentence S is to take the intersection of σ with the set of possible worlds in which S is true.

(H3) The notion of presuppositions is captured by taking a context change potential to be a partial function: presupposition failure occurs when an input context is not in the domain of the function in question.

The goal of this subsection is to see how this dynamic semantics works and what predictions it provides for the projection problem of presupposition. Indeed, Heim (1983) only sketched a formalization of her theory; subsequent authors implemented or reconstructed Heim's idea in various formal theories, including update semantics (Dekker 1992; Beaver 1992, 1994; van Eijck 1994; Veltman 1996), three-valued semantics (Krahmer 1998; Beaver and Karhmer 2001), and dynamic epistemic logic (van Eijck 2010). Here, we adopt the framework of Veltman-style update semantics (Veltman 1996) augmented with a binary operator to handle presuppositions, which we denote by $\mathbin{\setminus\!\!\setminus}$. This framework is closer to Heim's original proposal and neatly illustrates a general idea behind her theory. Throughout the following subsections, we refer to this framework simply as *dynamic semantics*.

We first look at the propositional fragment in Section 3.2.1 and address an inherent problem in the treatment of conjunction and conditional in Section 3.2.2. Then in Section 3.2.3, we introduce the mechanism of accommodation, which will be referred to throughout subsequent sections. Finally, in Section 3.2.4, we introduce the quantificational fragment of dynamics semantics and examine what predictions it makes for presuppositions of quantified sentences.

3.2.1 The propositional fragment

In the following, the propositional fragment of dynamic semantics is abbreviated as DS. The language of DS is a language of propositional logic augmented with a binary operator $\mathbin{\setminus\!\!\setminus}$, which is used to represent presuppositions. Intuitively, in a formula of the form $A \mathbin{\setminus\!\!\setminus} B$, A represents what is presupposed and B what is asserted.[2] For instance,

(34) Bill regrets that he lied to Mary. [=(14) in Section 2.2]

[2] Such a binary presupposition operator has not been unexplored in the logic and linguistics literature. Krahmer (1998) discusses it in the framework of three-valued semantics (see also Beaver and Krahmer 2001). Blamey (1986/2002) introduced a similar connective, *transplication*, denoted as A/B, in partial logic. It is also related to the operator for *conditional assertion* introduced in Belanp (1970), which is represented as A/B. An alternative to the binary presupposition operator is Beaver's unary presupposition operator, denoted by ∂A, which is formalized within dynamic semantics in a manner similar to that adopted in this section (cf. Beaver 1992, 1994, 2001).

is schematically represented as $p \setminus\!\!\setminus q$, where p represents the proposition that Bill lied to Mary and q the proposition that Bill regrets this. The first goal here is to provide the definition of context change potentials for formulas of this language.

Definition 3.1 (Language) We have a fixed set of propositional variables Φ whose elements are typically denoted by p, q, r (possibly with subscripts), and logical symbols $\neg, \wedge, \setminus\!\!\setminus$. The set of formulas, denoted by $\mathcal{L}_{\mathsf{DS}}$, is defined by the rule

$$A ::= p \mid \neg A \mid A \wedge B \mid A \setminus\!\!\setminus B$$

where p ranges over elements of Φ. We use A, B, C (possibly with subscripts) as meta-variables for formulas of this language.

In addition to the presupposition operator $\setminus\!\!\setminus$, we take negation and conjunction to be primitive logical operators. Conditional $A \to B$ is defined to be $\neg(A \wedge \neg B)$.

Notation. We save on parentheses by assuming that $\neg$ binds more strongly than $\wedge, \vee$, or $\setminus\!\!\setminus$, and that $\wedge, \vee$, and $\setminus\!\!\setminus$ bind more strongly than $\to$. Thus, $\neg A \setminus\!\!\setminus B$ is to be read as $(\neg A) \setminus\!\!\setminus B$ and $A \to B \setminus\!\!\setminus C$ as $A \to (B \setminus\!\!\setminus C)$.

Definition 3.2 (Model) A model of $\mathcal{L}_{\mathsf{DS}}$ is a pair $\mathcal{M} = \langle W, (\cdot)^*_{\mathcal{M}} \rangle$, where W is a non-empty set, whose elements are called *worlds* and denoted by $w_1, w_2, \ldots$, and $(\cdot)^*_{\mathcal{M}}$ is an interpretation function that assigns a non-empty subset $p^*_{\mathcal{M}}$ of W to each atomic formula p. A subset of W is called an *information state* and is typically denoted by σ, σ', and so on.

The notion of context as stated in (H1) is identified with this notion of an information state. As stated in (H2), the meaning of a formula is identified not with a truth-condition but with a *context change potential* or what is called an *update condition* (cf. Veltman 1996). The context change potential of a formula is defined as a partial function from information states to information states. We write the context change potential of a formula A as $[A]$. Following Veltman's (1996) postfix notation for update function, the result of applying $[A]$ to an information state σ (i.e. the result of updating

σ with $[A]$) is written as $\sigma[A]$, rather than $[A](\sigma)$. More generally, the result of updating σ with $[A_1], \ldots, [A_n]$ is written as $\sigma[A_1], \ldots, [A_n]$, rather than $[A_n](\ldots[A_1](\sigma)\ldots)$. When $\sigma[A]$ exists, we say that $[A]$ is defined for σ or, more simply, $\sigma[A]$ is defined.

Definition 3.3 (Context change potentials) Let $A \in \mathcal{L}_{\mathsf{DS}}$. The context change potential of A with respect to a DS-model $\mathcal{M} = (W, (\cdot)^*_{\mathcal{M}})$ is denoted by $[A]_{\mathcal{M}}$ and inductively defined as follows. To simplify the notation, we will omit reference to $\mathcal{M}$ and write p^* instead of $p^*_{\mathcal{M}}$, and $[A]$ instead of $[A]_{\mathcal{M}}$.

1. $\sigma[p]$ is always defined and $\sigma[p] = \sigma \cap p^*$ if p is atomic.
2. $\sigma[\neg A]$ is defined if $\sigma[A]$ is defined;
   if defined, $\sigma[\neg A] = \sigma \setminus \sigma[A]$.
3. $\sigma[A \wedge B]$ is defined if both $\sigma[A]$ and $\sigma[A][B]$ are defined;
   if defined, $\sigma[A \wedge B] = \sigma[A][B]$.
4. $\sigma[A \bbslash B]$ is defined if $\sigma[A]$ and $\sigma[B]$ are defined and $\sigma \subseteq \sigma[A]$;
   if defined, $\sigma[A \bbslash B] = \sigma[B]$.

A few comments are in order with respect to this definition. The context change potential of an atomic formula p is a function mapping an information state σ to $\sigma \cap p^*$. This amounts to excluding from the initial context σ all the worlds that are incompatible with p. For simplicity, we assume that atomic formulas do not introduce presuppositions by themselves.

In computing negation $\sigma[\neg A]$, σ is hypothetically updated with A, and then the result is subtracted from the original context σ.

The context change potential of $A \wedge B$ is a function that updates an initial context σ with A and successively with B. The point is that the second conjunct B is evaluated with respect to $\sigma[A]$, i.e., the context obtained by updating the initial context σ with A. It is evident that the specification of a local context for conjunction in (33) follows from this definition.

$A \bbslash B$ is associated with a *partial* function; hence, it is a possible source of presupposition failure. The function $[A \bbslash B]$ applied to an initial context σ can be regarded as a test to check whether or not σ already contains the information delivered by A. If $\sigma \subseteq \sigma[A]$ holds, that is, updating σ with A does not add any new information to σ, then we have the effect of updating σ with B; otherwise the function is undefined, resulting in a presupposition failure.

Given that $A \to B$ is defined as $\neg(A \wedge \neg B)$, one can specify the context change potential for implication as follows.

Fact 3.4 $\sigma[A \to B]$ is defined if both $\sigma[A]$ and $\sigma[A][B]$ are defined; if it is defined, $\sigma[A \to B] = \sigma \setminus (\sigma[A] \setminus \sigma[A][B])$.

The definedness condition of implication is the same as that of conjunction. Note that the definedness conditions of negation, conjunction, and implication are stated for explicitness. They can be read off from the respective main clauses, under the convention that if an argument of a function is undefined, then so is the value of that function.[3]

The crucial notions in DS, *acceptance* and *presupposition*, are defined as follows.

Definition 3.5 (Acceptance and presupposition) Let $A, B \in \mathcal{L}_{\mathsf{DS}}$.

1. An information state σ *accepts* A if $\sigma[A]$ is defined and $\sigma \subseteq \sigma[A]$.
2. A *presupposes* B if every information state σ for which $[A]$ is defined accepts B.

As is clear from Definition 3.3, applying the context change potential of a formula to an information state always yields a subset of that state; using

[3]The point that the definedness conditions of context change potentials can be read off from the definitions of the main effects is particularly emphasized in Heim (1983). In fact, Heim (1983) makes a stronger claim that the definedness condition (i.e., the so-called inheritance condition) of a sentence need not be specified separately from its main effects; rather, it logically follows from the specification of the main effects. However, given the objection that it is technically possible to define different context change potentials with the same main effects (cf. Soames 1989), Heim (1992) withdrawn this claim. Recently, Schlenker (2008, 2009) argued that Heim's original project is still tenable and showed that in a non-dynamic framework, the inheritance conditions of connectives can be derived from truth-conditions with the help of certain general pragmatic principles. Since this development is orthogonal to our interest, we do not discuss it any further in this thesis.

the terminology of e.g., Veltman (1996), the updates in DS are *eliminative* in the following sense.

Fact 3.6 For all information states σ and formulas A in $\mathcal{L}_{\mathsf{DS}}$, if $\sigma[A]$ is defined, then $\sigma[A] \subseteq \sigma$.

Consequently, when an information state σ accepts a formula A, we have $\sigma = \sigma[A]$; that is, σ is a fixed point of the function $[A]$.

An immediate consequence of Definition 3.3 is that it makes conjunction non-commutative. For instance, $p \wedge (p \setminus\!\!\setminus q)$ and $(p \setminus\!\!\setminus q) \wedge p$ denote different functions: if an initial context σ does not accept p, then $\sigma[(p \setminus\!\!\setminus q) \wedge p]$ is undefined, while $\sigma[p \wedge (p \setminus\!\!\setminus q)]$ is defined and computes to $\sigma[p][q]$. More generally, it is seen that $(p \setminus\!\!\setminus q) \wedge p$ presupposes p, while $p \wedge (p \setminus\!\!\setminus q)$ presupposes nothing; that is, $\sigma[p \wedge (p \setminus\!\!\setminus q)]$ is defined for any information state σ.

As a concrete example, consider a model in which $W = \{w_1, w_2, w_3\}$, $p^* = \{w_1, w_2\}$, and $q^* = \{w_1\}$. The following illustrates that the difference in information updates between $W[p \wedge (p \setminus\!\!\setminus q)]$ and $W[(p \setminus\!\!\setminus q) \wedge p]$ is caused by the presence of the presupposition operator:

$$\{w_1, w_2, w_3\} \xrightarrow{p} \{w_1, w_2\} \xrightarrow{p \setminus\!\!\setminus q} \{w_1\}$$
$$\{w_1, w_2, w_3\} \xrightarrow{p \setminus\!\!\setminus q} \textsf{presupposition failure}$$

Here, for the purpose of exposition, we write $\sigma \xrightarrow{A} \sigma'$ if $\sigma[A]$ is defined and computes to σ', and $\sigma \xrightarrow{A}$ presupposition failure if $\sigma[A]$ is undefined.

Arguably, this contrast is exemplified by the following pair of sentences:

(35) a. Someone solved the problem and it is John who solved it.
$p \wedge (p \setminus\!\!\setminus q)$
b. # It is John who solved the problem and someone solved it.
$(p \setminus\!\!\setminus q) \wedge p$

Here, p represents the proposition that someone solved the problem and q the proposition that John solved the problem. As noted before, Stalnaker (1974) attempted to explain such a contrast by appealing to general pragmatic considerations, specifically, Gricean maxim of manner. In Heim's dynamic semantics, by contrast, the oddity of (35b), which is marked by #, is directly accounted for in terms of the semantics of conjunction.

Although Heim (1983) does not provide a definition of validity in her dynamic semantics, one standard definition is as follows.[4]

Definition 3.7 (Validity) Let $A_1, \ldots, A_n, B \in \mathcal{L}_{\mathsf{DS}}$. B is a valid consequence of $A_1, \ldots, A_n$, written as $A_1, \ldots, A_n \models B$, if for any DS-model $\mathcal{M}$ and any information state σ based on $\mathcal{M}$ such that $\sigma[A_1], \sigma[A_1][A_2], \ldots$ and $\sigma[A_1] \ldots [A_n]$ are all defined, $\sigma[A_1], \ldots, [A_n]$ accepts B.

According to this definition, a formula B is a valid consequence of a sequence of formulas $A_1, \ldots, A_n$ if whenever an information state is sequentially updated with $A_1, \ldots, A_n$, the resulting state accepts B unless presupposition failure occurs along the way.

Fact 3.8 Let $A_1, \ldots, A_n, B \in \mathcal{L}_{\mathsf{DS}}$,

1. For all information states σ, $\sigma[A_1] \ldots [A_n]$ accepts B if and only if σ accepts $A_1 \wedge \cdots \wedge A_n \rightarrow B$.
2. $A_1, \ldots, A_n \models B$ if and only if $\models A_1 \wedge \cdots \wedge A_n \rightarrow B$.

With these definitions, it is easy to verify the following facts about the presupposition projection in DS.

Fact 3.9 Let p, q, r be atomic formulas. Then, $p \setminus\!\!\setminus q$, $\neg(p \setminus\!\!\setminus q)$ $(p \setminus\!\!\setminus q) \wedge r$, and $(p \setminus\!\!\setminus q) \rightarrow r$ all presuppose p.

These predictions agree with **Generalization 1** given in (21) in Section 2.2.

Fact 3.10 Let p' be an atomic formula such that $p' \models p$. Then, we have: $p \wedge (p \setminus\!\!\setminus q)$ and $p \rightarrow (p \setminus\!\!\setminus q)$ presuppose nothing; that is, for all information states σ, $\sigma[p \wedge (p \setminus\!\!\setminus q)]$, and $\sigma[p \rightarrow (p \setminus\!\!\setminus q)]$ are defined.

These predictions are clearly consistent with **Generalization 2** provided in (30) in Section 2.2.[5]

[4]There are several variations of definitions of validity in dynamic semantics. See Veltman (1996).

[5] It is not difficult to add disjunction and modal operators to Heim's framework. First, define $A \vee B$ as $\neg(\neg A \wedge \neg B)$ and add a unary operator $\Diamond$ to the language. The following

3.2.2 The proviso problem

Although Heim's dynamic semantics provide correct predictions for a wide range of constructions, it is known that there is a serious problem with the definition of the context change potentials for conjunction and implication—a problem called the "proviso problem" after Geurts (1999). Consider first the following pair of sentences:

(36) a. If someone solved the problem, it is John who solved it.
$p \to (p \setminus\!\!\setminus q)$

b. If the problem was difficult, it is John who solved it.
$r \to (p \setminus\!\!\setminus q)$

Abstracting away from the subsentential components, p represents "Someone solved the problem," q "John solved the problem," and r "The problem was difficult." The cleft sentence "It is John who solved the problem" is then represented as $p \setminus\!\!\setminus q$, and (36a) and (36b) are translated as indicated above. While Heim's dynamic semantics correctly predicts that (36a) presupposes nothing, it provides a wrong prediction for (36b). To see this point, observe first that the following holds in DS.

Fact 3.11 $r \to (p \setminus\!\!\setminus q)$ presupposes $r \to q$.

This can be proved as follows: for any information state σ,

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definitions of context change potentials are due to Beaver (2001):

1. $\sigma[A \vee B]$ is defined if both $\sigma[A]$ and $\sigma[\neg A][B]$ are defined;
if defined, $\sigma[A \vee B] = \sigma \setminus ((\sigma \setminus \sigma[A]) \setminus (\sigma \setminus \sigma[A][B])$

2. $\sigma[\Diamond A]$ is defined if $\sigma[A]$ is defined;
if defined, $\sigma[\Diamond A] = \sigma$ when $\sigma \subseteq \sigma[A]$; otherwise $\sigma[\Diamond A] = \emptyset$.

The unary operator $\Diamond$ is meant to represent epistemic possibility operators like *might*. The above definitions ensure that for atomic formulas p, q, r, (i) $(p \setminus\!\!\setminus q) \vee r$ presupposes p; (ii) $p \vee (p \setminus\!\!\setminus r)$ presupposes nothing; (iii) $\Diamond p$ presupposes p. (ii) corresponds to the scheme in (30c) mentioned in Section 2.2, (iii) means that epistemic modal *might* is a *hole* in Karttunen's sense (see (21b) in Section 2.2).

$$\begin{array}{llll} \sigma[r \to (p \mathbin{\backslash\!\!\backslash} q)] \text{ is defined} & \text{iff} & \sigma[r][p \mathbin{\backslash\!\!\backslash} q] \text{ is defined} & \text{(by Fact 3.4)} \\ & \text{iff} & \sigma[r] \subseteq \sigma[r][p] & \text{(by Definition 3.3.3)} \\ & \text{iff} & \sigma[r] \text{ accepts } p & \text{(by Definition 3.5)} \\ & \text{iff} & \sigma \text{ accepts } r \to p & \text{(by Fact 3.8.1)} \end{array}$$

Hence, by definition, $r \to (p \mathbin{\backslash\!\!\backslash} q)$ presupposes $r \to p$. Accordingly, it is predicted that (36b) presupposes that if the problem was difficult, someone solved it. Intuitively, however, this prediction is too weak: it is natural to interpret (36b) as presupposing that someone solved the problem, schematically represented as p. This intuition has been agreed on by various authors (e.g., Karttunen and Peters 1979; Soames 1989; Geurts 1999, among others).

The same point applies to Heim's definition of the context change potential of conjunction since, as shown in Fact 3.4, conjunction and implication are assigned the same definedness condition.

Fact 3.12 $r \wedge (p \mathbin{\backslash\!\!\backslash} q)$ presupposes $r \to q$.

Thus, (37) is predicted to presuppose that if the problem was difficult, someone solved it, schematically represented as $r \to p$, rather than that someone solved the problem, schematically represented as p.

(37) The problem was difficult and it is John who solved it.
$r \wedge (p \mathbin{\backslash\!\!\backslash} q)$

It might be argued that some additional pragmatic inferences could derive the desired presupposition p from $r \to p$.[6] That is, (36b) has basically the conditional presupposition $r \to p$, but some kind of pragmatic inferences strengthen $r \to p$ to p. However, as pointed out by Geurts (1996, 1999), pragmatic considerations alone are not enough to salvage Heim's definition of context change potentials of conjunction and implication. Consider the following pair of sentences:

(38) a. Peter knows that if the problem was difficult, someone solved it.
$(r \to p) \mathbin{\backslash\!\!\backslash} s$

[6] See Karttunen and Peters (1979) for an earlier proposal. See also van Rooij (2007) for a recent defense of strengthening inferences.

b. If the problem was difficult, then it isn't John who solved it.
$r \rightarrow \neg(p \mathbin{\backslash\!\backslash} q)$

Here p, q, r represent the same propositions as before, and s represents "Peter believes that if the problem was difficult." To simplify the matter concerning the semantics of *know*, we represent "x knows that A" as "$A \mathbin{\backslash\!\backslash}$($x$ believes that A)". Intuitively, (38a) has a conditional presupposition, "If the problem was difficult, someone solved it", namely, $r \rightarrow p$, whereas (38b) has an unconditional presupposition, "Someone solved the problem", namely, p. However, Heim's theory predicts that both have the conditional presupposition $r \rightarrow p$. Now, if some pragmatic inference is to derive p from $r \rightarrow p$ and the derivation solely depends on what proposition is literally presupposed—but not on *how* it is so presupposed—then it wrongly predicts that not only (38b) but also (38a) could have the unconditional presupposition p. Although various approaches to this problem have been proposed, it seems fair to say that there is no definitive solution in the literature.[7]

It should be noted here that as Geurts (1999) and Beaver (2001) point out, genuinely conditional presuppositions do arise in certain cases. Thus, in a normal context, (39a) has a conditional presupposition, as shown in (39b), rather than an unconditional presupposition "John has a wetsuit."

(39) a. If John is a diver, he'll bring his wetsuit on vacation.

b. If John is a diver, he has a wetsuit.

Dynamic semantics is good at handling such cases. The problem is, then, to explain why some sentences have a conditional presupposition, while others have an unconditional one. We will return to this issue in Section 3.4.

[7]See Karttunen and Peters (1979) for an earlier proposal based on pragmatic derivations. For arguments against Karttunen and Peters (1979), see Geurts (1999). Recently, Singh (2007) and Schlenker (2011b) proposed a new version of pragmatic derivations, which crucially depend on the syntactic structure of the sentences in question. We leave discussion of these new approaches for future occasion.

3.2.3 Accommodation and informative presuppositions

According to the dynamic view presented so far, a sentence S uttered in a conversation is intended to update the common ground σ of the conversation and the presuppositions associated with S are requirements placed on σ in order for the utterance of S to successfully update σ; if the common ground σ does not satisfy the presuppositions, then the update simply fails. As widely acknowledged, however, there seem to be clear counter-examples to this view. Consider:

(40) I am sorry I am late. My bike has a flat.

The utterance of the second sentence presupposes that the speaker has a bike. However, given the fact that the presupposed information is fairly uncontroversial, it appears that this sentence could successfully be uttered in a context where it is not already part of the common ground that the speaker has a bike. In this case, it would be quite unrealistic to suppose that the intended context update simply fails and communication breaks down at this point. Rather, such an example suggests that the presuppositions of an utterance are not always taken for granted but are sometimes informative to the participants in a conversation.

To account for this fact, it is standardly assumed by the proponents of the dynamic view of presuppositions that in such a case, the participants in the conversation quickly adjust the common ground so as to satisfy the presupposition in question. This process of adjustment has been called "accommodation" since the seminal work of Lewis (1979). A similar idea was earlier suggested by Karttunen (1974) and Stalnaker (1973). Thus, Karttunen (1974: 191) says:

> [O]rdinary conversation does not always proceed in the ideal orderly fashion described earlier. People do make leaps and shortcuts by using sentences whose presuppositions are not satisfied in the conversational context [...]. I think we can maintain that a sentence is always taken to be an increment to a context that satisfies its presuppositions. If the current conversational context does not suffice, the listener is entitled and expected to extend

> it as required. He must determine for himself what context he is supposed to be in on the basis of what was said and, if he is willing to go along with it, make the same tacit extension that his interlocutor appears to have made. This is one way in which we communicate indirectly, convey matters without discussing them.

Although there is a controversy about the existence and status of accommodation,[8] it is widely acknowledged among the proponents of dynamic semantics that it plays a crucial role in accounting for the flexible and context-dependent behavior of presupposition projection. In particular, with the help of the notion of local contexts, the mechanism of accommodation can be used to account for a certain sort of ambiguity that could be produced by a complex sentence with an embedded presupposition trigger.

As an illustration, consider the case of negation, discussed by Heim (1983). As stated above, $\neg(p \mathbin{\setminus\!\!\setminus} q)$ presupposes p. Now suppose that the common ground σ does not satisfy p. There are two ways in which accommodation could take place, which we call "global accommodation" and "local accommodation," respectively, following the standard terminology.

1. **Global accommodation**. The initial context σ is adjusted to the context that satisfies p, i.e., $\sigma \cap p^*$. Then, by definition, the context change potential for $\neg(p \mathbin{\setminus\!\!\setminus} q)$ with respect to the amended context $\sigma \cap p^*$ is defined; the following equation shows that it has the same effect as updating σ with $p \wedge \neg q$:

$$\begin{aligned}(\sigma \cap p^*)[\neg(p \mathbin{\setminus\!\!\setminus} q)] &= (\sigma \cap p^*) \setminus (\sigma \cap p^*)[p \mathbin{\setminus\!\!\setminus} q] \\ &= (\sigma \cap p^*) \setminus (\sigma \cap p^*)[q] \\ &= \sigma[p] \setminus \sigma[p][q] \\ &= \sigma[p][\neg q] \\ &= \sigma[p \wedge \neg q]\end{aligned}$$

[8] For recent useful overviews, see Beaver and Zeevat (2007) and von Fintel (2008).

2. **Local accommodation**. With the initial context σ, the context change potential for $\neg(p \bbslash q)$ is calculated as follows:

$$\sigma[\neg(p \bbslash q)] = \sigma \setminus \sigma[p \bbslash q].$$

For the presupposition p to be satisfied, it is conceivable to replace the second occurrence of σ, i.e., the input context for the embedded construction $p \bbslash q$, by $\sigma \cap p^*$. Then, we obtain:

$$\begin{aligned} \sigma \setminus (\sigma \cap p^*)[p \bbslash q] &= \sigma \setminus (\sigma \cap p^*)[q] \\ &= \sigma \setminus \sigma[p][q] \\ &= \sigma \setminus \sigma[p \wedge q] \\ &= \sigma[\neg(p \wedge q)] \end{aligned}$$

Hence the local accommodation ends up with the same result as updating σ with $\neg(p \wedge q)$.

As a concrete example, consider the following:

(41) John didn't stop smoking. $\neg(p \bbslash q)$

Here p = "John used to smoke" and q = "John stopped smoking." Now, suppose (41) is uttered in a context that does not entail the proposition that John used to smoke.[9] Then, the hearer could adjust the context either by global accommodation or local accommodation. In the former case, we end up with the reading in (42a). In the latter case, we obtain the reading in (42b).

(42) a. John used to smoke and he stopped smoking. $p \wedge \neg q$

b. It is not the case that John used to smoke and he stopped smoking. $\neg(p \wedge q)$

Potentially, both readings would be available, but the preferred reading would be the first one. Heim (1983) and many subsequent authors suggested that global accommodation is generally preferred to local accommodation.

[9] This sentence may have other presuppositions such as that concerning the existence of the entity denoted by *John*. But this can be ignored for the current purpose.

In fact, in the case of (41), local accommodation seems to be a highly marked option in that it only arises if the hearer has special reason to abandon the global accommodation reading. As Kadmon (2001: 172) points out, however, in examples such as (43) the local accommodation reading, as shown in (44b), is quite natural, although the global accommodation reading, as shown in (44a), is also certainly possible.

(43) If Sue stopped smoking yesterday, she will get a prize from the health bureau. $(p \setminus\!\!\setminus q) \rightarrow r$

(44) a. Sue used to smoke, and if she stopped smoking yesterday, she will get a prize from the health bureau. $p \wedge (q \rightarrow r)$

b. If John used to smoke and stopped smoking yesterday, she will get a prize from the health bureau. $p \wedge q \rightarrow r$

As we will see later, for certain cases, there can be more than one way of performing local accommodation. Proponents of discourse representation theory (e.g., van der Sandt 1992; Geurts 1999) emphasize this fact and, accordingly, the notion of accommodation plays a more prominent role in their theory (see Section 3.3.2 below).

3.2.4 The quantificational fragment

Most of the discussion on the presupposition projection in the literature before Heim (1983) centered around the question of how the presuppositions of a simple sentence (i.e., an elementary clause) are inherited to a complex sentence containing it. Heim (1983) emphasizes that her dynamic approach provides a way to handle presuppositions *below* the level of simple sentences. We have already seen some typical examples that need to be accounted for by such an extended theory, namely, Mates' examples of open sentences containing a variable bound by an outside quantifier; see examples (12a) and (12b) on page 150. Heim (1983) provides only a sketch of the extension of her dynamic framework to a quantificational language. Subsequent authors, including Dekker (1992, 1996), Beaver (1992, 1994, 2001), and Groenendijk, Stokhof and Veltman (1996), formally developed Heim's idea within the framework of dynamic (update) semantics. Building on these studies, we will reconstruct Heim's proposal and present a dynamic semantics that deals with the presuppositions of quantified sentences. We refer to this system by DS_q. The predictions made by this system with respect to the projection behavior of quantified sentences will be a basis for our subsequent discussions.

The language of DS_q, which we denote by $\mathcal{L}_{\mathsf{DS}q}$, is obtained by adding a binary presupposition operator $\between$ to the language of standard predicate logic. More specifically, the vocabularies of $\mathcal{L}_{\mathsf{DS}q}$ consist of a set of individual variables, typically denoted by $x_1, x_2, \ldots$, a set of n-place relation symbols, typically denoted by $F, G, \ldots$ (possibly with numerical subscripts), and logical operators $\neg, \wedge, \between, \exists$. The formulas (denoted by A, B, C) are defined by the following rule:

$$A ::= F(x_1, \ldots, x_n) \mid \neg A \mid A \wedge B \mid A \between B \mid \exists x A$$

To simplify the exposition, we do not consider individual constants and equality.[10] Disjunction $\vee$, implication $\rightarrow$, and universal quantification $\forall$ are introduced as defined symbols: $A \vee B$ is an abbreviation for $\neg(\neg A \wedge \neg B)$, $A \rightarrow B$ for $\neg(A \wedge \neg B)$, and $\forall x A$ for $\neg \exists x \neg A$. We omit parentheses by assuming that $\forall, \exists, \neg$ bind more strongly than $\wedge, \vee, \between$, and that $\wedge, \vee, \between$ bind more strongly than $\rightarrow$ (cf. the convention in DS introduced on page 163). Sometimes, we also omit parentheses by writing, e.g., Fx_1x_2 instead of $F(x_1, x_2)$.

If we take into account the contributions made by quantifiers and variables to the conversational contexts, it is crucial to distinguish two kinds of information, which we call *discourse information* and *information about the world*, following Groenendijk, Stokhof and Veltman (1996). The former kind of information is concerned mainly with what individuals are introduced in a given context, and the latter is concerned with what properties are held to be true of the individuals thus introduced. Correspondingly, there are two

[10]Also, modal operators are not considered here. Indeed, it is not a trivial matter to add modal operators such as epistemic *might* to the quantificational fragment of dynamic semantics with individual constants and equality; it causes additional complications in a dynamic setting. See Groenendijk, Stokhof, and Veltman (1996) for discussion.

ways in which a context is updated: one way is by introducing new individuals into discourse, and the other way is by picking out individuals that are already introduced and eliminating a possibility that is incompatible with the information delivered by a sentence used. Typically, existential quantifiers are responsible for the former way of updating a context, and open sentences (formulas involving free variables) for the latter. In DS_q, these two ways of information update are semantically explicated in terms of the notion of *partial* assignment functions or, equivalently, a *finite* sequence of individuals. An information state in DS_q is identified with a set of partial assignment functions.[11] Roughly, to update discourse information is to add a new assignment in the current state, and to update information about the world is to eliminate possible assignments in the current state.

The notions of *model* and *information state* in DS_q are defined as follows.

Definition 3.13 (Model) A model of $\mathcal{L}_{\mathsf{DS}q}$ is a pair $\mathcal{M} = \langle D, (\cdot)^*_{\mathcal{M}} \rangle$, where D is a non-empty set, called a domain of individuals, and $(\cdot)^*_{\mathcal{M}}$ is an interpretation function mapping relational symbol F with arity n to $F^*_{\mathcal{M}} \subseteq D^n$.

Definition 3.14 (Information state) Let $\mathcal{M} = \langle D, (\cdot)^*_{\mathcal{M}} \rangle$ be a DS_q-model.

[11] The idea of using partial assignment function in dynamics semantics goes back at least to Heim (1983). In Heim (1983), information states are identified with a set of pairs $\langle w, g \rangle$ of worlds w and (partial) assignment functions g, for the purpose of treating the interaction between quantifiers and modal and intensional expressions in context change potential models (such a treatment is given in Heim 1992; see Groenendijk, Stokhof, and Veltman 1996 for formal development of a similar idea). Since our focus is on the presupposition projection of quantified sentences in *extensional* contexts, we do not consider world variables here. This will simplify the formulation of DS_q. A similar approach is taken in EDPL of Dekker (1992, 1996), where a Veltman-style update semantics for Dynamic Predicate Logic (DPL) of Groenendijk and Stokhof (1991) is provided. But the main concern of DPL and EDPL is the interpretation of anaphoric pronouns (formalized as free variables) and their interaction with quantifiers; accordingly, presupposition operators, which concern us in this section, are not considered there. Beaver (1992, 1994) develops Heim's idea of information states as world-sequence pairs, proposing a dynamic semantics for a (modal) predicate logic with a unary presupposition operator (termed as ∂), but the definedness condition of the presupposition operator given by Beaver (1992, 1994) is essentially different from the treatment in Heim (1983). See also the discussion at the end of this subsection.

By an information state σ, we mean a set of partial assignment functions whose domain is a finite set of variables, $\{x_1, x_2, \ldots, x_n\}$ and whose range is D, satisfying the condition that each assignment function in σ has the same domain, i.e., if s and s' are in σ then $\text{dom}(s) = \text{dom}(s')$.

We denote information states by $\sigma, \sigma', \ldots$, and assignment functions by s, s', and so on. We will adopt the following notation:

- By $\text{dom}(s)$ we denote the domain of an assignment function s.
- By $s[x/d]$ we denote the assignment function s' such that $x \notin \text{dom}(s)$, $\text{dom}(s') = \text{dom}(s) \cup \{x\}$, and $s'(x) = d$. That is, $s[x/d]$ is the assignment function that differs from s in that its domain contains a new variable x to which an individual d is assigned.
- By $\sigma[x/d]$ we denote the information state $\sigma' = \{s[x/d] \mid s \in \sigma\}$.
- Given a fixed set of variables, $\{x_i \mid i \leq n\}$, we often identify an assignment function $\{(x_1, a_1), (x_2, a_2), \ldots, (x_n, a_n)\}$ with a finite sequence of individuals $a_1 a_2 \ldots a_n$.

We mentioned the two ways in which information states are updated: (i) discourse information is extended by introducing new individuals; and (ii) information about the worlds is extended by eliminating possibilities. This distinction is captured by the following definition, which is due to Groenendijk, Stokhof and Veltman (1996).

Definition 3.15 Let s, s' be assignment functions, and σ, σ' information states.

1. s' is an *extension* of s, written as $s \leq s'$, if $\text{dom}(s) \subseteq \text{dom}(s')$ and for all $x \in \text{dom}(s)$, $s(x) = s'(x)$.
2. σ' is an *update* of σ, written as $\sigma \preccurlyeq \sigma'$, if for all $s' \in \sigma'$ there exists $s \in \sigma$ such that $s \leq s'$.

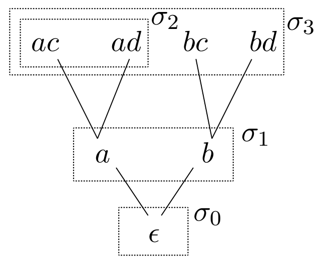
A state σ' is an update of a state σ if every assignment function in σ' is an extension of some assignment function in σ. Note that the definition allows some of the assignments in σ to be eliminated in σ'. Typically, when $\sigma \leq \sigma'$, some of the assignments in σ are eliminated in σ', and the remaining ones are extended with a new variable. It is easily seen that $\leq$ and $\preccurlyeq$ are partial orders. The minimal element with respect to $\leq$ is $\emptyset$, which we call *the empty assignment* (or *the empty sequence*) and denote by ϵ. The maximal element with respect to $\preccurlyeq$, i.e., the maximal information state, is also $\emptyset$, which we call *the absurd state* and denote by $\perp$.

It will be useful to introduce a notion of *non-eliminative* updates, namely, updates that may introduce new variables but do not eliminate any assignments.

Definition 3.16 Let σ and σ' be information states such that $\sigma \preccurlyeq \sigma'$, i.e., σ' is an update of σ. Then, σ' is a *non-eliminative* update of σ, written as $\sigma \sqsubseteq \sigma'$, if for all $s \in \sigma$ there exists $s' \in \sigma'$ such that $s \leq s'$; otherwise, σ' is an *eliminative* update of σ.

If σ' is a non-eliminative update of σ (i.e., $\sigma \sqsubseteq \sigma'$), then σ' is an update of σ (i.e., $\sigma \leq \sigma'$) and, furthermore, no possibilities (i.e., assignment functions) in σ are eliminated in σ'; only new variables to which some individuals are assigned can be introduced in σ'. The notion of non-eliminative update here is essentially the same as the notion of *subsistence* in Groenendijk, Stokhof, and Veltman (1996).

As an illustration, let a and b be individuals, and consider information states $\sigma_0 = \{\epsilon\}, \sigma_1 = \{a, b\}, \sigma_2 = \{ac, ad\}$, and $\sigma_3 = \{ac, ad, bc, bd\}$. The figure on the right illustrates the relationship between these states. Note that for all assignment functions s, $\epsilon \leq s$; hence, for all information states σ, $\{\epsilon\} \preccurlyeq \sigma$ and $\{\epsilon\} \sqsubseteq \sigma$. The state $\{\epsilon\}$ can be regarded as an initial state in which no discourse information has yet been established. Regarding the other states, we have: $\sigma_1 \preccurlyeq \sigma_2$, $\sigma_1 \preccurlyeq \sigma_3$, $\sigma_1 \not\sqsubseteq \sigma_2$, and $\sigma_1 \sqsubseteq \sigma_3$. Here, we see that σ_2 is an eliminative update of σ_1, and that σ_3 is a non-eliminative update of σ_1.



Now, the context change potentials in DS_q are defined as follows.

Definition 3.17 (Context change potentials) Let A in $\mathcal{L}_{\mathsf{DS}q}$, and let $\mathcal{M} = \langle D, (\cdot)^*_{\mathcal{M}} \rangle$ be a DS_q-model. The context change potential $[A]_{\mathcal{M}}$ of A is inductively defined as follows. Henceforth, we omit reference to $\mathcal{M}$ in $(\cdot)^*_{\mathcal{M}}$ and $[\cdot]_{\mathcal{M}}$.

1. $\sigma[F(x_1, \ldots, x_n)]$ is defined if $\{x_1, \ldots, x_n\} \subseteq \mathrm{dom}(\sigma)$;
   if defined, $\sigma[F(x_1, \ldots, x_n)] = \{s \in \sigma \mid \langle s(x_1), \ldots, s(x_n) \rangle \in F^*\}$.
2. $\sigma[\neg A]$ is defined if $\sigma[A]$ is defined;
   if defined, $\sigma[\neg A] = \{s \in \sigma \mid \text{there exists no } s' \in \sigma[A] \text{ such that } s \leq s'\}$.
3. $\sigma[A \wedge B]$ is defined if $\sigma[A]$ and $\sigma[A][B]$ are defined;
   if defined, $\sigma[A \wedge B] = \sigma[A][B]$.
4. $\sigma[A \setminus\!\!\setminus B]$ is defined if $\sigma[A]$ and $\sigma[B]$ are defined and $\sigma \sqsubseteq \sigma[A]$;
   if defined, $\sigma[A \setminus\!\!\setminus B] = \sigma[A][B]$.
5. $\sigma[\exists x A]$ is defined if $x \notin \mathrm{dom}(\sigma)$ and $\sigma[x/d][A]$ is defined for all $d \in D$;
   if defined, $\sigma[\exists x A] = (\bigcup_{d \in D} \sigma[x/d])[A]$.

The context change potential of an atomic formula $F(x_1, \ldots, x_n)$ is a partial function: if any one of the arguments $x_1, \ldots, x_n$ is not present in the input state σ, then the function is undefined and presupposition failure occurs.

In computing $\sigma[\neg A]$, σ is hypothetically updated with A; if $\sigma[A]$ is defined, then those possibilities that have an extension in $\sigma[A]$ are eliminated from σ. This definition of $[\neg A]$ differs from the one in DS because in DS_q, the formula A may introduce new variables in the course of the hypothetical update. (A typical example is a formula of the form $\neg\exists x F x$; see the example below.)

The context change potential of $A \setminus\!\!\setminus B$ is also a partial function; if in updating σ with A, any possibility in the initial context σ is eliminated (in other words, any information about the world is lost), then the function is undefined, and we end up with presupposition failure. Here, we see some differences from the definition of $[A \setminus\!\!\setminus B]$ in DS (cf. Definition 3.3.4). First, since A may introduce new variables, the definedness condition requires that $\sigma[A]$ be a non-eliminative extension of σ (i.e., $\sigma \sqsubseteq \sigma[A]$), rather than $\sigma \subseteq \sigma[A]$. Second, when the definedness condition is satisfied, the initial state σ is not directly updated with the assertive component B, because the presuppositional component A may introduce new variables. Rather, σ is first updated with component A, and then the resulting state $\sigma[A]$ is updated with B. Put differently, for the initial state σ to be successfully updated with $A \setminus\!\!\setminus B$, σ must contain all the information about the world encoded by A, but σ may be extended with some new discourse information contained in A.

Finally, in updating a context σ with $\exists xA$, σ is extended with a new variable x; when the variable x is already present in σ, the updating process comes to a halt. Thus a repetition of an existential quantifier with the same variable, say, $\exists xFx \wedge \exists xGx$, always results in presupposition failure.[12] It should be noted that as seen from the definition, $\exists x$ can be regarded as a separate unit; $\exists xA$ can be decomposed as $\exists x \wedge A$. The context change potential of $\exists x$ is then defined as follows: $\sigma[\exists x] = \bigcup_{d \in D} \sigma[x/d]$ if $x \notin \mathrm{dom}(\sigma)$; otherwise it is undefined. Although the syntax becomes less familiar, this separation makes it explicit that the role of an existential quantifier $\exists x$ in DS_q is to add new discourse information upon which the subsequent part of the formula makes a comment.

To sum up, there are three sources of undefinedness (i.e., presupposition failure) in DS_q: (i) atomic formulas, (ii) existential quantifiers, and (iii) presupposition operators. (i) and (ii) are concerned with discourse information: in the case of (i) the relevant variables must already be present in the initial context; in the case of (ii), they must not. (iii) is concerned with information about the world: the initial context must already contain the world information specified in the antecedent of the presupposition operator.

It is observed from Definition 3.17 that interpretations always give rise to an information state that is an *update* of the input state.

Fact 3.18 For all formulas $A \in \mathcal{L}_{DS_q}$ and all information states σ, $\sigma \leq \sigma[A]$.

[12] Gronendijk, Stokhof, and Veltman (1996) do not adopt this view; using the notion of *pegs* they allow an existential quantifier to be repeatedly used with the same variable. Our treatment of existential quantifiers is similar to that in Dekker (1992, 1996) and also close to the original proposal (with respect to indefinites) in Heim (1982, 1983).

Now, we recall that $A \to B$ is defined as $\neg(A \wedge \neg B)$ and $\forall x A$ as $\neg\exists x \neg A$. The context change potentials of implication and universal quantification are then derived as follows.

Fact 3.19 (Implication and universal quantifier)

1. $\sigma[A \to B]$ is defined if both $\sigma[A]$ and $\sigma[A][B]$ are defined;
   if defined, $\sigma[A \to B] = \{s \in \sigma \mid \text{for all } s', \text{ if } s \leq s' \text{ and } s' \in \sigma[A], \text{ then there is } s'' \text{ such that } s' \leq s'' \text{ and } s'' \in \sigma[A][B]\}$
2. $\sigma[\forall x A]$ is defined if $\sigma[x/d][A]$ is defined for all $d \in D$;
   if defined, $\sigma[\forall x A] = \{s \in \sigma \mid \text{for all } d \in D, \text{ there is } s' \text{ such that } s \leq s' \text{ and } s' \in \sigma[x/d][A]\}$

The definedness condition of implication $[A \to B]$ is the same as that of conjunction $[A \wedge B]$. After updating a state σ with $A \to B$, a possibility s in σ will remain if every extension of s in $\sigma[A]$ has an extension in $\sigma[A][B]$. If no $s \in \sigma$ has an extension in $\sigma[A]$, then $\sigma[A \to B] = \sigma$.

The definedness condition of universal quantification $[\forall x A]$ is the same as that of existential quantification $[\exists x A]$. In updating σ with $\forall x A$, a possibility s in σ will remain if s has an extension in $\sigma[x/d][A]$ for any $d \in D$.

Let us prove Fact 3.19.2 and see how the context change potential of $\forall x A$ is derived.

Proof. By definition, we have:

$$\begin{aligned} s \in \sigma[\forall x A] &\iff s \in \sigma[\neg\exists x \neg A] \\ &\iff \text{(i) } s \in \sigma \text{ and (ii) } s \text{ has no extension in } \sigma[\exists x \neg A]. \end{aligned}$$

(ii) means that s has no extension in $\sigma[x/d][\neg A]$ for any $d \in D$, i.e.,

> there is no s' such that $s \leq s'$, $s' \in \sigma[x/d]$, and s' has no extension in $\sigma[x/d][A]$ for any $d \in D$.

This is equivalent to the following:

(iii) For all $d \in D$ and all s', if $s \leq s'$ and $s' \in \sigma[x/d]$, s' has an extension in $\sigma[x/d][A]$.

It is clear, then, that $[\forall x A]$ has the same definedness condition as $[\exists x A]$. Now we claim that when $s \in \sigma$, (iii) is reduced to the following:

(iv) For all $d \in D$, s has an extension in $\sigma[x/d][A]$.

So let $s \in \sigma$ and $d \in D$.
(iii) $\Longrightarrow$ (iv): We have $s \leq s[x/d]$. Since $s \in \sigma$, we also have $s[x/d] \in \sigma[x/d]$. Then, from (iii), it follows that $s[x/d]$ has an extension in $\sigma[x/d][A]$. Since $\leq$ is transitive, this implies that s has an extension in $\sigma[x/d][A]$.
(iv) $\Longrightarrow$ (iii): Suppose that $s \leq s'$ and $s' \in \sigma[x/d]$. Certainly, $s' = s[x/d]$. By (iv), there is some s'' such that $s \leq s''$ and $s'' \in \sigma[x/d][A]$. Since $\sigma \leq \sigma[x/d] \leq \sigma[x/d][A]$, we have: for some $t \in \sigma$ and some $t' \in \sigma[x/d]$, $t \leq t' \leq s''$. But then, given $t \leq s''$, $s \leq s''$ and $s, t \in \sigma$, it clearly follows that $s = t$. So, $s \leq t'$. Since $t' \in \sigma[x/d]$, we have $t' = s[x/d]$. Hence, $s[x/d] \leq s'' \in \sigma[x/d][A]$, i.e., $s[x/d]$ has an extension in $\sigma[x/d][A]$. ■

Some simple examples may help understand how context change potentials work. Consider:

(45) a. A man walks. He whistles.
b. $\exists x_1(\mathsf{man}(x_1) \wedge \mathsf{walk}(x_1)) \wedge \mathsf{whistle}(x_1)$

Let us assume that (45a) is translated as (45b) in DS_q. Here, the occurrence of the variable x_1 in $\mathsf{whistle}(x_1)$ remains free; but it is, in a sense, allowed to be bounded by the initial quantifier $\exists x_1$. To see it, consider a model $\mathcal{M} = \langle D, (\cdot)^* \rangle$ where $D = \{a, b, c\}$, $\mathsf{man}^* = \{a, b, c\}$, $\mathsf{walk}^* = \{a, b\}$, and $\mathsf{whistle}^* = \{a\}$. Note that $\{a, b, c\}$ is an abbreviation for $\{(x_1, a), (x_1, b), (x_1, c)\}$, and so on. Suppose that the initial context σ_0 is $\{\epsilon\}$, where no discourse information is introduced. Then, we have:

$$\begin{aligned}
&\sigma_0[\exists x_1(\mathsf{man}\, x_1 \wedge \mathsf{walk}\ x_1) \wedge \mathsf{whistle}\, x_1] \\
&\quad = \sigma_0[\exists x_1(\mathsf{man}\, x_1 \wedge \mathsf{walk}\ x_1)][\mathsf{whistle}\, x_1] \\
&\quad = (\textstyle\bigcup_{d \in D} \sigma_0[x/d])[\mathsf{man}\, x_1 \wedge \mathsf{walk}\ x_1][\mathsf{whistle}\, x_1] \\
&\quad = (\textstyle\bigcup_{d \in D} \sigma_0[x/d])[\mathsf{man}\, x_1][\mathsf{walk}\ x_1][\mathsf{whistle}\, x_1] \\
&\quad = \{a, b, c\}\, [\mathsf{man}\, x_1][\mathsf{walk}\ x_1][\mathsf{whistle}\, x_1] \\
&\quad = \{a, b, c\}\, [\mathsf{walk}\ x_1][\mathsf{whistle}\, x_1] \\
&\quad = \{a, b\}\, [\mathsf{whistle}\, x_1] \\
&\quad = \{a\}
\end{aligned}$$

Note that the context change potential of (45b) is defined for the input σ_0, since $x_1 \notin \mathrm{dom}(\sigma_0)$, and for every intermediate context σ, since $x_1 \in \mathrm{dom}(\sigma_0)$. If we regard $\exists_1 x$ as a separate unit, the overall update process can be depicted in the following way:

$$\{\epsilon\} \xrightarrow{\exists x_1} \{a,b,c\} \xrightarrow{\mathsf{man}\, x_1} \{a,b,c\} \xrightarrow{\mathsf{walk}\, x_1} \{a,b\} \xrightarrow{\mathsf{whistle}\, x_1} \{a\}\,.$$

Next, consider the case in which an existential quantifier appears within the scope of negation:

(46) a. No man walks.

b. $\neg\exists x_1(\mathsf{man}\, x_1 \wedge \mathsf{walk}\, x_1)$

It can be easily verified that if a model is such that $\mathsf{man}^* \cap \mathsf{walk}^* = \emptyset$, then for any state σ, $\sigma[(46\mathrm{b})] = \bot$; otherwise $\sigma[(46\mathrm{b})] = \sigma$. Either way, no new variable is introduced by (46b); hence, continuing (46b) with a formula like $\mathsf{whistle}\, x_1$ results in undefinedness. In general, existential quantifiers inside the scope of negation cannot bind free variables in subsequent formulas. This accounts for the fact that noun phrases such as *every N* and *no N* cannot usually serve as an antecedent for pronouns that appear in the subsequent discourse.

(47) a. Every man walks in the park. *He whistles.

b. No man walks in the park. *He whistles.

Using the terminology of Groenendijk and Stokhof (1991), we can say that negation, implication, and universal quantifier are *externally static* in that they do not pass discourse information to the subsequent discourse. By contrast, conjunction and existential quantification are *externally dynamic* in that discourse information is carried over to the subsequent discourse.

We introduce a notion of equivalence between DS_q-formulas as follows.

Definition 3.20 Let $A, B \in \mathcal{L}_{\mathsf{DS}q}$. We say that A and B are equivalent, written as $A \equiv B$ if for all models $\mathcal{M}$ and all information states σ based on $\mathcal{M}$, the following conditions are satisfied:

1. $\sigma[A]$ is defined if and only if $\sigma[B]$ is defined.

2. If $\sigma[A]$ or $\sigma[B]$ is defined, then $\sigma[A] = \sigma[B]$.

Two formulas are equivalent when their context change potentials have the same definedness condition and are extensionally equivalent for all input states for which they are defined.

Fact 3.21 Let $A, B \in \mathcal{L}_{\mathsf{DS}q}$.

1. $\exists x A \wedge B \equiv \exists x(A \wedge B)$
2. $\exists x A \to B \equiv \forall x(A \to B)$
3. $\exists x A \setminus\!\!\setminus B \equiv \exists x(A \setminus\!\!\setminus B)$

Note that unlike the case in standard first-order logic, these equivalences hold even when B contains free variable x. Thus, in DS_q, $\exists x Fx \wedge Gx \equiv \exists x(Fx \wedge Gx)$ and $\exists x Fx \to Gx \equiv \forall x(Fx \to Gx)$. It is characteristic of dynamic semantics that existential quantifiers are externally dynamic in that they can bind variables outside their scope.[13] Such equivalences are used to account for cross-sentential anaphora, as in (45), and donkey anaphora, as typically shown in the equivalence between (48a) and (48b).[14]

(48) a. If a farmer owns a donkey, he beats it.
$\exists x_1 \exists x_2(\mathsf{farmer}\, x_1 \wedge \mathsf{own}\, x_1 x_2 \wedge \mathsf{donkey}\, x_2) \to \mathsf{beat}\, x_1 x_2$

b. Every farmer who owns a donkey beats it.
$\forall x_1 \forall x_2(\mathsf{farmer}\, x_1 \wedge \mathsf{own}\, x_1 x_2 \wedge \mathsf{donkey}\, x_2 \to \mathsf{beat}\, x_1 x_2)$

[13] That these equivalences hold in DS_q is related to the fact that for all DS_q-formulas A, $[A]$ is *distributive* in the sense that for all states σ, if $\sigma[A]$ is defined, then $\sigma[A] = \bigcup_{s \in \sigma} \{i\}[A]$. A typical example of a *non-distributive* operator is the epistemic *might* operator, symbolized as $\Diamond A$. Its context change potential is usually defined as follows (cf. Footnote 5 on page 167):

$$\sigma[\Diamond A] = \{s \in \sigma \mid \sigma[A] \neq \emptyset\} \text{ if } \sigma[A] \text{ is defined; otherwise, it is undefined.}$$

It can be easily verified that $\Diamond A$ is *not* distributive. If we extend DS_q with $\Diamond A$, Fact 3.21 no longer holds in this general form. For extensive discussion on this point, see Groenendijk, Stokhof, and Veltman (1996).

[14] For an analysis of donkey-anaphora in dynamic semantics, see, in particular, Groenendijk and Stokhof (1991).

In Fact 3.21.3, we see that binary presupposition operator $\setminus\!\!\setminus$ acts like conjunction in that any discourse referent introduced in the presupposed part A carries over to the assertive part B. This property is important when one tries to represent the existential presupposition associated with a definite description with the help of the $\setminus\!\!\setminus$ operator. Thus, focusing on the existential presupposition, a sentence of the form "the F is G" is translated into a DS_q-formula $\exists x Fx \setminus\!\!\setminus Gx$, which is guaranteed to be equivalent to $\exists x(Fx \setminus\!\!\setminus Gx)$ by Fact 3.21.3. In such a case, a binding relation holds between the existential quantifier occurring in a presupposed part and the variable occurring in the assertive part.

The notions of *acceptance* and *presupposition* in DS_q are defined as follows.

Definition 3.22 Let $A, B \in \mathcal{L}_{\mathsf{DS}q}$.

1. An information state σ *accepts* A if $\sigma[A]$ is defined and $\sigma \sqsubseteq \sigma[A]$.
2. A *presupposes* B if every information state σ for which $[A]$ is defined accepts B.

This definition differs from the one in DS (cf. Definition 3.5) in that acceptance and presupposition in DS_q are defined in terms of $\sqsubseteq$, i.e., non-eliminative updates, rather than $\subseteq$. The notion of validity is defined in the same manner as DS (cf. Definition 3.7). The following facts also hold in DS_q.

Fact 3.23 Let $A_1, \dots, A_n, B \in \mathcal{L}_{\mathsf{DS}}$,

1. For all information states σ, $\sigma[A_1]\dots[A_n]$ accepts B if and only if σ accepts $A_1 \wedge \cdots \wedge A_n \rightarrow B$.
2. $A_1, \dots, A_n \models B$ if and only if $\models A_1 \wedge \cdots \wedge A_n \rightarrow B$.

It can now be easily seen that all the facts about the projection properties of connectives $\wedge, \rightarrow$ and $\neg$ in DS also hold in DS_q. This means that the proviso problem, which we discussed in Section 3.2.2, also arises in DS_q.

We are now in a position to state the presuppositions of quantified formulas in DS_q. We will work through several important examples, which were taken up in Heim (1983) and discussd in the subsequent literature (e.g., Beaver 2001; Kadmon 2001). We will see that for each case, we obtain Heim's (1983) predictions in DS_q.

Example 3.24 $\forall x(Fx \to Gx \mathbin{\backslash\!\backslash} Hx)$ presupposes $\forall x(Fx \to Gx)$.

This is proved in the following way. To focus on the projection behavior of $\backslash\!\backslash$, let us assume that x is a new variable in input state σ, i.e., $x \notin \mathrm{dom}(\sigma)$. Then we have:

$$\begin{array}{lll}
& \sigma[\forall x(Fx \to Gx \mathbin{\backslash\!\backslash} Hx)] \text{ is defined} & \\
\iff & \sigma[x/d][Fx \to Gx \mathbin{\backslash\!\backslash} Hx] \text{ is defined for all } d \in D & \text{(by Fact 3.19.2)} \\
\iff & \sigma[x/d][Fx][Gx \mathbin{\backslash\!\backslash} Hx] \text{ is defined for all } d \in D & \text{(by Fact 3.19.1)} \\
\iff & \sigma[x/d][Fx] \text{ accepts } Gx \text{ for all } d \in D & \text{(by Def. 3.17.4)} \\
\iff & \sigma[x/d] \text{ accepts } Fx \to Gx \text{ for all } d \in D & \text{(by Fact 3.23.1)} \\
\iff & \sigma \text{ accepts } \forall x(Fx \to Gx) &
\end{array}$$

Hence, by Definition 3.22, it follows that $\forall x(Fx \to Gx \mathbin{\backslash\!\backslash} Hx)$ presupposes $\forall x(Fx \to Gx)$.

As a concrete example, consider the following example, where the presupposition trigger *stop* appears in the nuclear scope of quantifier *every*.[15]

(49) a. Every student stopped smoking.
$\forall x(Fx \to Gx \mathbin{\backslash\!\backslash} Hx)$

b. Every student used to smoke.
$\forall x(Fx \to Gx)$

Here, Fx = "x is a student," Gx = "x used to smoke," and Hx = "x stopped smoking." It is then predicted that (49a) presupposes (49b). Some support for the claim that in an intuitive sense, (49b) is a presupposition of (49a) comes from the fact that (49b) usually survives in the following contexts:

(50) a. If every student stopped smoking, I will be pleased.

b. It is not the case that every student stopped smoking.

[15] In sentences of the form "Q A is B," where Q is a determiner such as *every*, *no*, and *some*, A is called the *restrictor* of Q and B the *nuclear scope* of Q.

c. It might be the case that every student stopped smoking.

d. Did every student stop smoking?

e. Suppose that every student stopped smoking.

Further support comes from the fact that exchanges like the following sound odd (cf. Kadmon 2001, 193):

(51) a. A: Not every student used to smoke.

b. B: # But every student stopped smoking.

Indeed, since Karttunen and Peters (1979) and Heim (1983), it seems to have been widely agreed in the literature that sentences like (49a) have universal presuppositions as in (49b).

As a case in which a description appears in the nuclear scope of a quantifier, consider (52a), which is discussed in Heim (1983).

(52) a. Every nation cherishes its king.

b. $\forall x_1(\mathsf{nation}\, x_1 \rightarrow \exists x_2\, \mathsf{king}\, x_2 x_1 \mathbin{\backslash\!\backslash} \mathsf{cherish}\, x_1 x_2)$

In DS_q, (52a) can be represented as (52b), where $\mathsf{king}\, x_2 x_1$ is to be read as "x_2 is a king of x_1." Here, we regard the definite description *its king* as triggering the existential presupposition that there is a king, which corresponds to the subformula $\exists x_2\, \mathsf{king}\, x_2\, x_1$ in (52b).[16] It then follows that (52b) presupposes (53b) in DS_q.

(53) a. Every nation has a king.

b. $\forall x_1(\mathsf{nation}\, x_1 \rightarrow \exists x_2\, \mathsf{king}\, x_2 x_1)$

Hence, the prediction is that (52a) presupposes (53a), which coincides with the prediction in Heim (1983).

Next, consider cases in which presupposition triggers appear in the nuclear scope of a quantifier *no*. Notice that the following obtains in DS_q.

Example 3.25 $\neg\exists x(Fx \wedge (Gx \mathbin{\backslash\!\backslash} Hx))$ presupposes $\forall x(Fx \rightarrow Gx)$.

[16] Note that in (52b), the existential quantifier $\exists x_2$ can bind the variable occurrence x_2 in $\mathsf{cherish}\, x_1\, x_2$ by Fact 3.21.3.

This is proved in a similar way as Fact 3.24, because $\forall xA$ has the same definedness condition as $\exists xA$, which in turn has the same definedness condition as $\neg\exists xA$. As a result, it is predicted that a sentence like (54a) triggers a universal presupposition as shown in (54b):

(54) a. No student stopped smoking.
$\neg\exists x(Fx \wedge (Gx \mathbin{\backslash\!\backslash} Hx))$
b. Every student used to smoke.
$\forall x(Fx \rightarrow Gx)$

Let us also mention Heim's (1983) example, together with its representation in DS_q:

(55) a. No nation cherishes its king.
$\neg\exists x_1(\mathsf{nation}\, x_1 \wedge (\exists x_2\, \mathsf{king}\, x_2\, x_1 \mathbin{\backslash\!\backslash} \mathsf{cherish} x_1\, x_2))$
b. Every nation has a king.
$\forall x_1(\mathsf{nation}\, x_1 \rightarrow \exists x_2\, \mathsf{king}\, x_2\, x_1)$

Heim (1983) claims that sentence (55a) presupposes (55b), which coincides with the prediction of DS_q. Although there has been controversy about what sentences like (54a) and (55a) presuppose (see Kadmon 2001 and references therein), a recent experimental study of Chemla (2009) suggests that universal presuppositions are actually robust in such cases.

Now, it is easily observed that the following fact obtains in DS_q, since a formula A and its negation $\neg A$ presuppose the same thing.

Example 3.26 $\exists x(Fx \wedge (Gx \mathbin{\backslash\!\backslash} Hx))$ presupposes $\forall x(Fx \rightarrow Gx)$.

This predicts that sentences like (56a) also have a universal presupposition.

(56) a. Some student stopped smoking.
$\exists x(Fx \wedge (Gx \mathbin{\backslash\!\backslash} Hx))$
b. Every student used to smoke.
$\forall x(Fx \rightarrow Gx)$

Heim's (1983) famous example is the following:

(57) a. A fat man was pushing his bicycle.
$\exists x_1(\mathsf{fat\_man}\, x_1 \wedge (\exists x_2(\mathsf{bycycle}\, x_2 \wedge \mathsf{own}\, x_1\, x_2) \setminus\!\!\setminus \mathsf{push}\, x_1 x_2))$

b. Every fat man had a bicycle.
$\forall x_1(\mathsf{fat\_man}\, x_1 \rightarrow \exists x_2(\mathsf{bycycle}\, x_2 \wedge \mathsf{own}\, x_1\, x_2))$

Again, both Heim's theory and DS_q predict that (57a) presupposes (57b). At first sight, this prediction might be counter-intuitive; however, it is inevitable since $\neg\exists x A$ and $\exists x A$ have the same presupposition. See Kadmon (2001) for an extensive defense of this prediction.

There are at least two (independently motivated) methods with which one can supplement the predictions of dynamic semantics. One is to invoke *local accommodation*, which we introduced in Section 3.2.3. Heim (1983) suggests that if one performs local accommodation after processing the indefinite *a man* in (57a), one can resolve presupposition within the scope of the existential quantifier and obtain a reading like "There is a fat man who owns a bike and was pushing it." In terms of DS_q, this option amounts to saying that sentence (57a) ends up having the following interpretation:

(58) $\exists x_1(\mathsf{fat\_man}\, x_1 \wedge \exists x_2(\mathsf{bycycle}\, x_2 \wedge \mathsf{own}\, x_1\, x_2 \wedge \mathsf{push}\, x_1 x_2))$

Another method is to invoke *domain restriction*. It is well known that the domain of a quantifier in natural language is often narrowed down in context. It is at least conceivable that the presuppositions of a given sentence serve as a *clue* to fix some intended domain of the quantifier in question. For example, in the case of sentence (56a), the universal presupposition that every student used to smoke could help the hearer identify an intended context, namely, a context in which this presupposition obtains. This option should amount to interpreting (56a) as meaning "among the students who used to smoke, there is a student who stopped smoking." Arguably, the resulting interpretation is (truth-conditionally) equivalent to the one predicted by the option of local accommodation. Either way, we can obtain the desired result. Although we do not go into discussing how to formalize these two options, they at least suggest that the universal presuppositions predicted for existential expressions like *some* are tenable ones.[17]

[17]For an attempt to formalize the process of accommodation within Heim's dynamic

Finally, consider cases in which presupposition triggers appear in the restrictor of universal quantifiers like *every*. We first observe that the following holds in DS_q.

Example 3.27 $\forall x(Fx \wedge (Gx \mathbin{\backslash\!\backslash} Hx) \to Kx)$ presupposes $\forall x(Fx \to Gx)$.

Hence, again, it is predicted that sentences like (59a) have a universal presupposition as shown in (59b).

(59) a. Every student who stopped smoking will be rewarded.
$\forall x(Fx \wedge (Gx \mathbin{\backslash\!\backslash} Hx) \to Kx)$
b. Every student used to smoke.
$\forall x(Fx \to Gx)$

This agrees with Heim's (1983) prediction. The following is her example, associated with representations in DS_q.

(60) a. Every man who serves his king will be rewarded.
$\forall x_1(\mathsf{man}\, x_1 \wedge (\exists x_2\, \mathsf{king}\, x_2\, x_1 \mathbin{\backslash\!\backslash} \mathsf{served}\, x_1\, x_2) \to \mathsf{rewarded}\, x_1)$
b. Every man has a king.
$\forall x_1(\mathsf{man}\, x_1 \to \exists x_2\, \mathsf{king}\, x_2\, x_1)$

It might be argued that universal presuppositions like (59b) and (60b) are too strong.[18] However, it would be reasonable to suppose that these universal presuppositions act as a clue to specify the contexts in which each utterance is evaluated, in a similar way to the cases of (56) and (57). Such interpretations could be guaranteed either by local accommodation (*every student who used to smoke but has stopped doing it will be rewarded*) or by domain restriction (*among those students who used to smoke, every student who stopped smoking will be rewarded*). Although the presuppositions of quantified sentences are still open to empirical debate, we will henceforth assume that those predictions that we have seen so far are basically correct and we will explore how they can be accounted for by other theories, such as discourse representation theory and our proof-theoretic approach.

framework, see Zeevat (1992). For discussions on domain restriction, see Chapter 3 of this thesis.

[18]See, e.g., Beaver (1994).

3.3 Discourse representation theory

As we mentioned in Section 2.1, Discourse Representation Theory (DRT) shares with dynamic semantics the dynamic conception of meaning, according to which the meaning of an expression is explicated in terms of its potential to change the context. Specifically, both theories build on the following observation: an utterance of a sentence changes the context in which it is used and creates an updated context against which subsequent utterances are evaluated, and, in particular, different subparts of a complex expression may be evaluated in different contexts. In dynamic semantics, the notions of content and context are explicated in terms of propositions that are identified denotationally with a set of possible worlds. By contrast, what plays the central role in DRT is the notion of *discourse representation structure* (DRS). Thus, the basic assumptions of dynamic approach to meaning, as summarized in (D1), (D2), and (D3) on page 160, are captured in DRT as follows:

(K1) The notion of context is explicated in terms of DRS, which is a kind of structured entity.
(K2) The meaning of the assertoric utterance of a sentence S is also identified with the DRS that can be compositionally derived from S; updating a DRS given as a context with the DRS encoded by an uttered sentence consists of merging the two DRSs and creating a new one.
(K3) The notion of presupposition is captured by introducing the notion of *preliminary DRS*, which is a DRS with some preconditions that need to be satisfied for its effective use.

We will first outline the basic framework of discourse representation theory (Section 3.3.1) and then proceed to see how the notion of presuppositions is handled within this framework (Section 3.3.2). Finally, we will discuss some problems in the treatment of presuppositions in discourse representation theory (Section 3.3.3).

3.3.1 Basic account of DRT

We start with introducing the language of DRT. It contains a set of *discourse referents*, typically denoted by $x_1, \ldots, x_n$, a set of predicate symbols, the equality symbol $=$, and logical symbols $\neg$ and $\Rightarrow$. A DRS K is a pair $\langle U, Con\rangle$ consisting of a finite and possibly empty set U of discourse referents, often called the *universe* of the DRS, and a set Con of *conditions.* The conditions γ are defined by the rule

$$\gamma ::= P^n(x_1, \ldots, x_n) \mid x_1 = x_2 \mid \neg K \mid K_1 \Rightarrow K_2$$

where P^n is a predicate symbol with arity n and K, K_1, K_2 are meta-variables for DRSs.

Like the information states in DS_q, DRSs deliver two kinds of information—namely, what entities are introduced in a discourse and what conditions hold of them, respectively. We illustrate how DRSs work by some examples. Consider sentence (61a). This sentence corresponds to the DRS in (61b).

(61) a. A woman catches a cat.

b. $\langle \{x_1, x_2\}, \{\mathsf{woman}(x_1), \mathsf{cat}(x_2), \mathsf{catch}(x_1, x_2)\}\rangle$

Such a DRS can also be pictorially represented in box notation:

(62)

$x_1\ x_2$
$\mathsf{woman}(x_1)$
$\mathsf{cat}(x_2)$
$\mathsf{catch}(x_1, x_2)$

Roughly, this DRS models a situation that involves (at least) two individuals satisfying the three conditions indicated in the box. Such DRSs are built from the structure of sentences with the help of a construction algorithm. Several different versions of construction algorithms have been proposed in the literature, building on various syntactic frameworks (see Kamp and Reyle 1993). For our purpose, it suffices to assume that some construction algorithm is given; we will not discuss the choice of a particular algorithm.

Now, suppose that sentence (61) is followed by sentence (63a), whose DRS is shown in (63b).

(63) a. She smiles.

b. $$\begin{array}{|l|}\hline x_3 \\ \mathsf{smile}(x_3) \\ \hline \end{array}$$

The DRS in (62) is used as a context in which (63a) is interpreted. Specifically, the discourse referents in (62) serve as an antecedent for the anaphoric NP *she* in (63a). We represent the process of extending the DRS in (62) with (63b) in the following way:

(64) $$\begin{array}{|l|}\hline x_1\ x_2 \\ \mathsf{woman}(x_1) \\ \mathsf{cat}(x_2) \\ \mathsf{catch}(x_1, x_2) \\ \hline \end{array} \oplus \begin{array}{|l|}\hline x_3 \\ \mathsf{smile}(x_3) \\ \hline \end{array} = \begin{array}{|l|}\hline x_1\ x_2\ x_3 \\ \mathsf{woman}(x_1) \\ \mathsf{cat}(x_2) \\ \mathsf{catch}(x_1, x_2) \\ \mathsf{smile}(x_3) \\ \hline \end{array}$$

Here the operation of combining two DRSs is represented by the *merge* operation $\oplus$. This operation combines two DRSs by taking the union of the two universes and the two sets of conditions, i.e., $\langle U_1, Con_1 \rangle \oplus \langle U_2, Con_2 \rangle = \langle U_1 \cup U_2, Con_1 \cup Con_2 \rangle$.

Now, suppose that the anaphoric pronoun *she* in (63a) is anaphorically linked to *a woman* in (61a). We can express this anaphoric relationship by means of an equational condition, i.e., $x_1 = x_3$. Adding this equational condition to the final representation in (64) leads to the DRS in (65b), which is equivalent to the one in (65c).

(65) a. *A woman* catches a cat. *She* smiles.

b. $$\begin{array}{|l|}\hline x_1\ x_2\ x_3 \\ \mathsf{woman}(x_1) \\ \mathsf{cat}(x_2) \\ \mathsf{catch}(x_1, x_2) \\ \mathsf{smile}(x_3) \\ x_1 = x_3 \\ \hline \end{array}$$

c. $$\begin{array}{|l|}\hline x_1\ x_2 \\ \mathsf{woman}(x_1) \\ \mathsf{cat}(x_2) \\ \mathsf{catch}(x_1, x_2) \\ \mathsf{smile}(x_1) \\ \hline \end{array}$$

The resulting DRS models a situation in which there are at least two individuals, say, a and b, such that a is a woman, b is a cat, a catches b, and a smiles.

A universal sentence like (66a) is represented as shown in (66b).

(66) a. Every woman catches a cat.

b.
$$\boxed{\boxed{\begin{array}{l} x_1 \\ \mathsf{woman}(x_1) \end{array}} \Rightarrow \boxed{\begin{array}{l} x_2 \\ \mathsf{cat}(x_2) \\ \mathsf{catch}(x_1, x_2) \end{array}}}$$

As we saw before, it is usually not possible to use the anaphoric pronoun *she* as in (63) to refer back to *every woman* in (66). Such a constraint on anaphora resolution can be captured by the following constraint on the availability of the addition of equality conditions:

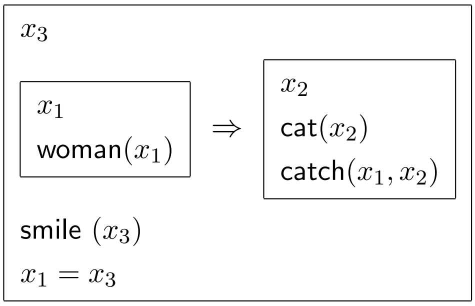
(67) **The accessibility constraint** Let x_1 be a discourse referent in a DRS K_1 and x_2 a discourse referent in a DRS K_2. Then, the equality condition $x_1 = x_2$ can be added to K_1 if K_1 is *accessible* from K_2. When K_1 is accessible from K_2, we also say that x_1 is accessible from x_2.

We define the accessibility relation in the following way. First, we say that DRS K_1 *subordinates* K_2, written as $K_2 < K_1$, if (i) K_1 contains a condition of the form $\neg K_2$; or (ii) K_1 contains a condition of the form $K_2 \Rightarrow K$ for some DRS K; or (iii) $K_1 \Rightarrow K_2$ is a condition in some DRS K. The accessibility relation is defined as the reflexive transitive closure of $<$. That is, K_1 is accessible from K_2 if (i) $K_1 = K_2$; or (ii) $K_2 < K_1$; or (iii) there is some DRS K such that K_1 is accessible from K and K is accessible from K_2.

The accessibility constraint in (67) correctly predicts that an anaphoric link between *a woman* and *she* is permitted in the case of (65): since x_1 and x_3 are in the same universe, clearly x_1 is accessible to x_3; hence, we can add the equational condition $x_1 = x_3$. By contrast, in the case of (68), the NP *a woman* cannot be anaphorically linked to the pronoun *it*, since the discourse referent x_1, which is associated with *a woman*, is not accessible from the discourse referent x_3, which is associated with *she*. Thus, the DRS in (68b) violates the accessibility constraint.

(68) a. *Every woman* catches a cat. # *She* smiles.

b.



We can see that the notion of the accessibility relation plays the same role as the notion of local context in dynamic semantics. We will see later that the accessibility relation also plays an important role in the interpretations of presuppositions.

To provide a precise interpretation for DRSs, we define a translation $(\cdot)^\bullet$ from the language of DRT into the language of first-order logic with equality.[19] The translation is shown in Figure 3.1.

$$\begin{aligned}
\langle\{x_1,\dots,x_n\},\{\gamma_1,\dots,\gamma_n\}\rangle^\bullet &= \exists x_1\dots\exists x_n(\gamma_1^\bullet \wedge \cdots \wedge \gamma_n^\bullet) \\
(P(x_1,\dots,x_n))^\bullet &= P(x_1,\dots,x_n) \\
(x_1 = x_2)^\bullet &= x_1 = x_2 \\
(\neg K)^\bullet &= \neg K^\bullet \\
(K_1 \Rightarrow K_2)^\bullet &= \forall x_1\dots\forall x_n(\gamma_1^\bullet \wedge \cdots \wedge \gamma_n^\bullet \to K_2^\bullet) \\
&\qquad \text{where } K_1 = \langle\{x_1,\dots,x_n\},\{\gamma_1,\dots,\gamma_n\}\rangle
\end{aligned}$$

Fig. 3.1 A translation from DRT to first-order logic

For example, the DRSs in (65c) and (66b) are translated as (69a) and (69a), respectively.

(69) a. $\exists x_1 \exists x_2(\mathsf{woman}(x_1) \wedge \mathsf{cat}(x_2) \wedge \mathsf{catch}(x_1, x_2) \wedge \mathsf{smile}(x_1))$

b. $\forall x_1(\mathsf{woman}(x_1) \to \exists x_2(\mathsf{cat}(x_2) \wedge \mathsf{catch}(x_1, x_2)))$

[19] It is also straightforward to give a model theoretic semantics for DRSs. See Kamp and Reyle (1993) for a textbook treatment.

3.3.2 Presuppositions in DRT

The systematic treatment of presupposition within the framework of DRT was put forward by van der Sandt (1992), and it was further developed by Geurts (1999), and Kamp, van Genabith, and Reyle (2011). The basic observation motivating their approach is that there are striking parallels between anaphoric expressions and presupposition triggers.[20] Recall the paradigm examples of presupposition projection in (10) on page 149, repeated here:

(10) a. John has children and his children are wise.
b. If John has children, his children are wise.
c. Either John does not have any children or his children are wise.

The problem posed by these examples was to account for the fact that while a simple sentence *John's children are wise* presupposes that John has children, none of (10a–c) inherits this presupposition. Now, compare these examples with the paradigm examples of anaphora resolution, which motivated dynamic semantics and discourse representation theory:

(70) a. John owns a donkey and he beats it.
b. If John owns a donkey, he beats it.
c. Either John does not own a donkey or he beats it.

The problem posed by these sentences was to explain how a pronoun *it* can be anaphorically linked to a quantificational expression *a donkey*, despite the fact that syntactically the pronoun *it* is out of the scope of *a donkey*. Traditionally, these two problems, i.e., the problems of presupposition projection and anaphora resolution, were formulated in different terms and were considered to be accounted for by different theories.

According to van der Sandt (1992), however, the parallels between (10) and (70) suggests that a similar mechanism underlies both anaphora resolution and presupposition projection. Indeed, he claims that presupposition triggers are simply anaphoric expressions that search for suitable an-

[20]The observation that there is a close connection between anaphora and presupposition was also found in Kripke (2009), which was originally delivered as a lecture in 1990. See also Soames (1989: 614) for a report on Kripke's observation.

tecedents. This means that the two principal motivations for the dynamic conception of meaning, namely, donkey anaphora and presupposition projection, are two sides of the same coin: both phenomena are explained by a unified account of how discourse representation structures evolve as the hearer processes a sequence of sentences in an incremental way. In van der Sandt's account, there are two ways in which presuppositional elements are interpreted in a given discourse representation structure:

1. **Binding** When a presuppositional element β finds an antecedent then β will be *bound* to it; the descriptive content associated with β will then be transferred to the site where β is bound.

2. **Accommodation** When a presuppositional element cannot be bound, it will be *accommodated* at the highest possible level of discourse representation structure.

According to van der Sandt (1992) and Geurts (1999), the only difference between standard presupposition triggers, such as definite descriptions and factive verbs, and pronominal expressions, such as *she* and *it*, is that the former has a richer descriptive content than the latter; as a result, presuppositional anaphors are normally easier to accommodate. In other words, presuppositional anaphors have sufficient descriptive content to establish a discourse referent when the context does not provide one.

Let us see how this accounts works with some typical examples. First, we need to extend the language of DRSs to handle those sentences that contain presupposition triggers. In the extended version of DRT, a DRS K is defined as a triple $\langle U, Con, \mathcal{A}\rangle$ where U and Con are a set of discourse referents and a set of conditions as before, and $\mathcal{A}$ is a finite and possibly empty set of DRSs. The members of $\mathcal{A}$ of a DRS K indicate all the anaphoric elements of K. In box notation, each member K of $\mathcal{A}$ is indicated by ∂K and called ∂-structure. For example, consider the sentence in (71a). The DRS for this sentence is shown in (71b), and it can be pictorially represented as in (71c).

(71) a. The cat collapses.

b. $\langle \emptyset, \{\mathsf{collapese}(x_1)\}, \{\langle\{x_1\}, \{\mathsf{cat}(x_1)\}, \emptyset\rangle\}\rangle$

c. $$\begin{array}{|l|}\hline \\ \mathsf{collapse}(x_1) \\ \partial\begin{array}{|l|}\hline x_1 \\ \mathsf{cat}(x_1) \\ \hline\end{array} \\ \hline\end{array}$$

This DRS is like an ordinary DRS except that it is marked as being unresolved with respect to the presupposition indicated by the ∂-structure. Following Kamp, van Genabith, and Reyle (2011), we call such a DRS a *preliminary* DRS. Intuitively, by ∂K we indicate that a DRS K is an anaphor that needs to be bound or accommodated for the entire DRS to be properly evaluated. Note that one sentence may contain more than one anaphoric element, in which case the corresponding DRS will contain more than one ∂-structure in $\mathcal{A}$. Note also that an anaphoric expression may embed other anaphoric expressions so that one ∂-structure may contain other ∂-structures. For example, the NP *the cat's tail* will trigger the following ∂-structure:

(72) $$\partial\begin{array}{|l|}\hline x_1 \\ \mathsf{tail}(x_1, x_2) \\ \partial\begin{array}{|l|}\hline x_2 \\ \mathsf{cat}(x_2) \\ \hline\end{array} \\ \hline\end{array}$$

Now, suppose that sentence (71) is uttered in the context shown in (73a). Merging the DRS in (71b) with the DRS established by the first sentence leads to a new DRS as shown in (73b).

(73) a. A woman catches a cat. The cat collapses.

b. $$\begin{array}{|l|}\hline x_2\ x_3 \\ \mathsf{woman}(x_2) \\ \mathsf{cat}(x_3) \\ \mathsf{catch}(x_2, x_3) \\ \hline\end{array} \oplus \begin{array}{|l|}\hline \\ \mathsf{collapse}(x_1) \\ \partial\begin{array}{|l|}\hline x_1 \\ \mathsf{cat}(x_1) \\ \hline\end{array} \\ \hline\end{array} = \begin{array}{|l|}\hline x_2\ x_3 \\ \mathsf{woman}(x_2) \\ \mathsf{cat}(x_3) \\ \mathsf{catch}(x_2, x_3) \\ \mathsf{collapse}(x_1) \\ \partial\begin{array}{|l|}\hline x_1 \\ \mathsf{cat}(x_1) \\ \hline\end{array} \\ \hline\end{array}$$

Note that the merge operation $\oplus$ is extended for preliminary DRSs in the following way:

(74) $\langle U_1, Con_1, \mathcal{A}_1\rangle \oplus \langle U_2, Con_2, \mathcal{A}_2\rangle = \langle U_1 \cup U_2, Con_1 \cup Con_2, \mathcal{A}_1 \cup \mathcal{A}_2\rangle\,.$

To resolve the presupposition indicated by a ∂-structure, we first try to find its antecedent. More specifically, we try to match the condition in the ∂-structure with the conditions contained in its superordinated DRSs. In the case of (73b), the condition $\mathsf{cat}(x_1)$ in the ∂-structure is matched with $\mathsf{cat}(x_3)$ in the main DRS, so that we move the discourse referent x_1 to the universe of the main DRS and add the equational condition $x_1 = x_3$. In such a case, we say that x_3 is *bound* to x_1. Then, we obtain the DRS in (75), which is equivalent to the one in (76). This result is intuitively correct as an interpretation of the discourse in (73).

(75) $$\begin{array}{|l|}\hline x_1\ x_2\ x_3 \\ \mathsf{woman}(x_2) \\ \mathsf{cat}(x_3) \\ \mathsf{catch}(x_2, x_3) \\ \mathsf{collapse}(x_1) \\ x_1 = x_3 \\ \hline \end{array}$$

(76) $$\begin{array}{|l|}\hline x_1\ x_2 \\ \mathsf{woman}(x_2) \\ \mathsf{cat}(x_1) \\ \mathsf{catch}(x_2, x_1) \\ \mathsf{collapse}(x_1) \\ \hline \end{array}$$

Next, suppose that sentence (71a) is uttered in a context in which one cannot find any suitable antecedent for the presuppositional element. In such a case, the DRS in (71b) is *accommodated* so that the discourse referent x_1 and its associated condition $\mathsf{cat}(x_1)$ in the ∂-structure are moved to the main DRS. This yields the following result.

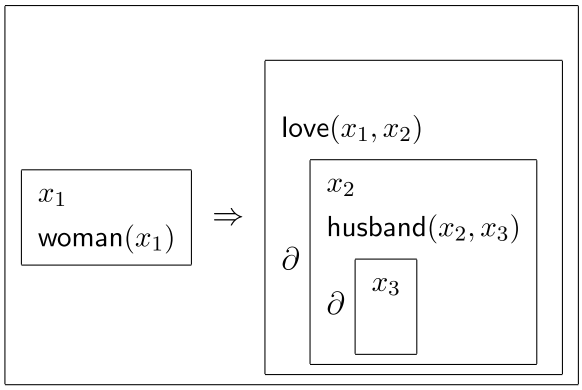
(77) $$\begin{array}{|l|}\hline x_1 \\ \mathsf{cat}(x_1) \\ \mathsf{collapse}(x_1) \\ \hline \end{array}$$

In general, there can be various "landing sites" of ∂-structures; as a result, there can be various ways of performing accommodation. In Section 3.2.3, we saw that Heim's dynamic semantics distinguishes between global and local accommodation. It is clear that the examples discussed in that section can also be handled by the mechanism of accommodation within DRT. Interestingly, for some cases, DRT predicts more readings than dynamic semantics does.[21] As an illustration, consider the following example:

(78) Every woman loves her husband.

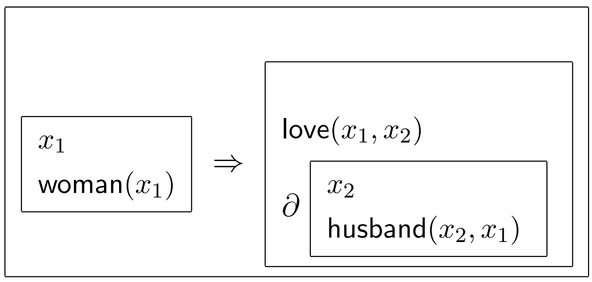
The initial DRS looks as follows:

(79)



We start with processing the most deeply embedded ∂-structure, namely, the one having the discourse reference x_3, which corresponds to the pronoun *her*. We then check whether a suitable antecedent can be found in its superordinate DRSs. In this case, we assume that x_3 is bound to x_1. The resulting DRS is equivalent to the following:

(80)



Since there is no candidate DRS for matching the remaining ∂-structure, we make use of accommodation. Here, we have three possible landing sites of

[21]Some readings are ruled out by constraints on adequate interpretation, for example, by the constraints requiring that the resulting DRSs must be consistent and informative. See Geurts (1999) and Beaver and Zeevat (2007) for a detailed discussion on various constraints.

the ∂-structure, which lead to three possible ways of accommodation: (a) global, (b) intermediate, and (c) local accommodation.

(81) a. **Global**:

x_2
$\mathsf{husband}(x_2, x_1)$
[x_1 $\mathsf{woman}(x_1)$] $\Rightarrow$ [$\mathsf{love}(x_1, x_2)$]

b. **Intermediate**:

[x_1 x_2 $\mathsf{woman}(x_1)$ $\mathsf{husband}(x_2, x_1)$] $\Rightarrow$ [$\mathsf{love}(x_1, x_2)$]

c. **Local**:

[x_1 $\mathsf{woman}(x_1)$] $\Rightarrow$ [x_2 $\mathsf{love}(x_1, x_2)$ $\mathsf{husband}(x_2, x_1)$]

The DRS obtained by global accommodation shown in (81a) is ruled out, since the occurrence of variable x_1 in $\mathsf{husband}(x_2, x_1)$ of the main DRS remains free; hence, the whole DRS is not interpretable. (This is what Beaver and Zeevat (2007) called the *trapping constraint* for accommodation.) The other two options lead to proper DRSs, namely, DRSs that contain neither free variables nor ∂-structures; hence, these are interpretable. The following translation makes clear the final interpretations: (81b) is translated into (82a) and (81c) into (82b).

(82) a. $\forall x_1 \forall x_2 (\mathsf{woman}(x_1) \wedge \mathsf{husband}(x_2, x_1) \rightarrow \mathsf{love}(x_1, x_2))$

b. $\forall x_1 (\mathsf{woman}(x_1) \rightarrow \exists x_2 \, (\mathsf{love}(x_1, x_2) \wedge \mathsf{husband}(x_2, x_1)))$

Arguably, both interpretations are possible. One natural generalization posited by the proponents of DRT is that presuppositions tend to be projected (i.e., bound or accommodated) to the highest possible DRS (cf. Geurts

1999). It can then be predicted that the intermediate reading in (81b) is preferable to the local reading in (81c). Note that the local reading in (81c) entails that every woman has a husband. In this respect, this reading is similar to that predicted by dynamic semantics for the example in (49). On the other hand, the intermediate reading is not directly accounted for by Heim's dynamic semantics. However, a suitable mechanism of accommodation or domain restriction, as discussed in Section 3.2.4, would also make it possible to generate readings like (82b) within DRT.

In Section 3.2.2, we saw that dynamic semantics suffers from the proviso problem: roughly speaking, when a presupposition triggers appears in the consequent of a conditional or the second conjunct of a conjunctive sentence, dynamic semantics always generates *conditional* presuppositions; however, in some cases, it is intuitively clear that these constructions have *unconditional* presuppositions. For example, dynamic semantics predicts that the sentence in (83a) is associated with the conditional presupposition in (83b).

(83) a. If John works hard, his wife is happy.
b. If John works hard, he has a wife.

Now, it is easy to see how DRT avoids the proviso problem. In the case of (83a), when the previous discourse contains the information that John has a wife, the presupposition triggered by *his wife* is simply bound to it, and this gives rise to an intuitively correct interpretation. If the previous discourse does not contain this information, the hearer needs to perform accommodation. In this case, there are three possible ways of accommodation with respect to the presupposition associated with *his wife*:

(84) a. **Global**. John has a wife, and if he works hard, his wife is happy.
b. **Intermediate**. If John has a wife and he works hard, his wife is happy.
c. **Local**. If John works hard, he has a wife and his wife is happy.

As before, it is predicted that the most preferable option is global accommodation. This yields the reading in (84a), which is an intuitively correct interpretation of (83a).

3.3.3 Problems for DRT

There are at least two advantages of DRT over dynamic semantics. One is that it offers a unified account of presupposition projection and anaphora resolution, and the other is that it provides a natural solution to the proviso problem. However, it is known that DRT's account of presupposition projection faces some empirical problems, which threaten the advantages of DRT over dynamic semantics.

Problem 1: Quantified sentences

In Section 3.2.4, we observed that sentences containing the determiner *no* such as (54a) have the universal presupposition as shown in (54b).

(54) a. No student stopped smoking.
b. Every student used to smoke.

However, DRT fails to generate the universal presupposition in this case. To observe this, let us assume that (54a) has the initial representation in (85a). Then, the only interpretation available is that due to local accommodation, as shown in (85b).[22]

(85) a.

¬ [x_1 student(x_1)   stopped smoking (x_1) ∂ [used to smoke (x_1)]]

b.

¬ [x_1 student(x_1)   stopped smoking (x_1)   used to smoke (x_1)]

This interpretation amounts to saying that there is no student who used to smoke but stopped doing so. Clearly, this is weaker than the desired reading, namely, the reading that every student used to smoke but none of them stopped.[23] Note that the option of global accommodation — that is, of

[22] It is now standard within DRT to analyze the determiner *no* as a binary quantifier (cf. Kamp and Reyle 1993). Such an analysis allows the option of intermediate accommodation, which copies the conditions in ∂-structures into the restrictor of *no*. But, obviously, the problem cannot be solved by such a move.

[23] See Chemla (2009) for discussions on the presupposition of *no*.

moving the condition in the ∂-structure to the main DRS — is not available in this case, since in the resulting DRS, the occurrence of variable x_1 in the condition used to smoke(x_1) becomes free.

Furthermore, Schlenker (2011a) points out that DRT fails to predict universal presupposition even for universal sentences when they appear in non-assertive positions. For instance, sentence (86a) and (86b) intuitively presuppose the proposition that every student used to smoke, as is the case with the simple sentence *Every student stopped smoking*.

(86) a. If every student stopped smoking, I will be surprised.
b. Did every student stop smoking?

However, this is not predicted by any strategy of accommodation in DRT: global accommodation leads to an uninterpretable DRS and local accommodation yields an undesired reading.

Problem 2: Conditional presuppositions

As we saw in the last section, DRT avoids the proviso problem by generating *unconditional* presuppositions for those conditional sentences that contain a presupposition trigger in their consequent. Indeed, the standard framework of DRT simply cannot generate conditional presuppositions for such constructions. However, it has been argued that genuine conditional presuppositions do arise in certain cases.[24] The following example is taken from Schlenker (2011a):

(87) a. If this applicant is 64 years old, he knows that we cannot hire him.
b. If this applicant is 64 years old, does he know that we cannot hire him?
c. **Conditional presupposition**: If this applicant is 64 years old, we cannot hire him.

[24] It should be added that the existence of conditional presuppositions is still open to empirical debate. For proponents of conditional presuppositions, see Beaver (2001) and Schlenker (2011a, 2011b).

From the utterance of (87a), one can naturally infer the conditional proposition in (87c). This inference cannot be an entailment, since it survives in a question like (87b). Schlenker (2011a) also points out that example (88a) has the conditional presupposition in (88b).

(88) a. If this applicant is 64 years old and realizes that we cannot hire him, he won't be disappointed by a rejection letter.
b. **Conditional presupposition**: If this applicant is 64 years old, we cannot hire him.

Again, the fact that an inference like (88b) survives in the antecedent of a conditional suggests that it is not a mere entailment but a presupposition.[25]

Schlenker (2011b) also emphasizes the relevance of examples involving presupposition triggers such as *too*.

(89) a. If John calls you a Republican, his wife too will insult you.
b. **Unconditional**: Someone other than his wife (namely, John) will insult you.
c. **Conditional**: If John calls you a Republican, someone other than his wife (namely, John) will insult you.

Schlenker argues that intuitively (89a) gives rise to the conditional inference in (89c), rather than the unconditional one in (89b). Given the fact that anaphoric triggers like *too* generally resist local accommodation (cf., e.g., Beaver and Zeevat 2007), it seems difficult to handle such conditional inferences within the framework of DRT. Another example due to Schlenker (2011b) is the following:

(90) a. If Ann decides to study abroad, her brother too will make a stupid decision.
b. **Unconditional**: Ann will make a stupid decision.
c. **Conditional**: Studying abroad would be stupid of Ann.

[25] In DS, the sentence (88a) is schematically represented as $p \wedge (q \setminus\setminus r) \rightarrow s$; hence, it is correctly predicted that this presupposes that $p \rightarrow q$.

Again, (90a) intuitively presupposes the conditional proposition in (90c), rather than the unconditional one in (90b).

Problem 3: Semi-conditional presuppositions

Geurts (1999) points out that examples such as (91a) give rise to the "semi-conditional" presupposition as in (91b), rather than the "fully conditional" presuppositions as in (91c).

(91) a. If John is a scuba diver and he wants to impress his girlfriend, he'll bring his wetsuit.
b. If John is a scuba diver, he has a wetsuit.
c. If John is a scuba diver and he wants to impress his girlfriend, he has a wetsuit.

In the framework of DRT, however, it is not clear how to account for this kind of example. Note that when the presupposed information is copied in the consequent of a conditional via local accommodation, we can obtain a conditional inference — cf. the local accommodation reading in (81c) for the sentence in (78). However, as argued in Schlenker (2011b), it is the fully conditional inference, not the semi-conditional one we desired, since all the antecedents stay in the same position after performing local accommodation.

Problem 4: Interaction with implicit assumptions

It is often the case that assumptions that are not explicitly mentioned in a discourse play a role in presupposition projection. As a simple example, consider the following example.

(92) If John is married, his wife is happy.

In this example, the definite description *his wife* does not have an antecedent, but a suitable antecedent is easily *inferred* using the commonplace assumption that if John is married, he has a wife. Such examples are not exceptions at all; they are a central case to be accounted for by any theory of presupposition projection. A serious problem with the framework of DRT is that it does not handle such cases. Moreover, it is not clear how to extend the framework of DRT with components to handle inferences with presupposed information and implicit assumptions.

There are two other important cases that involve the interaction of presupposition projection and implicit assumptions. One is the so-called *bridging* inference (Clark 1975).

(93) Mary got some picnic supplies out of the car. The beer was warm.

The definite description *the beer* in the second sentence does not have an overt antecedent, but the hearer can make connection between the two sentences by inferring the existence of beer from the information conveyed by the first sentence. Here again, inferences with world knowledge play a crucial role in identifying the antecedent of a presupposition.

The other is the case of pronoun resolution. So far, pronouns have been treated as expressions that introduce discourse referents but do not have conditions by themselves. This is only for expository purposes, and it is quite natural to assume that pronouns introduce conditions such as gender information.[26] Thus, the first example we considered in Section 3.3.1 can be analyzed as follows:

(94) a. She smiles.

b.
$$\begin{array}{|l|}\hline \\ \mathsf{smile}(x_1) \\ \partial \begin{array}{|l|}\hline x_1 \\ \mathsf{female}(x_1) \\ \hline \end{array} \\ \hline \end{array}$$

This analysis enables us to encode both pronominal and presuppositional anaphora as ∂-structures and thus to simplify the whole architecture of DRT.[27] However, to combine the DRS like (94b) with the DRS of a precedent discourse and identify the antecedent of *she* in a successful way, we need to perform inferences, making use of relevant background assumptions. For

[26] The treatment of gender information of pronoun as presuppositions goes back at least to Cooper (1983). See Heim and Kratzer (1998) for a textbook treatment.

[27] Such an analysis is developed in Kamp, van Genabith, and Reyle (2011).

example, if (94a) follows the sentence *Mary walks*, then obviously the assumption that *Mary is female* plays a role in identifying the antecedent of *she* with *Mary*.

Presuppositions are resolved in various ways. In simple cases, the presupposed information is merely identified with some element present in the previous discourse or copied in a suitable place via accommodation. These possibilities are correctly accounted for within the framework of DRT. However, the examples such as (92) and (93) suggest that in complex cases, the antecedents of presuppositions need to be *inferred* using implicit assumptions, that is, assumptions that are not explicitly established in a previous discourse. Moreover, such implicit assumptions play a role even when the antecedent is explicitly given, as in (94a). Resolving presuppositions is not simply a matter of matching or adding information as is standardly assumed in the framework of DRT; rather, it crucially involves inferences with assumptions that are not directly provided in a discourse. This is not a surprising claim at all; indeed, it has been recognized among various authors.[28] However, it is fair to say that the question of how to incorporate the interaction between presupposition projection and ordinary inferences into the formal theory of presupposition has largely been unexplored.

3.4 Summary and discussion

Let us summarize what we have said so far. We started with the dynamic conception of meaning; according to this, asserting a sentence leads to a change of context and presuppositions are viewed as requirements imposed on the context in which the sentence is used. We have seen that dynamic semantics (reconstructed as DS and DS_q in Section 3.2) and DRT implement this conception in different ways.

In dynamic semantics, the notion of contexts is identified with a set of worlds, i.e., what we call *information states*, and the content of an utterance

[28] Geurts (1999: 72–79) admits the importance of world knowledge and inferences with them in resolving presuppositions, but provides no clues on how to incorporate additional inferential architectures into the framework of DRT.

is formalized as a partial function that takes an input context and returns an updated context if the presupposition is satisfied: see (H1) (H2) (H3) on page 161. Dynamic semantics provides correct predictions for the presuppositions of quantified sentences, but it faces the proviso problem: it fails to make correct predictions for a range of sentences involving conjunction and conditionals.

In DRT, both the notion of contexts and the notion of contents of utterances are formalized in terms of discourse representation structures (DRSs). Presuppositions are encoded as ∂-structures in preliminary DRSs; such ∂-structures must be resolved in a suitable way for the entire DRS to be interpretable. DRT successfully avoids the proviso problem, but it fails to provide correct readings for a certain class of quantified and conditional sentences, for which dynamic semantics can make correct predictions.

Comparing the predictions made by the two systems, we can see that the predictions made by dynamic semantics are often too weak; specifically, regarding the constructions causing the proviso problem, dynamic semantics makes the prediction that $p \rightarrow q$, though the desired presupposition is q. On the other hand, the predictions made by DRT are often too strong; in particular, when the conditional presupposition, schematically represented as $p \rightarrow q$, is required, DRT makes the prediction that q. These facts suggest that the information provided in the antecedent p may or may not be relevant to resolving the presuppositions in the consequent q, depending upon background assumptions held about p and q. It should be noticed that the problem of semi-conditional presuppositions also suggests this kind of flexibility of antecedently provided information. Dynamic semantics is committed to the view that, at least semantically, the information provided in previous contexts is *always* involved in the satisfaction of presuppositions in subsequent contexts; accordingly, some pragmatic strengthening mechanism needs to be supplied to make correct predictions. DRT, on the other hand, takes another extreme position in that the antecedently given information is relevant to resolving presuppositions only when it is identical to the presupposed information. What is required is a flexible theory, in which information previously given in context plays a role only when it is relevant to resolving the presuppositions in question.

We have also seen that DRT is unable to handle the interaction of presupposition resolution and ordinary inferences with implicit assumptions held by the participants in a conversation. Indeed, the same problem arises with dynamic semantics. For example, to process the sentence in (92) within DS_q, we need to extend the assignment of the input information state with a new individual satisfying the description *John's wife*; otherwise, the presupposition of the consequent would not be satisfied. Unfortunately, it is unclear how to account for such a case in the framework of dynamic semantics. Clearly, this kind of extension of assignment functions is not due to processes of accommodation in the usual sense; the antecedent of the conditional (i.e., *John is married*) and the implicit premise should play a role in extending the assignment.

These difficulties suggest that both DRT and dynamic semantics need to be augmented with components to handle *inferences* within the systems. Checking the validity of an inference is also needed to account for such tasks as checking informativeness and consistency of semantic representations — the tasks that play a crucial role in resolving anaphora and presupposition.[29] Such a component is usually provided by setting up a proof system for a given representation system, in particular, the language of dynamic semantics and the language of DRSs in DRT. However, in the case of dynamic semantics, only a semantic definition of logic is usually given.[30] Indeed, there has been no established proof system, mainly because of the unusual treatment of variables in dynamic semantics.[31] Accordingly, there is no direct, syntactic way of checking the validity of an inference. In the case of DRT, several attempts have been made to add a proof system to DRT (cf. Saurer 1993;

[29]See Blackburn and Bos (2005) for extensive discussions from the perspective of computational linguistics.

[30]This applies to various authors working on dynamic semantics we mentioned in Section 3.2, including Dekker (1992, 1996), Groenendijk, Stokhof, and Veltman (1996), and Beaver (1992, 1994, 2001).

[31]For example, in the case of DS_q, $\exists x Fx \wedge \exists x Fx$ is always undefined, and $\exists x Fx$ is not equivalent to $\exists y Fy$. Such facts will make it difficult to apply well-established techniques in logic. We add that Groenenveld and Veltman (1994) discuss several attempts to formulate a Gentzen-style proof system for Veltman's (1996) update semantics.

Kamp, van Genabith, and Reyle 2011); however, currently, there are no detailed proposals for combining such a proof system with a component to handle presupposition resolutions. In addition, the way in which DRSs are presented makes comparisons difficult with usual logical systems.[32] From a methodological point of view, adopting an existing, well-understood proof system would be preferable to creating a new theory.

Finally, we observe that dynamic semantics and DRT differ in their treatment of *indexing*. Dynamic semantics, formulated as DS_q, requires a full coindexing of anaphoric elements and their antecedents prior to interpretation. Thus, representing a sentence like (48a) with its intended meaning in DS_q requires the task of coindexing the variables with their antecedents, as indicated below:

(48a) If a farmer owns a donkey, he beats it.
$\exists x_1 \exists x_2 (\mathsf{farmer}\, x_1 \wedge \mathsf{own}\, x_1 x_2 \wedge \mathsf{donkey}\, x_2) \rightarrow \mathsf{beat}\, x_1 x_2$

On the other hand, DRT does not require this kind of external coindexing. For example, using standard DRSs, rather than preliminary DRSs containing ∂-structures, the donkey sentence in (48a) is initially represented as in (95), where noun phrases and pronouns are assigned different variables.

(95)
$$\boxed{\boxed{\begin{array}{l} x_1\, x_2 \\ \mathsf{farmer}(x_1) \\ \mathsf{own}(x_1, x_2) \\ \mathsf{donkey}(x_2) \end{array}} \Rightarrow \boxed{\begin{array}{l} \\ \mathsf{beat}(x_3, x_4) \end{array}}}$$

In this case, the natural interpretation requires that x_3 is bound to x_1 and x_4 to x_2, which satisfies the accessibility constraint we discussed before; then, we can obtain the intended interpretation of the sentence in (48). We

[32] Instead of providing DRSs with a proof system in a direct way, one can make use of the proof system of first-order logic via the translation of DRSs into first-order languages as seen in Section 3.2.1. This possibility is suggested in Blackburn and Bos (1999). But then, the level of DRSs would look as if they served merely as an auxiliary device to facilitate the translation into first-order logic.

may say that the process of anaphora resolution is *internal* to the construction of a DRS. Both approaches are technically tenable, but the method of internal coindexing as in DRT is methodologically preferable because of its simplicity and explicitness: in particular, the constraints on anaphora resolutions, which are applied to presupposition resolution as well, can be explicitly formulated as a process of constructing legitimate DRSs.[33]

We have accumulated desiderata for a formal theory of presupposition projection. Such a theory should satisfy the following requirements:

- Be flexible enough to account for the fact that presuppositions may or may not depend upon the antecedently given information;
- Provide correct predictions for quantified sentences;
- Provide a mechanism to handle the interaction between presupposed information and reasoning about implicit assumptions.

Furthermore, our methodological reflection suggests that:

- a well-developed proof system is needed to account for the fact that presupposition resolutions in general require us to perform inference;
- it is preferable to combine such a proof system with some mechanism of internal coindexing.

The two major approaches to presuppositions in formal semantics — dynamic semantics and discourse representation theory — are not well-suited to satisfy these desiderata. It is time to take steps toward an alternative approach.

[33]See Muskens (2011) for an elaboration of the method of external coindexing within the framework of compositional DRT of Muskens (1996).

4. A proof-theoretic framework

If we try to find a formal theory of presupposition projection satisfying the desiderata listed in the last section, it may be worthwhile changing the perspective: rather than starting with a semantic (i.e., model-theoretic) definition of logic, we may start with a logic equipped with a well-established proof system and explore its applications to the projection problem. Throughout this section, we will concentrate on the treatment of definite descriptions, which is a typical instance of presupposition triggers as we saw in Section 2. Later, we will briefly discuss how the methods developed here can be applied to other presupposition triggers.

4.1 Descriptions in proof theory

The initial observation to motivate our approach is that the use of descriptive phrases is quite common even in mathematics and computer science, and that the phenomena of presuppositions we have seen so far also arise in mathematical practice; in particular, in contexts where we reason about partial functions or programs that are not defined for certain arguments. Examples such as the ones in (96) could count as typical instances of definite descriptions used in mathematics.

(96) a. The inverse of d is less than 5.
b. $d^{-1} < 5$.

Our first observation here is that it is quite unusual to assert a sentence like (96) when the descriptive phrase does not refer to anything. Thus, unless it is guaranteed that a number d has an inverse, it would be odd to assert a sentence like (96). This observation also applies to the case in which the sentence in question is embedded under negation or supposition.

(97) a. $d^{-1} \not< 5$. NEGATION

b. Suppose that $d^{-1} < 5$. SUPPOSITION

Furthermore, we observe that even if the number d is not known to have an inverse, we can safely assert sentences like the following:

(98) a. d has an inverse and $d^{-1} < 5$.

b. If d has an inverse, then $d^{-1} < 5$.

c. If d is invertible, then $d^{-1} < 5$.

It is clear from these examples that whether such a description is well-defined (i.e., has a value) is sensitive to the environment in which it occurs. Indeed, these examples suggest that descriptive phrases that are a commonplace in mathematical practice show a pattern of projection behavior similar to that in examples discussed in Section 2.2.[1]

The fact that undefined or partial terms abound in mathematics and computer science has stimulated the study of logic with undefinedness, which is expected to be closer to mathematical practice than a standard logic.[2] Among others, Stenlund (1973, 1975) develops a proof-theoretic analysis of definite descriptions within the framework of classical first-order logic. Stenlund's view is particularly suitable for our purpose. It consists of two ideas:

(P1) a definite description is a referring expression, not a quantificational expression as in Russellian analysis;

[1]It has been sometimes argued that Strawson's presupposition theory of definite descriptions could be correctly applied to natural languages but not to languages used in mathematics. Russell himself argued this way in his reply to Strawson (1950). See Russell (1957). As we have seen in Section 2.1, Mate (1973) raised a similar objection to Strawson's view. Historically, however, it was Frege (1892) who first proposed the presupposition theory of descriptions, and it is worth emphasizing that Frege was mainly concerned with the use of descriptions in mathematics.

[2]See Feferman (1995) for an illuminating survey.

(P2) it can be properly used under the *assumption* that it refers to an individual.

Comparisons with the other two views with respect to the treatment of empty descriptions will be useful here. Consider the following classical example:

(99) The present king of France is bald.

Under the Russellian analysis, which is summarized in (R1) and (R2) on page 145 in Section 2.1, (99) is simply false. Under the presuppositional analysis of Strawson, as summarized in (S1) and (S2) on page 146, (99) fails to express a proposition. Under Stenlund's proof-theoretic analysis, which could be regarded as a refinement of the presuppositional analysis, (99) can be used to express a proposition under the assumption that there is a king of France. On this account, a sentence involving an empty description can express a proposition simply because we can use any assumption we want in the course of an inference. As we will see later, the notion of assumption plays a crucial role in accounting for the projection problem of presuppositions.

Stenlund (1973) introduced a natural deduction system of classical first-order logic with $\imath$-operators, where $\imath x A(x)$ is intended to give a logical analysis of "the x such that $A(x)$" in mathematical reasoning.[3] Roughly, the idea is that a description like $\imath x A(x)$ can be properly used when the existence

[3]It should be added that Stenlend (1973) is skeptical about the possibility of applying his theory to ordinary discourse, claiming that "ordinary usage seems to give little guidance on questions of the truth or falsity of many sentences containing empty descriptions" (Stenlund 1973: 2). However, if we focus on the projection problem of presuppositions rather than on the problem of empty descriptions (or more generally, the problem of *presupposition failure*), then we find that there is much in common between mathematical and ordinary uses of descriptions. Over a past few decades, the formal theories of presupposition has centered around the projection problem rather than the problem of presupposition failure, and this shift of focus led to a variety of novel proposals in formal semantics and pragmatics as we have seen in Section 2. Of course, this does not mean that formal semantics and pragmatics have nothing to say about the problem of presupposition failure; for recent discussions of truth-value judgements about empty descriptions and related examples, see Lasersohn (1993), von Fintel (2004), and Schoubye (2009).

condition $\exists x A(x)$ and uniqueness condition $\forall x \forall y(A(x) \wedge A(y) \rightarrow x = y)$ are both satisfied. Additionally, Stenlund (1975) introduces an intuitionistic version of the system. Carlström (2005) presents a similar natural deduction system of intuitionistic first-order logic with ε-operators, where $\varepsilon x A(x)$ is associated only with the existence condition $\exists x A(x)$ and is intended to capture the use of indefinite descriptions like "an x such that $A(x)$" in mathematical reasoning. Interestingly, Carlström (2005) suggests that definite descriptions in ordinary discourse could sometimes be represented by ε-terms.

To summarize, depending on whether one takes the uniqueness implication to be part of the meaning of definite descriptions, there are at least two possibilities that one could develop a proof system for definite descriptions.

1. If we decide to take uniqueness implications to be part of the meaning of definite descriptions, then we could analyze descriptions of the form "the F" as "$\iota x F(x)$."
2. Alternatively, if we do not incorporate uniqueness implications into the semantics of definite descriptions, then we could analyze descriptions of the form "the F" as "$\varepsilon x F(x)$."

The former view is in accordance with the recent Fregean analyses of definite descriptions (and related referring expressions such as pronouns and proper names) like the one developed in Elbourne (2005). A long-standing problem with such a view is how to deal with cases in which uniqueness implications associated with definite descriptions are not in fact satisfied. The analysis in (ii) is in agreement with the view in the philosophy and linguistics literature that uniqueness implications associated with definite descriptions should be derived pragmatically.[4] Although the question whether uniqueness implications are to be explained in terms of semantics or pragmatics is admittedly still open to debate, we will choose to pursue (ii) over (i), mainly because it allows us to keep the semantics of definite descriptions simpler. Thus,

[4]Under this analysis, such a notion as *salience* (cf. Lewis 1979; McCawley 1979) or *familiarity* (cf. Heim 1982) plays a dominant role in determining the referent of a description. For a recent discussion, see Szabó (2000, 2005).

in what follows, we will set aside the uniqueness condition and analyze a definite description "the F" as "$\varepsilon x\, F(x)$."

The considerations in Section 2 suggest that the issue between the various theories of presuppositions ultimately turns upon the proper treatment of logical operators (i.e., connectives and quantifiers) in natural languages. We must then determine which logic is suitable to represent reasoning about presuppositions in natural languages. A common assumption that is implicitly or explicitly held by proponents of the dynamic view of meaning is that classical logic is unsuitable for handling certain aspects of discourse phenomena, in particular, anaphora and presupposition. Recall a typical case motivating dynamic predicate logic and DRT, taken from Kamp, van Genabith, and Reyle (2011).

(100) A student arrived.
$\exists x(\mathsf{student}\, x \wedge \mathsf{arrived}\, x)$

Given that $\exists x A$ and $\neg\forall x \neg A$ are equivalent in classical logic, it is expected that (100) behaves the same way as sentences like (101).

(101) It is not the case that every student failed to arrive.
$\neg\forall x(\mathsf{student}\, x \rightarrow \neg\mathsf{arrived}\, x)$

As we already saw in Section 3.2.4, however, negation acts as an externally static operator in that it does not pass a discourse referent to the subsequent discourse. Thus, while (100) can be extended as in (102a), (101) cannot be extended as in (102b).

(102) a. A student arrived. She/The student registered.
b. It is not the case that every student failed to arrive. # She/The student registered.

These examples suggest that the antecedent of an anaphoric or presuppositional element has to be explicitly provided in a suitable context. To account this fact, dynamic semantics and DRT adopt the view that classical truth-conditions alone are insufficient to capture the dynamic dimension of meaning, as we have already seen in Section 2. Thus, classical logical laws such as $\neg\neg A \equiv A$ and $\neg\forall x A \equiv \exists x \neg A$ no longer hold in various systems of dynamic semantics, including DS_q and related systems that we discussed in Section 3.2.4.[5] Additionally, in the case of DRT, (100) and (101) are assigned different DRSs that have different accessibility relations for subsequent sentences.[6]

Agreeing with these views, we take it that classical logic is not good at accounting for the fact that the antecedent of an anaphoric element must be explicitly given in a context. A natural alternative that is consistent with the observations above is, we would claim, intuitionistic logic. As is well known, logical connectives and quantifiers of intuitionistic logic are given *constructive* interpretations (i.e., the so-called BHK interpretation; see below), and consequently, classical laws such as $\neg\neg A \equiv A$ and $\neg\forall x A \equiv \exists x \neg A$ fail in intuitionistic logic. In particular, to assert existential formulas $\exists x A(x)$, one has to find a particular individual satisfying $A(x)$. As we will see in detail later, this interpretation of existential quantifiers is well suited for explaining the anaphoric linking as indicated in the examples above.

Indeed, applying constructive logic to the analysis of discourse phenomena in natural languages is not new. In particular, building on earlier work by Sundholm (1986, 1989), Ranta (1994) shows that Constructive Type Theory developed by Martin-Löf (henceforth, abbreviated as CTT) provides us with an alternative framework to the standard dynamic theories such as DRT and DPL.[7] Recently, CTT and type-theoretic methods have been

[5]These systems include dynamic predicate logic (Groenendijk and Stokhof 1991), eliminative dynamic predicate logic (Dekker 1993, 1996), and update semantics (Veltman 1996).

[6]The matter is a little complicated in the case of DRT, since as we saw in Section 3.3.1, DRSs can be translated into formulas of classical first-order logic. This means that DRSs deliver two aspects of meaning: one is the dynamic aspect, which can be captured by defining proper accessibility relations; the other is the informational or truth-conditional aspect, which can be captured by the recursive definition of semantics or by the translation into first-order logic. However, in the last section, we observed that resolving presuppositions sometimes requires inferences, which are performed across the two aspects of meaning. This suggests that a certain uniform system that is responsible for both informational and dynamic aspects of meaning would be preferred.

[7]There are several versions of constructive type theory. One original system, called the "polymorphic" type system, is presented in detail in Martin-Löf (1984) and Part I of Nordström, et al. (1990) An alternative, the "monomorphic" type system, is presented in

applied to various problems in natural language interpretations.[8] Interestingly, Martin-Löf (1984) suggests that ε-terms can be naturally interpreted in terms of the existential quantification of CTT. Following this suggestion, Carlström (2005) showed that his system (i.e., the natural deduction system of intuitionistic first-order logic with ε-operators) can be translated into CTT. The ε-calculus can then serve as an intermediate language in translating natural language sentences into formulas in CTT. As we will see later, this translation has several nice properties that lead to a better formulation of presupposition and anaphora resolution. For this reason, we will use the intuitionistic (first-order) logic with ε-operators as a logic that handles presupposition projection.

Before moving on, let us make a cautionary note. We take it that the presuppositional analysis adopted here applies only to a definite description occurring in the argument position of a predicate, as in (103a) below. It is widely observed in the literature that there are at least two types of constructions in which definite descriptions have a "non-referential" use.[9] One is the case in which a definite description occurs in the predicative position as in (103b), and another is the case in which a definite description occurs in an existential construction as in (103c).

(103) a. The king of France is wise.

b. John is the king of France.

Part III of Nordström, et al. (1990). In what follows, we will work with the polymorphic type system.

[8]See Fernando (2001, 2009), Cooper (2005a, 2005b), and Asher (2011). Applications of CTT to the analysis of presuppositions in natural language are found in Krahmer and Piwek (1999).

[9]For three types of uses of noun phrases, the standard reference in formal semantics is Partee (1986). Strawson (1950: 320) has already observed that the predicational use of definite descriptions does not carry any existential presupposition. For the predicative use of descriptions, see also Fara (2000) and references given there. Geach (1950) also advocated the presuppositional theory of definite descriptions, raising objections to Russell's theory. Geach remarked that while Russell's theory makes incorrect predictions for sentences like (103a), it works adequately for the predicative and existential uses of descriptions as in (103a) and (103b). Such "non-referential" uses of noun phrases will be discussed in detail in Chapter 3 of this thesis.

c. The king of France does not exist.

It seems that *the king of France* in (103b) and (103c) are best analyzed as predicates, rather than referential terms.[10] The ε-terms employed here only capture the referential use of definite descriptions. Thus, it is important to recognize that by saying that descriptions like "the A" are translated as $\varepsilon x A(x)$, we mean that certain *occurrences* of descriptions, or more specifically occurrences as an argument of a predicate as in (103a), can be analyzed as such ε-terms; it is not a claim about the lexical properties of descriptions that can be described independently of their occurrence.

4.2 Natural deduction system with ε-operators

Building on Carlström (2005), we introduce a natural deduction system of intuitionistic first-order logic with ε-terms, abbreviated as $\mathsf{IL}\varepsilon$. In the next section, we review how $\mathsf{IL}\varepsilon$ can be translated into CTT.

The vocabularies of $\mathsf{IL}\varepsilon$ consists of those of a standard first-order logic augmented with ε-operators. We assume that there are predicate symbols and individual constants corresponding to expressions appearing in the examples that we deal with in the subsequent discussion:

- Logical symbols: $\wedge$, $\rightarrow$, $\bot$, $\forall$, $\exists$, ε
- 1-place predicate symbols: $\mathsf{man}, \mathsf{wise}, \mathsf{donkey}$, etc.
- 2-place predicate symbols: $\mathsf{king}, \mathsf{wife}, \mathsf{beat}$, etc.
- individual constants: $\mathsf{john}, \mathsf{france}$, etc.

[10] This is the view suggested by Geach (1950). We add that (103b) is ambiguous among (at least) three readings. First, it can be used as an answer to the question "What is John like?". This is the predicative reading we have in mind here. In this reading, the description *the king of France* seems to act as a predicate. Second, it can be used as an answer to the question "Who is the king of France?". This is what Higgins (1973) called the *specificational* reading. Third, it has a reading equivalent to the identity statement, "John is identical to the king of France." All three readings must be carefully distinguished. The presuppositional analysis seems to apply to the identity reading; for some discussion of a related problem, see Section 3.3 of Chapter 3. Whether it can be applied to the specificational reading as well is more intricate, and we leave this question for future work.

- individual variables: $x, y, z, \dots$

The terms and formulas of $\mathsf{IL}\varepsilon$ are defined in a mutually dependent way. The set of formulas, denoted by $\mathcal{L}_{\mathsf{IL}\varepsilon}$, is defined by the rule

$$A ::= P(t_1, \dots, t_n) \mid \bot \mid A \wedge B \mid A \to B \mid \forall x A \mid \exists x A,$$

where P ranges over the set of n-place predicate symbols, t_i over the set of terms with $1 \leq i \leq n$, and A, B over the set of formulas. The set of terms of $\mathsf{IL}\varepsilon$ is defined by the rule

$$t ::= x \mid c \mid \varepsilon x A(x),$$

where x ranges over the set of individual variables, c over the set of individual constants, and A over the set of formulas.

Negation $\neg A$ is defined to be $A \to \bot$. We exclude disjunction from our considerations, since it introduces additional complexity to our discussion.[11] Free and bound variables are defined in the usual way, and we adopt the usual convention of omitting parentheses.

To formalize the presupposition theory of definite descriptions introduced in the last section, we need to distinguish between (i) formulas and (ii) formulas that express a proposition, and between (i′) terms and (ii′) terms that denote an individual.[12] We then need to formally represent inferences like

[11] It seems that disjunction in natural languages works in a different way from disjunction in intuitionistic logic. The constructive (BHK) interpretation of disjunction tells us that to assert $A \vee B$, one has to be in a position to assert A or to assert B. This interpretation is reflected by the so-called *disjunction property* of intuitionistic proof systems, according to which either A or B is provable whenever $A \vee B$ is provable. It seems clear that disjunctive sentences in natural languages do not behave this way. Thus, we can correctly assert the sentence *John or Mary came* even when we are not in a position to assert either *John came* or *Mary came*. One possibility of handling natural language disjunction within intuitionistic logic is to translate "A or B" as $\neg A \to B$. This then predicts the same project property for disjunction as dynamic semantics of Beaver (2001), which we saw in footnote 5 on page 167. See also Klinedinst and Rothschild (2012) for arguments in favor of treating disjunctions as conditionals with negative antecedents. We leave a detailed analysis of disjunction within our framework for future work.

[12] Stenlund (1973) refers to (i′) as *formula-expressions*, and to (ii′) as *term-expressions*.

(104) Suppose that "There is a king of France" is true. Then "the king of France" refers to an individual. Hence, "The king of France is bald" expresses a proposition.

This is written in a tree form as follows:

(105)
$$\dfrac{\dfrac{\text{“There is a king of France” is true}}{\text{“the king of France” refers to an individual}}}{\text{“The king of France is bald” expresses a proposition.}}$$

Here we assert that a sentence expresses a proposition, under the assumption that a sentence is true. We call each statement appearing in this inference a *judgement*, following the terminology of Stenlund (1973).[13] Judgements make explicit what is usually taken to be implicit in the meta-language of a logical system. For our purpose, we need three kinds of judgements as shown in Figure 4.1.

Judgement	Reading
A true	a formula A is true
A : prop	a formula A expresses a proposition
t : ind	a term t refers to an individual

Fig. 4.1 Three forms of judgements in ILε

In accordance with the notation in standard logic, henceforth, we write A, instead of A true, in a formal derivation. However, it is always important to keep in mind the distinction between a proposition A and a judgement A true.[14]

Now the informal derivation in (105) can be represented as follows.

[13]The distinction between propositions and judgements has been widely adopted in type theory since the work of Martin-Löf (1975, 1982, 1984).

[14]The distinction between propositions and judgements is made fully explicit in Frege's (1893) *Begriffschrift*, where notation is introduced to distinguish judgements from propositions; i.e., *thoughts* (*Gedanken*) in Frege's terminology. As noted by Aczel (1980) and Martin-Löf (1983), a judgement of the form A : prop in our notation is analogous to a judgement of the form —A in Frege's notation, and a judgement of the form A true is analogous to a judgement of the form $\vdash\!\!-\!\!-A$, where the sign $\vdash\!\!-\!\!-$, called a *judgement-stroke*, is composed of a vertical line and a horizontal line (Frege 1893: 38). Frege's notation for

$$\text{(106)} \quad \cfrac{\cfrac{\exists x\, \mathsf{king}(x, \mathsf{france})}{\varepsilon x\, \mathsf{king}(x, \mathsf{france}) : \mathsf{ind}}}{\mathsf{bald}(\varepsilon x\, \mathsf{king}(x, \mathsf{france})) : \mathsf{prop}}$$

Here we assume that *king* is a relational noun and represent *the king of France* as $\varepsilon x\, \mathsf{king}(x, \mathsf{france})$. In this case, we make a *hypothetical* derivation. In general, we may have more than one assumption; hence, a hypothetical derivation has the form

$$\begin{array}{c} J_1 \ \cdots \ J_n \\ \vdots \\ J \end{array} \quad ,$$

where the judgements $J_1, \ldots, J_n$ are unproven assumptions and the judgement J is the conclusion. Here J is asserted *hypothetically*; that is, at the final step of the derivation, we assert that J holds provided all the assumptions $J_1, \ldots, J_n$ hold. In the case of (106), the final step of the derivation hypothetically asserts the following: $\varepsilon x\, \mathsf{king}(x, \mathsf{france}))$ expresses a proposition provided $\exists x\, \mathsf{king}(x, \mathsf{france})$ is true.

Note that types ind and prop are standardly expressed by types e and t in Montague's notion, where e is used for the type of individuals and t for the type of truth values. Since we are working within intuitionistic logic, where propositions are not interpreted as Boolean truth-values, we prefer our notation.

Another important difference from the standard practice in formal semantics is that in our framework, judgements of the forms $A : \mathsf{prop}$ and $t : \mathsf{ind}$, on the one hand, and judgements of the form $A\ \mathsf{true}$, on the other, may be mutually dependent in a derivation. In the standard setting, it is usually assumed that the question of whether a given sentence A is true is resolved only after the question of whether A expresses a proposition (i.e., A is of the type t) is settled. In our framework, by contrast, the question of whether a given sentence expresses a proposition can be resolved after resolving the question of whether a sentence is true, and such dependency is crucial for representing reasoning about presuppositions.

judgement-strokes presupposes that what is asserted to be true must be a proposition (i.e., a bearer of truth-values in Frege's case). Indeed, an analogous property holds for $\mathsf{IL}\varepsilon$. See Proposition 4.3 below.

Now we present a natural deduction system of $\mathsf{IL}\varepsilon$, which is obtained by extending and modifying a standard system of natural deduction developed by Gentzen (1934) and Prawitz (1965). As in standard systems, a *derivation* has a tree form: it starts with an axiom or an assumption and proceeds by means of inference rules.

Two kinds of assumptions are allowed in this system.

1. We may introduce $x : \mathsf{ind}$ as an assumption.
2. When we have a derivation of $A : \mathsf{prop}$, we may introduce A as an assumption.

In other words, we are always allowed to assume that a variable x ranges over individuals, and when we know that a formula A expresses a proposition, we are allowed to assume that A is true. In making assumptions of the latter kind. we omit indicating the derivation of $A : \mathsf{prop}$ to simplify the whole derivation. It should be understood that each assumption of the form A in a derivation is associated with a derivation of $A : \mathsf{prop}$.[15]

The inference rules are divided into three groups: formation rules, introduction rules, and elimination rules.

Formation rules. There are two kinds of formation rules: those for forming a term denoting an individual and those for forming a formula expressing a proposition.

$$\frac{}{c : \mathsf{ind}}\ cF \qquad \frac{\exists x A(x)}{\varepsilon x A(x) : \mathsf{ind}}\ \varepsilon F$$

$$\frac{t_1 : \mathsf{ind} \quad \cdots \quad t_n : \mathsf{ind}}{P(t_1, \ldots, t_n) : \mathsf{prop}}\ PF \qquad \frac{}{\bot : \mathsf{prop}}\ \bot F \qquad \frac{A : \mathsf{prop} \quad \begin{matrix}[A] \\ \vdots \\ B : \mathsf{prop}\end{matrix}}{A \wedge B : \mathsf{prop}}\ \wedge F$$

[15] A rule for making assumptions of this kind can be explicitly formulated in a sequent-style natural deduction system in the following way:

$$\frac{\Gamma \vdash A : \mathsf{prop}}{\Gamma, A \vdash A}\ .$$

Although the sequent-style presentation enables us to represent operations on contexts in an explicit way, the resulting derivations would be more involved. To simplify the presentation of a derivation, we adopt the Gentzen–Prawitz style system.

$$\frac{A : \mathsf{prop} \quad \begin{array}{c}[A]\\ \vdots \\ B : \mathsf{prop}\end{array}}{A \rightarrow B : \mathsf{prop}} \rightarrow F \qquad \frac{\begin{array}{c}[x : \mathsf{ind}]\\ \vdots \\ A(x) : \mathsf{prop}\end{array}}{\forall x A(x) : \mathsf{prop}} \forall F \qquad \frac{\begin{array}{c}[x : \mathsf{ind}]\\ \vdots \\ A(x) : \mathsf{prop}\end{array}}{\exists x A(x) : \mathsf{prop}} \exists F$$

The rules cF and PF are schematic rules: for every n-place predicate symbol P and for every individual constant c, there is such a formation rule. Note that the rule cF is to be understood as an axiom; i.e., a rule with no premises. The rules $\forall F$ and $\exists F$ are subject to the restriction that x must not occur free in any non-discharged assumption in the derivation of the conclusion of the rule.

The rule εF is the only source of presuppositions; it states that $\varepsilon x A(x)$ refers to an individual when $\exists x A(x)$ is true. The rule PF states that an atomic formula expresses a proposition when all terms appearing in it refer to an individual. The rules $\wedge F$ and $\rightarrow F$ are in agreement with the idea behind dynamic semantics and DRT that each new sentence is interpreted in the context provided by the sentences preceding it. The rules $\forall F$ and $\exists F$ will play a role in predicting universal presuppositions for quantified sentences.

Since negation $\neg A$ is defined as $A \rightarrow \bot$, the formation rule for negation, the rule $\neg F$, can be derived in the following way:

$$\frac{A : \mathsf{prop}}{\neg A : \mathsf{prop}} \neg F.$$

Introduction and elimination rules. The introduction and elimination rules are the following.

$$\frac{A \quad B}{A \wedge B} \wedge I \qquad \frac{A \wedge B}{A} \wedge El \qquad \frac{A \wedge B}{B} \wedge Er \qquad \frac{A : \mathsf{prop} \quad \bot}{A} \bot E$$

$$\frac{A \quad A \rightarrow B}{B} \rightarrow E \qquad \frac{A : \mathsf{prop} \quad \begin{array}{c}[A]\\ \vdots \\ B\end{array}}{A \rightarrow B} \rightarrow I \qquad \frac{\exists x A(x)}{A(\varepsilon x A(x))} \varepsilon I$$

$$\dfrac{\begin{matrix}[x : \mathsf{ind}] \\ \vdots \\ A(x)\end{matrix}}{\forall x A(x)}\ \forall I \qquad \dfrac{\forall x A(x) \quad t : \mathsf{ind}}{A(t)}\ \forall E \qquad \dfrac{\begin{matrix}[x : \mathsf{ind}] \\ \vdots \\ A(x) : \mathsf{prop}\end{matrix} \quad t : \mathsf{ind} \quad A(t)}{\exists x A(x)}\ \exists I$$

The rules $\forall I$ and $\exists I$ are subject to the usual variable restriction: x must not occur free in any non-discharged assumption in the derivation of the conclusion of the rule. We also need certain special restrictions that apply to $\rightarrow E$ and $\exists I$, as stated in Carlström (2005).

Compared with standard first-order logic, the rules $\forall E$ and $\exists I$ have a judgement of the form $t : \mathsf{ind}$ as an additional premise, in an analogous way to standard systems of free logic.[16] In particular, the restriction on the rule $\exists I$ is crucial when we want to combine intuitionistic logic with ε-operators. The addition of the rule εI to the standard first-order intuitionistic logic is non-conservative in the sense that, in the extended system, we can prove ε-free formulas that cannot be proved in the original system. A crucial example is $\exists y(\exists x A(x) \rightarrow A(y))$, which is not provable in standard intuitionistic logic, but is provable if we admit both the standard introduction rule for existential quantification and the rule εI.

$$\dfrac{\dfrac{\dfrac{[\exists x A(x)]^1}{A(\varepsilon x A(x))}\ \varepsilon I}{\exists x A(x) \rightarrow A(\varepsilon x A(x))}\ \rightarrow I,\ 1}{\exists y(\exists x A(x) \rightarrow A(y))}\ \spadesuit$$

In $\mathsf{IL}\varepsilon$, the step indicated by $\spadesuit$ is illegitimate, since to apply $\exists I$ at this step, we need the judgement $\varepsilon x A(x) : \mathsf{ind}$ as an additional premise; i.e., we need to ensure that $\varepsilon x A(x)$ has a reference.[17]

It is easily seen that the elimination rule $\exists E$ in standard systems, shown

[16] See Lambert (1991, 2003) for an overview of free logic.

[17] See DeVidi (2004) and Carlström (2005) for overviews of attempts to block this inference within systems of intuitionistic ε-calculus.

below, can be derived using the rules εF and εI.

$$\dfrac{\exists x A(x) \qquad \begin{matrix}[x:\mathsf{ind}][A(x)]\\ \vdots \\ B\end{matrix}}{B}\ \exists E$$

The elimination rule $\exists E$, which is adopted by Gentzen (1934) and Prawitz (1965), has been criticized for the reason that it does not agree with the fact that it is normally only after deriving $\exists x A(x)$ that we introduce t such that $A(t)$ holds.[18] It can be argued that the rule εI is an improvement in this respect: see Example 4.2 below for an illustration of the use of εI.

Example 4.1 Each step of the derivation in (106) can be annotated with a label as follows.

$$\dfrac{\dfrac{\exists x\,\mathsf{king}(x,\mathsf{france})}{\varepsilon x\,\mathsf{king}(x,\mathsf{france}):\mathsf{ind}}\ \varepsilon F}{\mathsf{bald}(\varepsilon x\,\mathsf{king}(x,\mathsf{france})):\mathsf{prop}}\ \mathsf{bald}F$$

It should be noted that the assumption $\exists x\,\mathsf{king}(x,\mathsf{france})$ is properly introduced since we have a derivation of $\exists x\,\mathsf{king}(x,\mathsf{france}):\mathsf{prop}$ as shown below.

$$\dfrac{\dfrac{[x:\mathsf{ind}]^1 \qquad \dfrac{}{\mathsf{france}:\mathsf{ind}}\ \mathsf{france}F}{\mathsf{king}(x,\mathsf{france}):\mathsf{prop}}\ \mathsf{king}F}{\exists x\,\mathsf{king}(x,\mathsf{france}):\mathsf{prop}}\ \exists F,\ 1$$

Example 4.2 The following is a derivation of $\forall x\exists y P(x,y)$ from $\exists y\forall x P(x,y)$.

$$\dfrac{\dfrac{\dfrac{[x:\mathsf{ind}]^2 \quad [y:\mathsf{ind}]^1}{P(x,y):\mathsf{prop}}\ PF \qquad \dfrac{\exists y\forall x P(x,y)}{\varepsilon y\forall x P(x,y):\mathsf{ind}}\ \varepsilon F \qquad \dfrac{\dfrac{\exists y\forall x P(x,y)}{\forall x P(x,\varepsilon y\forall x P(x,y))}\ \varepsilon I \quad [x:\mathsf{ind}]^2}{P(x,\varepsilon y\forall x P(x,y))}\ \forall E}{\exists y P(x,y)}\ \exists I,\ 1}{\forall x\exists y P(x,y)}\ \forall I,\ 2$$

In $\mathsf{IL}\varepsilon$ we cannot derive a judgement $A:\mathsf{prop}$ from a judgement A true. However, interestingly, this inference is *admissible* in the following sense.

[18] See Cellucci (1995) for a useful overview of this issue.

Proposition 4.3 *Let A be a formula of $\mathcal{L}_{IL\varepsilon}$. If there is a derivation $\mathcal{D}$ of A* true *from a set of assumptions Γ, then there is a derivation $\mathcal{D}^*$ of A* : prop *from Γ.*

For a proof, see Carlström (2005).[19] It is then ensured that whenever we have a derivation of A true, we have a derivation of A : prop. This property plays a particularly important role when A contains descriptions of the form $\varepsilon x B(x)$. As will be explained below, what $\varepsilon x B(x)$ refers to is fixed, in some sense, by a derivation of A : prop.

Comparisons between Stenlund's system and earlier systems of free logic, including those of Hintikka (1959), Smiley (1960), Scott (1967), and Schock (1968), are found in Stenlund (1973). Additionally, Lambert (2003) discussed differences between Stenlund's system and ε- and $\imath$-calculus of Hilbert and Bernays (1934). There are various proof systems of intuitionistic first-order logic enriched with ε-terms in the literature: a sequent calculus of Shirai (1971), which follows the work of Maehara (1955); E-logic of Scott (1979); ε-calculus of Mints (1977), among others. Comparisons with these systems are found in Carlström (2005). As stated above, an advantage of Stenlund and Carlström's system that we adopt here is that it can be translated into CTT, and the latter has been well studied in the context of natural language semantics. In the next section, we will review CTT and see how the translation from ILε into CTT works.

4.3 Constructive type theory

As we mentioned in the previous section, Ranta (1994) applied CTT to the area of natural language semantics and showed that it serves as an alternative framework to discourse semantics such as DRT (Kamp 1981) and DPL (Groenendijk and Stokhof 1991). One problem that arises when we apply CTT to natural language semantics is how to formulate the translation procedure from natural language sentences or discourses into formal expressions

[19]This proposition was originally proved by Stenlund (1973) for a natural deduction system with $\imath$-operators.

in CTT. In particular, the problem is how to deal with the resolution process of presupposition and anaphora. Ranta (1994) handles this problem by adopting the generation-based approach, according to which the translation proceeds from formulas in CTT to natural language sentences. In contrast, ILε that we presented in the last section enables us to preserve a usual parsing-based translation procedure, where natural language sentences are mapped into formulas in CTT. The basic idea is to use formulas of ILε as an intermediate language in the translation procedure from natural languages into CTT. From the linguistic point of view, a formula of ε-calculus plays the role of an *underspecified* representation of what the utterance of a sentence conveys. This makes it possible to separate the entire translation process into deterministic and non-deterministic parts, and thereby to provide internal coindexing, as we desired.

The basic idea of CTT is to understand the *truth* of a proposition in terms of the existence of a *proof*, where a proof is taken to be an individual. Thus, a judgement of the form

$$A \ \mathsf{true}$$

is replaced by a judgement of the form

$$p : A,$$

which reads as "a proposition A has a proof p." Here, following the so-called *Curry–Howard correspondence* ("propositions-as-sets" interpretation), the meaning of a proposition is identified with the set of proofs of that proposition.[20]

In addition to judgements of the form $p : A$, we need *hypothetical* judgements.[21] Meanwhile, in ILε, a hypothetical judgement is implicitly employed

[20] See Curry and Feys (1958) and Howard (1980). In applications to natural language semantics, such proofs are sometimes viewed as *situations* (Cooper 2005a) or *events* (Ranta 1994), in connection with situation semantics and event semantics in formal semantics.

[21] Here and henceforth we are working within the monomorphic type theory of Nordström et al. (1990) in contrast to polymorphic type theory as used in Carlström (2005). This is because it is easier to make comparisons with standard first-order logic, and nothing substantial depends upon this choice.

in a hypothetical derivation; here we need to represent it explicitly. We need hypothetical judgements of the form

$$f(x) : B\ (x : A),$$

which can be read as "$f(x) : B$ under the assumption that $x : A$." This form of hypothetical judgement introduces a function from A to B, and allows us to derive that $f(a)$ is an element of B whenever a is an element of A. This meaning is captured by the *substitution rule* of the following kind.[22]

$$\frac{f(x) : B\ (x : A) \qquad a : A}{f(a) : B}\ S$$

In general, hypothetical judgements have the form

$$f(x_1, \ldots, x_n) : B\ (x_1 : A_1, \ldots, x_n : A_n).$$

In particular, corresponding to each n-place predicate symbol P in $\mathsf{IL}\varepsilon$, we will allow hypothetical judgements of the form

$$P(x_1, \ldots, x_n) : \mathsf{prop}\ (x_1 : \mathsf{ind}, \ldots, x_n : \mathsf{ind}),$$

which introduce an n-ary propositional function on the set ind of individuals. This form of hypothetical judgement is taken as being primitive; i.e., judgements that require no further justification. Note that every application of the rule PF for an n-place predicate P in $\mathsf{IL}\varepsilon$ can be simulated as n-times applications of the substitution rule in CTT. Thus, we admit

$$\frac{a_1 : \mathsf{ind} \ \cdots \ a_n : \mathsf{ind}}{P(a_1, \ldots, a_n) : \mathsf{prop}}$$

as a derived rule, in accordance with the rule PF in $\mathsf{IL}\varepsilon$.

[22] Note that in sequent-style systems where all the assumptions are indicated, the substitution rule can be written as follows.

$$\frac{\Gamma, x : A \vdash f(x) : B \qquad \Gamma \vdash a : A}{\Gamma \vdash f(a) : B}\ S$$

In contrast to standard first-order logic, the domains of quantification are always made explicit in CTT. We write $(\Sigma x : A)\, B(x)$ for existential quantification and $(\Pi x : A)\, B(x)$ for universal quantification, where A is a set term denoting the domain of quantification. In particular, $\exists x A(x)$ in ILε is replaced by $(\Sigma x : \mathsf{ind})\, A(x)$, and $\forall x A(x)$ by $(\Pi x : \mathsf{ind})\, A(x)$. Given the identification between propositions and sets, A in $(\Sigma x : A)\, B(x)$ and $(\Pi x : A)\, B(x)$ may be a proposition, in which case x ranges over proofs of A. As will be explained below, this plays a role in accounting for representing donkey and cross-sentential anaphora.

As stated above, the meaning of a proposition is given in terms of what counts as a proof of it.[23] The intended semantics here is the so-called Brouwer–Heyting–Kolmogorov (BHK) interpretation.[24] According to this interpretation, the following holds:

1. a proof of $A \to B$ is a construction that returns a proof of B, given a proof of A;
2. a proof of $A \wedge B$ consists of a proof of A and a proof of B;
3. a proof of $\forall x A(x)$ is a function converting c into a proof of $A(c)$;
4. a proof of $\exists x A$ is a pair (a, b) where b is a proof of $A(a)$.

In line with this interpretation, we have *formation rules* for each logical connective and quantifier. As in ILε, the formation rules tell us when a judgement of the form $A : \mathsf{prop}$ can be derived. We also need introduction and elimination rules. In what follows, we will present the rules for existential and universal quantification, which are most relevant to our purpose.[25] It will turn out that implication and conjunction are definable in terms of universal and existential quantifiers.

Existential quantification. The meaning of an existential quantification $(\Sigma x : A)\, B(x)$ is given by a pair (a, b) consisting of an individual a belonging to set A and a proof b of the corresponding proposition $B(a)$. This is captured by the following introduction and elimination rules:

[23] This conception of meaning is best explained in Martin-Löf (1983/1996). Martin-Löf's approach to logic has been particularly successful for applications in computer science; see Nordström et al. (1990).

[24] See Troelstra and van Dalen (1988).

[25] See Nordström, et al. (1990) for other inference rules.

$$\frac{\begin{array}{c}[x:A]\\ \vdots\\ B(x):\mathsf{prop}\end{array}\quad a:A\quad b:B(a)}{(a,b):(\Sigma x:A)\,B(x)}\,\Sigma I$$

$$\frac{c:(\Sigma x{:}A)\,B(x)}{\pi_l(c):A}\,\Sigma El\qquad\qquad \frac{c:(\Sigma x{:}A)\,B(x)}{\pi_r(c):B(\pi_l(c))}\,\Sigma Er$$

Here π_l and π_r denote the left and right projections, respectively. This is captured by the following computation rules:

$$\pi_l(a,b)\longrightarrow a\qquad \pi_r(a,b)\longrightarrow b.$$

In addition to introduction, elimination and computation rules, we have a formation rule for existential quantification:

$$\frac{A:\mathsf{prop}\quad \begin{array}{c}[x:A]\\ \vdots\\ B(x):\mathsf{prop}\end{array}}{(\Sigma x:A)B(x):\mathsf{prop}}\,\Sigma F.$$

Universal quantification. The meaning of a universal quantification $(\Pi x{:}A)\,B(x)$ is given by a function that maps any entity x in A to a corresponding proof of $B(x)$. The introduction and elimination rules for universal quantification are

$$\frac{\begin{array}{c}[x:A]\\ \vdots\\ b(x){:}B(x)\end{array}}{\lambda x b(x):(\Pi x{:}A)\,B(x)}\,\Pi I,\qquad \frac{b:(\Pi x{:}A)\,B(x)\qquad a:A}{\mathrm{app}(b,a):B(a)}\,\Pi E.$$

The corresponding computation rule is

$$\mathrm{app}(\lambda x b(x),a)\longrightarrow b(a).$$

The formation rule for universal quantification is

$$\frac{A:\mathsf{prop}\quad \begin{array}{c}[x:A]\\ \vdots\\ B(x):\mathsf{prop}\end{array}}{(\Pi x{:}A)\,B(x):\mathsf{prop}}\,\Pi F.$$

Conjunction is defined in terms of existential quantification, and implication is defined in terms of universal quantification. That is, we have

$$A \wedge B \equiv (\Sigma x : A)\, B$$
$$A \rightarrow B \equiv (\Pi x : A)\, B,$$

provided B does not depend on x.

As an illustration of a derivation in CTT, let us consider some crucial examples that led to the development of discourse semantics as we have seen in the previous section.

Example 4.4 (Cross-sentential anaphora) As a case of cross-sentential anaphora, we consider the following:

(107) a. A man walks. He whistles. [= (45a) at page 182]

b. $(\Sigma x : (\Sigma y : \mathsf{ind})(\mathsf{man}(y) \wedge \mathsf{walk}(y)))\, \mathsf{whistle}(\pi_l(x))$

Here (107a) is represented in CTT as (107b), where the first sentence of (107a) corresponds to $(\Sigma y : \mathsf{ind})(\mathsf{man}(y) \wedge \mathsf{walk}(y))$, the second sentence corresponds to $\mathsf{whistle}(\pi_l(x))$, and these two are connected by existential quantifier Σx.[26] Here the variable x ranges over proofs of the proposition $(\Sigma y : \mathsf{ind})(\mathsf{man}(y) \wedge \mathsf{walk}(y))$, which consists of a pair (a, b) of an individual a and a proof b of the proposition that a is a student who arrived. The existential quantifier Σx passes such a proof to the subsequent sentence. In this case, the left element of x, i.e., $\pi_l(x)$, is a student, and it is asserted that this student has registered. This nicely captures the fact that existential quantification is externally dynamic in the sense of Groenendijk and Stokhof (1991).

[26] An alternative way of translating sentences like (107a), which is adopted in Ranta (1994), is the following:

$$(\Sigma x : (\Sigma y : \mathsf{man})\, \mathsf{walk}(y))\, \mathsf{registered}(\pi_l(x)).$$

In this analysis, the common noun *man* is translated into a set term man, while in (107b) it is treated as a propositional function $\mathsf{man}(y)$. For our purpose, we pursue the analysis in (107b), since it is in accordance with the analysis in ILε. A detailed comparison of the two approaches is left for future work.

On the other hand, in the case of sentences like (108a), there is no way to link the pronoun *he* to the antecedent discourse.

(108) Every man walks. # He whistles.

The first sentence can be translated as $(\Pi x\!:\!I)(\mathsf{man}(x) \to \mathsf{walk}(x))$, but what this expression introduces is a function, which does not serve as a suitable antecedent of the pronoun *he* in the second sentence.

Example 4.5 (Donkey anaphora) Donkey sentences are naturally represented in CTT.

(109) a. Every farmer who owns a donkey beats it.

b. $(\Pi x\!:\!I)(\Pi z\!:\!\mathsf{farmer}(x) \wedge (\Sigma y\!:\!I)(\mathsf{donkey}(y) \wedge \mathsf{own}(x,y)))$
$\mathsf{beat}(x, \pi_l(\pi_r(z))))$

(110) a. If a farmer owns a donkey, he beats it.

b. $(\Pi x\!:\!(\Sigma y\!:\!I)(\mathsf{farmer}(y) \wedge (\Sigma z\!:\!I)(\mathsf{donkey}(z) \wedge \mathsf{own}(y,z)))$
$\mathsf{beat}(\pi_l(x), \pi_l(\pi_r(x)))$

The treatment of donkey anaphora in CTT was first observed by Martin-Löf and Sundholm (1986, 1989) and extensively developed by Ranta (1994).[27] A natural question to be raised here is how to translate natural language sentences into formal expressions in CTT in a systematic way. In particular, it must be explained how to translate *it* in (109a) as $\pi_l(\pi_r(z))$ and *he* and *it* in (110a) as $\pi_l(x)$ and $\pi_l(\pi_r(x))$, respectively. Although our intuition suggests that these are correct interpretations, we need a systematic method to map natural language sentences into corresponding formal representations. As mentioned at the beginning of this subsection, we will solve this problem using ILε as an intermediate language in the translation procedure. The rest of this subsection is devoted to explaining how to translate ILε to CTT.

The translation from ILε to CTT is an extension of the well-known translation of intuitionistic (Heyting) first-order logic to CTT, which was originally provided by Martin-Löf (1975). See also Nordström et al. (1990) and Ranta (1994).

[27] The analysis of donkey sentences here is different from that by Ranta (1994) for the same reason as stated in footnote 26.

Proposition 4.6 (Translation from $\mathsf{IL}\varepsilon$ to CTT) *Let $\mathcal{D}$ be a derivation of A* true *in $\mathsf{IL}\varepsilon$ from a set of assumptions Γ, and let $\mathcal{D}^*$ be a derivation of A :* prop *in $\mathsf{IL}\varepsilon$ from Γ. Then we have:*

1. *$\mathcal{D}$ can be translated into a derivation of $p : A'$ in CTT from Γ' for some p;*
2. *$\mathcal{D}^*$ can be translated into a derivation of A' :* prop *in CTT from Γ'.*

The proof is by induction on the complexity of a derivation $\mathcal{D}$ of A in $\mathsf{IL}\varepsilon$. As will be seen below, it is important to recognize that when an formula A of $\mathsf{IL}\varepsilon$ is translated into a formula A' in CTT, A' is determined by a derivation of A : prop in $\mathsf{IL}\varepsilon$, not by the formula A itself. The translation of the inference rules of $\mathsf{IL}\varepsilon$ basically follows the patterns described in Martin-Löf (1975), and a sketch is found in Carlström (2005). Here we will see how to translate into derivations in CTT the formation rules of $\mathsf{IL}\varepsilon$ that are relevant to our later discussion.

We fix a set ind, which corresponds to the domain of individuals indicated by **ind** in $\mathsf{IL}\varepsilon$. Given the identification of propositions and sets, we assume a judgement ind : prop as an axiom in CTT.

When a judgement J appears as an assumption, i.e., J is in Γ, we have two cases. If J is of the form x : ind, then it is translated simply as x : ind in CTT. If J is of the form A true, we have a derivation of A : prop in $\mathsf{IL}\varepsilon$. Suppose that this derivation is translated into A' : prop in CTT. The assumption J is then translated into $p : A'$, where p is a fresh variable.

The formation rules are now translated as follows.

1. cF. For each individual constant c in $\mathsf{IL}\varepsilon$, we need an individual in ind, which we write as c : ind in CTT, using the same symbol as in $\mathsf{IL}\varepsilon$.

2. εF. The rule εF is translated into the rule ΣEl.

$$\frac{\exists x A(x)}{\varepsilon x A(x) : \mathsf{ind}}\,\varepsilon F \qquad \rightsquigarrow \qquad \frac{c : (\Sigma x : \mathsf{ind})\, A(x)}{\pi_l(c) : \mathsf{ind}}\,\Sigma El$$

3. PF. As stated before, each predicate symbol P corresponds to an n-ary propositional function on ind, which is introduced by a hypothetical

judgement of the form:

$$P : \mathsf{prop}\ (x_1 : \mathsf{ind}, \ldots, x_n : \mathsf{ind}).$$

The rule PF is then simulated by means of n-times applications of the substitution rule (S).

4. $\wedge F$ and $\rightarrow F$. The rules $\wedge F$ and $\rightarrow F$ are translated into the rules ΣF and ΠF, respectively.

$$\dfrac{A : \mathsf{prop} \quad \begin{matrix}[A] \\ \vdots \\ B : \mathsf{prop}\end{matrix}}{A \wedge B : \mathsf{prop}}\,\wedge F \quad \rightsquigarrow \quad \dfrac{A : \mathsf{prop} \quad \begin{matrix}[x : A] \\ \vdots \\ B(x) : \mathsf{prop}\end{matrix}}{(\Sigma x : A)B(x) : \mathsf{prop}}\,\Sigma F$$

$$\dfrac{A : \mathsf{prop} \quad \begin{matrix}[A] \\ \vdots \\ B : \mathsf{prop}\end{matrix}}{A \rightarrow B : \mathsf{prop}}\,\rightarrow F \quad \rightsquigarrow \quad \dfrac{A : \mathsf{prop} \quad \begin{matrix}[x : A] \\ \vdots \\ B(x) : \mathsf{prop}\end{matrix}}{(\Pi x : A)\, B(x) : \mathsf{prop}}\,\Pi F$$

5. εI. The rule εI is translated in terms of the rule ΣEr.

$$\dfrac{\exists x A(x)}{A(\varepsilon x A(x))}\,\varepsilon I \quad \rightsquigarrow \quad \dfrac{c : (\Sigma x : \mathsf{ind})\, A(x)}{\pi_r(c) : A(\pi_l(c))}\,\Sigma Er$$

6. The rules $\forall F$ and $\exists F$ are translated in terms of the rules $\forall F$ and $\exists F$ in an obvious way.

$$\dfrac{\begin{matrix}[x : \mathsf{ind}] \\ \vdots \\ A(x) : \mathsf{prop}\end{matrix}}{\forall x A(x) : \mathsf{prop}}\,\forall F \quad \rightsquigarrow \quad \dfrac{\mathsf{ind} : \mathsf{prop} \quad \begin{matrix}[x : \mathsf{ind}] \\ \vdots \\ A(x) : \mathsf{prop}\end{matrix}}{(\Pi x : \mathsf{ind})\, A(x) : \mathsf{prop}}\,\Pi F$$

$$\dfrac{\begin{matrix}[x : \mathsf{ind}] \\ \vdots \\ A(x) : \mathsf{prop}\end{matrix}}{\exists x A(x) : \mathsf{prop}}\,\exists F \quad \rightsquigarrow \quad \dfrac{\mathsf{ind} : \mathsf{prop} \quad \begin{matrix}[x : \mathsf{ind}] \\ \vdots \\ A(x) : \mathsf{prop}\end{matrix}}{(\Sigma x : \mathsf{ind})\, A(x) : \mathsf{prop}}\,\Sigma F$$

The crucial part is the translation of εF and εI in terms of $\exists El$ and $\exists Er$, respectively.[28] To see how it works, let us look at some examples.

Example 4.7 The derivation in $\mathsf{IL}\varepsilon$ shown in Example 4.1 is translated into the following derivation in CTT.

$$\dfrac{\mathsf{bald}(x) : \mathsf{prop}\ (x : \mathsf{ind}) \qquad \dfrac{p : (\exists x : \mathsf{ind})\,\mathsf{king}(x, \mathsf{france})}{\pi_l(p) : \mathsf{ind}}\ \exists El}{\mathsf{bald}(\pi_l(p)) : \mathsf{prop}}\ S$$

As stated before, we abbreviate such a derivation as follows.

$$\dfrac{\dfrac{p : (\exists x : \mathsf{ind})\,\mathsf{king}(x, \mathsf{france})}{\pi_l(p) : \mathsf{ind}}\ \exists El}{\mathsf{bald}(\pi_l(p)) : \mathsf{prop}}$$

As is the case in $\mathsf{IL}\varepsilon$, the assumption $p : (\exists x : \mathsf{ind})\,\mathsf{king}(x, \mathsf{france})$ must be accompanied by a derivation of $(\exists x : \mathsf{ind})\,\mathsf{king}(x, \mathsf{france}) : \mathsf{prop}$. To simplify the presentation, we omit such a subderivation throughout.

Example 4.8 $\forall x \exists y P(x, y)$ is translated into

$$(\Pi x : \mathsf{ind})(\Sigma y : \mathsf{ind})\, P(x, y),$$

and $\exists y \forall x P(x, y)$ is translated into

$$(\Sigma y : \mathsf{ind})(\Pi x : \mathsf{ind}) P(x, y).$$

The derivation shown in Example 4.2 can then be translated into a derivation in CTT in the following way.

$$\dfrac{\dfrac{\dfrac{[x : \mathsf{ind}]^2 \quad [y : \mathsf{ind}]^1}{P(x, y) : \mathsf{prop}} \qquad \dfrac{p : (\Sigma y : \mathsf{ind})(\Pi x : \mathsf{ind}) P(x, y)}{\pi_l(p) : \mathsf{ind}} \qquad \dfrac{\dfrac{p : (\Sigma y : \mathsf{ind})(\Pi x : \mathsf{ind}) P(x, y)}{\pi_r(p) : (\Pi x : \mathsf{ind}) P(x, \pi_l(p))} \quad [x : \mathsf{ind}]^2}{\mathrm{app}(\pi_r(p), x) : P(x, \pi_l(p))}}{(\pi_l(p), \mathrm{app}(\pi_r(p), x)) : (\Sigma y : \mathsf{ind})\, P(x, y)}\ \Sigma I, 1}{\lambda x(\pi_l(p), \mathrm{app}(\pi_r(p), x)) : (\Pi x : \mathsf{ind})(\Sigma y : \mathsf{ind})\, P(x, y)}\ \Pi I, 2$$

[28] This translation is described in Stenlund (1975) and Martin-Löf (1984).

5. Linguistic applications

We are now in a position to see how presupposition resolution is handled within our proof-theoretic framework. We will show that a process of presupposition resolution is naturally formulated as a process of searching for a derivation in $\mathsf{IL}\varepsilon$, and that it accounts for problems that are difficult to handle within dynamic semantics and DRT.

5.1 Presupposition resolution

From the linguistic point of view, a formula A in ε-calculus $\mathsf{IL}\varepsilon$ can be regarded as an *underspecified* representation of the content of an utterance. Such a representation is disambiguated in a derivation of $A : \mathsf{prop}$ in $\mathsf{IL}\varepsilon$. Here the process of finding a derivation of $A : \mathsf{prop}$ can be understood to be a process of presupposition resolution. The point of the translation into CTT is to make the anaphoric dependencies explicit.

We can distinguish two kinds of tasks involved in utterance interpretations: (i) the task of understanding an utterance, i.e., specifying the content of the utterance, and (ii) the task of evaluating its content, i.e., determining whether its content is true. These two kinds of tasks are naturally represented as tasks of finding derivations in $\mathsf{IL}\varepsilon$. Let A be a formula of $\mathsf{IL}\varepsilon$ that formalizes a given sentence S. The basic ideas are then summarized as follows.

1. A task of specifying the content of an utterance of S in a context (through presupposition resolution) is taken to be a task of searching for a derivation of a judgement $A : \mathsf{prop}$.

2. What is presupposed by an utterance of S is taken to be open assumptions in a derivation of $A : \mathsf{prop}$.
3. What is asserted by an utterance of S is taken to be a judgement A, and a task of evaluating the content of an utterance of S is taken as a task to search for a derivation of A.

In $\mathsf{IL}\varepsilon$, presuppositions are triggered by the occurrences of ε-terms. When a formula A contains ε-terms, a derivation of $A : \mathsf{prop}$ depends upon a derivation of some truth-judgement of the form $\exists xB$ true. This means that a task of specifying the content of the utterance may involve a task of evaluating and verifying the content of an existential sentence.

A task of specifying and evaluating the content of an utterance is done in a particular context. Such a context can be formally represented by a sequence of judgements, which work as assumptions in a derivation. In procedural terms, the hearer's task of understanding an utterance of a sentence S in a context Γ can be taken as a process of searching for a derivation of $A : \mathsf{prop}$ in the bottom-up way under the context Γ, where the process starts with the judgement $A : \mathsf{prop}$, and stops when all the assumptions upon which $A : \mathsf{prop}$ depends are closed or contained in Γ.

To see how this conception works, we begin with typical examples of presupposition projection that we have discussed in earlier sections.

Example 5.1 The following is a typical case in which presupposition projection is blocked.

(111) If John has a wife, John's wife is happy.

This sentence is translated into the following formula of $\mathsf{IL}\varepsilon$ by way of a certain semantic decoding procedure.[1]

(112) $\exists x\, \mathsf{wife}(x, \mathsf{john}) \to \mathsf{happy}(\varepsilon x\, \mathsf{wife}(x, \mathsf{john}))$

The definite description *John's wife* corresponds to $\varepsilon x\, \mathsf{wife}(x, \mathsf{john})$, where $\mathsf{wife}(x, y)$ is to be read as "x is a wife of y." At this stage of translation,

[1]Since such a decoding procedure into $\mathsf{IL}\varepsilon$ requires no special treatments, we are not concerned with how to formulate it throughout our discussion.

the anaphoric link between *John's wife* and the individual introduced in the antecedent is not yet established. Then the formula in (112) can be proved to express a proposition from no assumptions in the following way.

(113)

$$\dfrac{\dfrac{\dfrac{[x:\mathsf{ind}]^1 \qquad \dfrac{}{\mathsf{john}:\mathsf{ind}}\,\mathsf{john}F}{\mathsf{wife}(x,\mathsf{john}):\mathsf{prop}}\,\mathsf{wife}F}{\exists x\,\mathsf{wife}(x,\mathsf{john}):\mathsf{prop}}\,\exists F,\ 1 \qquad \dfrac{\dfrac{[\exists x\,\mathsf{wife}(x,\mathsf{john})]^2}{\varepsilon x\,\mathsf{wife}(x,\mathsf{john}):\mathsf{ind}}\,\varepsilon F}{\mathsf{happy}(\varepsilon x\,\mathsf{wife}(x,\mathsf{john})):\mathsf{prop}}\,\mathsf{happy}F}{\exists x\,\mathsf{wife}(x,\mathsf{john}) \to \mathsf{happy}(\varepsilon x\,\mathsf{wife}(x,\mathsf{john})):\mathsf{prop}}\,{\to}F,\ 2$$

This means that the sentence in (111) as a whole does not have presuppositions. This prediction is correct.[2] Interestingly, the antecedent of *John's wife* is determined in this derivation. By way of translating the derivation into one in CTT, as shown below, we obtain the desired formula of CTT, where the anaphoric link between *John's wife* and its antecedent is made explicit.

(114)

$$\dfrac{\dfrac{\dfrac{[x:\mathsf{ind}]^1 \qquad \dfrac{}{\mathsf{john}:\mathsf{ind}}}{\mathsf{wife}(x,\mathsf{john}):\mathsf{prop}}}{(\Sigma x:\mathsf{ind})\,\mathsf{wife}(x,\mathsf{john}):\mathsf{prop}}\,\exists F,\ 1 \qquad \dfrac{\dfrac{[y:(\Sigma x:\mathsf{ind})\,\mathsf{wife}(x,\mathsf{john})]^2}{\pi_l(y):\mathsf{ind}}\,\Sigma El}{\mathsf{happy}(\pi_l(y)):\mathsf{prop}}}{(\Pi y:(\Sigma x:\mathsf{ind})\,\mathsf{wife}(x,\mathsf{john}))\,\mathsf{happy}(\pi_l(y)):\mathsf{prop}}\,\Pi F,\ 2$$

The overall process of mapping sentences into CTT-formulas is summarized in Figure 5.1. Here the overall process is divided into deterministic and non-deterministic parts. Descriptions are uniformly translated as ε-terms. A different derivation of a judgement of the form $A:\mathsf{prop}$ in ILε then leads to a different translation. More specifically, the outcome of the translation is determined by how to derive a truth-judgement of the form $\exists xB$ that is triggered by a judgement of the form $\varepsilon xB:\mathsf{ind}$. We can see that a process of resolving presuppositions is realized *internally* as a process of constructing a formal derivation.

[2]Alternatively we may count the axiom $\mathsf{john}:\mathsf{ind}$ as a presupposition (cf. Mineshima 2008). This is in agreement with Frege's (1892) famous analysis of presuppositions, according to which (111) presupposes that the proper name *John* has a reference. Since the presuppositions of proper names are not our main concern here, we do not discuss this issue any further here.

$$\begin{array}{l}
\text{SENTENCE: If John has a wife, John's wife is happy}\\
\quad\downarrow \text{ semantic decoding: deterministic}\\
\text{IL}\varepsilon\text{-FORMULA: } \exists x\, \mathsf{wife}(x, \mathsf{john}) \rightarrow \mathsf{happy}(\varepsilon x\, \mathsf{wife}(x, \mathsf{john}))\\
\quad\downarrow \text{ presupposition resolution : non-deterministic}\\
\text{IL}\varepsilon\text{-DERIVATION of } \exists x\, \mathsf{wife}(x, \mathsf{john}) \rightarrow \mathsf{happy}(\varepsilon x\, \mathsf{wife}(x, \mathsf{john})) : \mathsf{prop}\\
\quad\downarrow \text{ translation into } \mathsf{CTT}\text{: deterministic}\\
\text{CTT-FORMULA: } (\Pi y : (\Sigma x : \mathsf{ind})\, \mathsf{wife}(x, \mathsf{john}))\, \mathsf{happy}(\pi_l(y))
\end{array}$$

Fig. 5.1 Translating sentence (111) into CTT via ILε

By inspection of the rule $\wedge F$ in ILε, it is clear that we can obtain a similar derivation for sentences containing a conjunction like (115).

(115) John has a wife and John's wife is happy.

This sentence is represented in ILε as shown in (116a), and then translated into CTT as shown in (116b).

(116) a. $\exists x\, \mathsf{wife}(x, \mathsf{john}) \wedge \mathsf{happy}(\varepsilon y\, \mathsf{wife}(x, \mathsf{john}))$

b. $(\Sigma y : (\Sigma x : \mathsf{ind})\, \mathsf{wife}(x, \mathsf{john}))\, \mathsf{happy}(\pi_l(y))$

Example 5.2 As a typical case in which presupposition resolution interacts with reasoning about implicit knowledge (cf. Problem 4 of Section 3.3.3), consider the following example:

(117) If John is married, John's wife is happy.

A derivation in which the existential presupposition triggered by *John's wife* does not project must look as follows:

(118)

$$\dfrac{\dfrac{\dfrac{}{\mathsf{john} : \mathsf{ind}}}{\mathsf{married}(\mathsf{john}) : \mathsf{prop}} \qquad \dfrac{\dfrac{\begin{array}{c}[\mathsf{married}(\mathsf{john})]^1\\ \vdots\ \mathcal{D}\\ \exists x\, \mathsf{wife}(x, \mathsf{john})\end{array}}{\varepsilon x\, \mathsf{wife}(x, \mathsf{john}) : \mathsf{ind}}\ \varepsilon F}{\mathsf{happy}(\varepsilon x\, \mathsf{wife}(x, \mathsf{john})) : \mathsf{prop}}}{\mathsf{married}(\mathsf{john}) \rightarrow \mathsf{happy}(\varepsilon x\, \mathsf{wife}(x, \mathsf{john})) : \mathsf{prop}}\ \rightarrow F,\ 1$$

Hence, at the initial stage of the interpretation, the utterance of (117) is taken to presuppose that there is a derivation $\mathcal{D}$ from the judgement

$$\mathsf{married}(\mathsf{john})$$

to the judgement

$$\exists x\, \mathsf{wife}(x, \mathsf{john}).$$

That is to say that the hearer is required to perform an inference to fill the gap $\mathcal{D}$ in this derivation. In this case, the hearer may rely on the lexical knowledge concerning the predicate married that licenses us to infer that for any individual x, if x is married then x has a wife. We can formulate such a lexical entry in the form of an inference rule. More specifically, we can use the following kind of elimination rule for the predicate married.

$$\frac{\mathsf{married}(y) \quad y : \mathsf{ind}}{\exists x\, \mathsf{wife}(x, y)}\ \mathsf{married}E$$

Let us assume that the hearer's lexical and encyclopedic knowledge is formally represented as a set of such inference rules.[3] Now with the help of this inference rule, we can obtain a closed derivation of the following form:

(119)

$$\frac{\dfrac{\dfrac{}{\mathsf{john} : \mathsf{ind}}}{\mathsf{married}(\mathsf{john}) : \mathsf{prop}} \quad \dfrac{\dfrac{\dfrac{[\mathsf{married}(\mathsf{john})]^1 \quad \dfrac{}{\mathsf{john} : \mathsf{ind}}}{\exists x\, \mathsf{wife}(x, \mathsf{john})}\ \mathsf{married}E}{\varepsilon x\, \mathsf{wife}(x, \mathsf{john}) : \mathsf{ind}}\ \varepsilon F}{\mathsf{happy}(\varepsilon x\, \mathsf{wife}(x, \mathsf{john})) : \mathsf{prop}}}{\mathsf{married}(\mathsf{john}) \rightarrow \mathsf{happy}(\varepsilon x\, \mathsf{wife}(x, \mathsf{john})) : \mathsf{prop}}\ {\rightarrow}F,\ 1$$

This means that with the help of a suitable assumption about the hearer's implicit (lexical) knowledge, the utterance of (117) is predicted to presuppose nothing. This prediction is correct, as we have seen earlier.

[3] We do not enter into a debate concerning the nature of such knowledge; see Sperber and Wilson (1986/95) for a proponent of the view that lexical entries are to be formulated as a kind of elimination rule. Instead of inference rules, we may also adopt an axiom of the form $\forall y\, (\mathsf{married}(y) \rightarrow \exists x\, \mathsf{wife}(x, y))$. It is clear that such an axiom is derivable from the corresponding inference rule, and vice versa. However, we prefer the current approach since it simplifies the presentation of a derivation.

The treatment of the cases like (117) shows an essential difference between the proof-theoretic approach adopted here and the model-theoretic approach of DRT. As we saw in Section 3.3.3, a problem inherent to DRT is that the standard analysis of presuppositions within DRT does not involve reasoning about semantic representations like DRSs, and thus, the process of ordinary inference and the process of presupposition resolution must be treated at different levels of the system; i.e., the levels of meta-language and object-language (cf. Problem 5 of Section 3.3.3). This separation would make it difficult to handle examples such as (117) in DRT, as many writers have pointed out (e.g., Beaver 2001). In our proof-theoretic framework, by contrast, both processes can be represented at the same level by making explicit the distinction between propositions and judgements, and the interaction between the two processes can be captured in a formal derivation.

Let us see how to handle within our framework some other problems that we raised for dynamic semantics and DRT in the last section, specifically, the proviso problem (Section 3.2.2), the problem of conditional presupposition (Problem 2 of Section 3.3.3), and the problem of semi-conditional presupposition (Problem 4 of Section 3.3.3). These problems arise from constructions that can be schematically represented as follows.

(120) a. If there is a P, then the P is Q.
$\exists x P(x) \rightarrow Q(\varepsilon x P(x))$
b. If A, then the P is Q.
$A \rightarrow Q(\varepsilon x P(x))$
c. If A and B, then the P is Q.
$A \wedge B \rightarrow Q(\varepsilon x P(x))$

Both dynamic semantics and DRT can make correct predictions for (120a). A problem arises for sentences of the form in (120b). Sentences of this form can give rise to conditional or unconditional presuppositions:

(i) **Conditional presupposition**: If A then there exists a P.
$A \rightarrow \exists x P(x)$

(ii) **Unconditional presupposition**: There exists a P.
$\exists x P(x)$

For instance, sentences like *If John is married, his wife is happy* gives rise to conditional inferences, while sentences like *If John works hard, his wife is happy* gives rise to unconditional ones.[4] As we saw in Section 3.3.3, the conditional inference in (i) can be correctly handled by dynamic semantics but not by DRT. This is the problem of conditional presuppositions (Problem 2 of Section 3.3.3). On the other hand, the unconditional inference in (ii) can be correctly handled by DRT but not by dynamic semantics. This is the proviso problem discussed in Section 3.2.2. Finally, both dynamic semantics and DRT have difficulty in accounting for cases like (120c), where the presupposition triggered by the consequent may depend upon one of the conjuncts of the antecedent, so that it is presupposed that $A \to \exists x P(x)$, not that $A \wedge B \to \exists x P(x)$. This is the problem of semi-conditional presupposition (Problem 5 of Section 3.3.3).

Our proof-theoretic framework can make correct predictions for all the cases. To see this, it is crucial to understand how the formation rules of ILε for conjunction and implication, repeated here, work for these cases.

$$\dfrac{A : \mathsf{prop} \quad \begin{array}{c}[A]\\ \vdots \\ B : \mathsf{prop}\end{array}}{A \to B : \mathsf{prop}} \to F \qquad \dfrac{A : \mathsf{prop} \quad \begin{array}{c}[A]\\ \vdots \\ B : \mathsf{prop}\end{array}}{A \wedge B : \mathsf{prop}} \wedge F$$

In applications of the rules $\to F$ and $\wedge F$, the judgement $B : \mathsf{prop}$ may or may not depend upon the assumption A.[5] In the case of sentences of the form (120b), when A is irrelevant to deriving $\exists x P(x) : \mathsf{prop}$, the derivation is as seen on the left below, which corresponds to the case of unconditional

[4] In the case of *If John is married, his wife is happy*, the conditional inference that *If John is married, he has a wife* can be derived from the hearer's constant knowledge. In such a case, the inference would not be felt as being presupposed. On the other hand, in the case of the examples discussed in Problem 2 of Section 3.3.3, the conditional inferences in question cannot be derived from constant knowledge. Consequently, the inferences would be felt as being presupposed in the sense that the hearer must accept them to be true to make sense of the utterance.

[5] It should be noted that the *weakening* rule, which is implicit in our presentation of ILε but can be explicitly handled in a sequent-style system, is applied when A does not appear in a derivation of $B : \mathsf{prop}$.

presuppositions. On the other hand, when A is relevant as in Example 5.2, the derivation is as on the right, which accounts for the case of conditional presuppositions.

$$\dfrac{\begin{array}{c}\vdots\\ A : \mathsf{prop}\end{array} \quad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \exists x P(x)\end{array}}{\varepsilon x P(x) : \mathsf{ind}}\,\varepsilon F}{Q(\varepsilon x P(x)) : \mathsf{prop}}}{A \to Q(\varepsilon x P(x))}\,{\to}F \qquad \dfrac{\begin{array}{c}\vdots\\ A : \mathsf{prop}\end{array} \quad \dfrac{\dfrac{\dfrac{[A]^1 \quad \begin{array}{c}\vdots\\ A \to \exists x P(x)\end{array}}{\exists x P(x)}}{\varepsilon x P(x) : \mathsf{ind}}\,\varepsilon F}{Q(\varepsilon x P(x)) : \mathsf{prop}}}{A \to Q(\varepsilon x P(x))}\,{\to}F,\ 1$$

Similarly, in the case of (120c), where the antecedent of a conditional consists of $A \wedge B$, there is no need to use both A and B in deriving $\exists x P(x)$.

It should be emphasized that there is an essential difference between the proof-theoretic approach and dynamic semantics. Although there is a certain similarity between the inference rules like $\to F$ and $\wedge F$ and the definition of context change potentials for implication and conjunction in dynamic semantics (cf. Fernando 2001), these two theories make different predictions. To be concrete, consider formulas like $p \to (q \mathbin{\backslash\!\backslash} r)$ in DS. What is presupposed by this formula is that the initial information state σ entails the implication $p \to q$; consequently, some additional mechanism that can derive q from $p \to q$ is needed to account for the case of unconditional presuppositions. On the other hand, our proof-theoretic approach predicts that what is presupposed by examples like (120b) is that there is a derivation of $\exists x P(x)$ from an initial set of assumptions Γ together with an assumption that A is true, where it is understood that not all the assumptions need to be used in the derivation. The predictions made by dynamic semantics lack this flexibility.

A question remains as to how those assumptions that are relevant to deriving the presuppositions are identified. To answer this kind of question, we need an additional theory. Obviously, we need a pragmatic theory of inferences that accounts for questions as to how hearers draw inferences from the speaker's utterance, what direction they take their inferences, and when they stop. Such a theory would be largely driven by considerations external to formal semantics. The notion of relevance that plays a role in identifying assumptions in presupposition resolution should be considered from such a general perspective of the role of inferences in communication. Our goal here was to describe what the semantic representations of sentences contribute to the presupposition resolution. It is not unreasonable to expect that our proof-theoretic or deductive perspective could naturally connect to a pragmatic theory of utterance interpretation in which inferences play a dominant role.[6]

5.2 Pronominal anaphora

In Section 3.3.2, we saw that there are striking similarities between anaphora resolution and presupposition projection, and that this parallel can be nicely captured within the framework of DRT (van der Sandt 1992; Geurts 1999) by treating anaphora resolution as a special case of presupposition resolution. Now if we take it that a pronoun refers to an individual satisfying some condition associated with its gender information, we can formally represent a pronoun as an ε-term, in the same way as definite descriptions, and then carry over the presuppositional analysis of pronominal anaphora within DRT into our proof-theoretic framework. For our purpose, we will assume that *he* is represented as $\varepsilon x\,\mathsf{male}(x)$, *she* as $\varepsilon x\,\mathsf{female}(x)$, and *it* as $\varepsilon x\,\mathsf{non\text{-}human}(x)$, and so on. We also need to make use of inference rules (elimination rules) like the following.[7]

$$\frac{\mathsf{man}(x) \quad x : \mathsf{ind}}{\mathsf{male}(x)}\,\mathsf{man}E \qquad \frac{\mathsf{donkey}(x) \quad x : \mathsf{ind}}{\mathsf{non\text{-}human}(x)}\,\mathsf{donkey}E$$

Such lexical information is implicitly assumed in the standard presentation of DRT, as we have seen in Section 3.3.3. In our framework, these infer-

[6] A well-developed pragmatic theory of this kind is Relevance Theory (Sperber and Wilson 1986/95). Chapter 3 is devoted to developing a relevance-theoretic approach to a certain class of inferences in utterance interpretation.

[7] Although we cannot go into details here, such a treatment of lexical information as formulated by elimination rules is found in Relevance Theory (Sperver and Wilson 1986/95).

ence rules will be explicitly used to provide a bridge between a pronominal element and its antecedent.

Example 5.3 (Cross-sentential anaphora) Let us consider the example we considered in Section 4.3.

(107) A man walks. He whistles.

This sequence of sentences is first translated into the following formula of $\mathsf{IL}\varepsilon$.

(121) $\exists x(\mathsf{man}(x) \wedge \mathsf{walk}(x)) \wedge \mathsf{whistle}(\varepsilon x\, \mathsf{male}(x))$

Here the pronoun *he* corresponds to $\varepsilon x\, \mathsf{male}(x)$. Note that the choice of variable x in this ε-term is arbitrary, and that at this stage of translation the "antecedent" of $\varepsilon x\, \mathsf{male}(x)$ is not yet determined. A derivation of the judgement that (121) expresses a proposition can be obtained in a similar way to the derivation for (117) that we have seen before. Abbreviating $\varepsilon x(\mathsf{man}(x) \wedge \mathsf{walk}(x))$ as t, the relevant part of the derivation is as follows:

(122)

$$\dfrac{\begin{matrix}\vdots\\ \exists x(\mathsf{man}(x)\wedge\mathsf{walk}(x)) : \mathsf{prop}\end{matrix} \qquad \dfrac{\dfrac{\dfrac{\dfrac{[x : \mathsf{ind}]}{\mathsf{male}(x) : \mathsf{prop}} \quad \dfrac{[\exists x(\mathsf{man}(x)\wedge\mathsf{walk}(x))]^1}{t : \mathsf{ind}}\,\varepsilon F \quad \dfrac{\dfrac{\dfrac{[\exists x(\mathsf{man}(x)\wedge\mathsf{walk}(x))]^1}{\mathsf{man}(t)\wedge\mathsf{walk}(t)}\,\varepsilon I}{\mathsf{man}(t)}\,\wedge E}{\mathsf{male}(t)}\,\mathsf{man}E}{\exists x\, \mathsf{male}(x)}\,\exists I}{\varepsilon x\, \mathsf{male}(x) : \mathsf{ind}}\,\varepsilon F}{\mathsf{whistle}(\varepsilon x\, \mathsf{male}(x)) : \mathsf{prop}}\,\mathsf{whistle}F}{\exists x(\mathsf{man}(x)\wedge\mathsf{walk}(x)) \wedge \mathsf{whistle}(\varepsilon x\, \mathsf{male}(x)) : \mathsf{prop}}\,\wedge F,\ 1$$

It is easily checked that this derivation can be translated into a derivation of the following judgement in CTT.

(123) $(\Sigma x : (\Sigma y : \mathsf{ind})(\mathsf{man}(y) \wedge \mathsf{walk}(y)))\, \mathsf{whistle}(\pi_l(x)) : \mathsf{prop}$

This result corresponds to the desired translation of the original sentence into a CTT-formula as shown in (107b).

Example 5.4 (Donkey anaphora) An analysis of donkey anaphora within our framework can be given in a similar way to that of cross-sentential anaphora that we have just seen above. The donkey sentence in (124a) is represented as (124b).

(124) a. If a man owns a donkey, he beats it.

b. $\exists x\,(\mathsf{man}(x) \wedge \exists y\,(\mathsf{donkey}(y) \wedge \mathsf{own}(x, y)))$
$\to \mathsf{beat}(\varepsilon z\,\mathsf{male}(z), \varepsilon w\,\mathsf{non\text{-}human}(w))$

We can then derive the judgement that (124*b*) expresses a proposition with the help of the lexical inference rules $\mathsf{man}\,E$ and $\mathsf{donkey}\,E$ as shown before. We omit the derivation since it is similar to that for (121). Then again, by translating the $\mathsf{IL}\varepsilon$-derivation into a CTT-derivation, we can obtain the following correct translation in CTT (cf. (109b) on page 234).

(125) $(\Pi x : (\Sigma y : \mathsf{ind})(\mathsf{man}(y) \wedge (\Sigma z : \mathsf{ind})(\mathsf{donkey}(z) \wedge \mathsf{own}(y, z)))$
$\mathsf{beat}(\pi_l(x), \pi_l(\pi_r(x)))$

Similarly, (126a) is represented as (126b) in $\mathsf{IL}\varepsilon$, and proved to express a proposition; then the translation into CTT yields the translation in (126c), which coincides with the desired translation in (109a).

(126) a. Every man who owns a donkey beats it.

b. $\forall x\,(\mathsf{man}(x) \wedge \exists y\,(\mathsf{donkey}(y) \wedge \mathsf{own}(x, y)))$
$\wedge\,\mathsf{beat}(x, \varepsilon z\,\mathsf{non\text{-}human}(z))$

c. $(\Pi x : \mathsf{ind})(\Pi z : \mathsf{man}(x) \wedge (\Sigma y : \mathsf{ind})(\mathsf{donkey}(y) \wedge \mathsf{own}(x, y)))$
$\mathsf{beat}(x, \pi_l(\pi_r(z))))$

5.3 Embedded descriptions

Special consideration is required for cases in which one definite description is embedded within another, as in (127).

(127) John's aunt's cousin disappeared.

This sentence presupposes that John has an aunt and that John's aunt has a cousin. To account for this fact, we will translate the description *John's aunt's cousin* as a whole into the following.

(128) $\varepsilon x\,(\exists y\,\mathsf{aunt}(y, \mathsf{john}) \wedge \mathsf{cousin}(x, y))$

Here both *aunt* and *cousin* are analyzed as relational nouns. The crucial point is that the outmost description *John's aunt's cousin* corresponds to an ε-term, whereas the embedded description *John's aunt* corresponds to an existentially quantified expression, rather than an ε-term like $\varepsilon y\,\mathsf{aunt}(y, \mathsf{john})$.[8] Now (127) is represented as follows.

(129) $\mathsf{disappeared}(\varepsilon x(\exists y\,\mathsf{aunt}(y, \mathsf{john}) \wedge \mathsf{cousin}(x, y))$

We then have the following derivation, which yields the desired result.

(130)

$$\dfrac{\dfrac{\exists x(\exists y\,\mathsf{aunt}(y, \mathsf{john}) \wedge \mathsf{cousin}(x, y))}{\varepsilon x(\exists y\,\mathsf{aunt}(y, \mathsf{john}) \wedge \mathsf{cousin}(x, y)) : \mathsf{ind}}\,\varepsilon F}{\mathsf{disappeared}(\varepsilon x(\exists y\,\mathsf{aunt}(y, \mathsf{john}) \wedge \mathsf{cousin}(x, y)) : \mathsf{prop}}\,\mathsf{disappeared}F$$

Let us consider how to deal with a more complex example involving a pronoun. We consider the following example:

(131) His aunt's cousin disappeared.

This sentence is translated as follows.

(132) $\mathsf{disappeared}(\varepsilon x\,(\exists y\,\mathsf{male}(y) \wedge (\exists z\,\mathsf{aunt}(z, y) \wedge \mathsf{cousin}(x, z))))$

Here the embedded pronoun and description, *his* and *his aunt*, are translated as existentially quantified expressions. In this analysis, it can be shown that (132) presupposes the following judgement:

(133) $\exists z\,(\exists y\,(\exists x\,\mathsf{cousin}(x, y) \wedge \mathsf{aunt}(y, z)) \wedge \mathsf{male}(z))$.

[8]This will cause certain complications in a procedure to translate English sentences into formulas of $\mathsf{IL}\varepsilon$ in a systematic way: we need to translate all the descriptions embedded in a description into generalized quantifiers with existential force and the outmost description into an ε-term.

5.4 Informative presuppositions

Sometimes the use of a presupposing sentence introduces novel information into a discourse, in particular when the presupposed information is uncontroversial. As we saw in Section 3.2.3, such a case of informative presupposition has been analyzed in terms of a mechanism called *accommodation*, since the seminal works of Lewis (1979) and Heim (1983). To deal with informative presuppositions within our proof-theoretic framework, we introduce the following procedure that transforms one derivation into another derivation.

(134) **Accommodation as transformations of derivations**. Given a derivation in which a judgement of the form $B : \mathsf{prop}$ depends on an open assumption of the form $A\,\mathsf{true}$, we may transform the derivation into one in which $A : \mathsf{prop}$ is located at some stage in the derivation so that the assumption $A\,\mathsf{true}$ is discharged.

We call this transformation simply *accommodation*. In our terms, accommodation is a process of adjusting the *content*, rather than the *context*, of an utterance. From the hearer's point of view, to perform accommodation is to add an implicitly understood proposition at some stage in the course of a derivation. The procedure of accommodation as stated in (134) allows us the following transformation:

$$\begin{array}{c} A \\ \vdots \\ B : \mathsf{prop} \end{array} \quad \text{is transformed into} \quad \dfrac{A : \mathsf{prop} \quad \begin{array}{c}[A] \\ \vdots \\ B : \mathsf{prop}\end{array}}{A \wedge B : \mathsf{prop}} \wedge F$$

Using this transformation, we can simulate within our framework the processes of global and local accommodation that are adopted in standard dynamic theories of presuppositions, including Heim's (1983) dynamic semantics (cf. Section 3.2.3) and van der Sandt's (1992) DRT (cf. Section 3.3.2).

Example 5.5 (Accommodation) Let us illustrate with the classical example in (135) how the transformation works.

(135) The king of France is not bald.

This sentence is translated into an $\mathsf{IL}\varepsilon$-formula as follows.

(136) $\neg\,\mathsf{bald}(\varepsilon x\,\mathsf{king}(x, \mathsf{france}))$

We can then derive the following:

(137)

$$\cfrac{\cfrac{\cfrac{\exists x\,\mathsf{king}(x, \mathsf{france})}{\varepsilon x\,\mathsf{king}(x, \mathsf{france}) : \mathsf{ind}}\ \varepsilon F}{\mathsf{bald}(\varepsilon x\,\mathsf{king}(x, \mathsf{france})) : \mathsf{prop}}\ \mathsf{bald}F}{\neg\,\mathsf{bald}(\varepsilon x\,\mathsf{king}(x, \mathsf{france})) : \mathsf{prop}}\ \neg F$$

Given this derivation, the hearer can proceed in at least three different ways.

First, suppose that it is within the common ground that someone, say John, is the king of France. This is to say, the context Γ contains the judgement

$$\mathsf{king}\,(\mathsf{john}, \mathsf{france}).$$

In this context, the presupposition would be simply satisfied. This reading can be captured in the following derivation.

(138)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[x : \mathsf{ind}]^1 \quad \cfrac{}{\mathsf{france} : \mathsf{ind}}}{\mathsf{king}(x, \mathsf{france}) : \mathsf{prop}}\ \mathsf{king}F \qquad \cfrac{}{\mathsf{john} : \mathsf{ind}} \quad \mathsf{king}(\mathsf{john}, \mathsf{france})}{\exists x\,\mathsf{king}(x, \mathsf{france})}\ \exists F,\ 1}{\varepsilon x\,\mathsf{king}(x, \mathsf{france}) : \mathsf{ind}}\ \varepsilon F}{\mathsf{bald}(\varepsilon x\,\mathsf{king}(x, \mathsf{france})) : \mathsf{prop}}\ \mathsf{bald}F}{\neg\,\mathsf{bald}(\varepsilon x\,\mathsf{king}(x, \mathsf{france})) : \mathsf{prop}}\ \neg F$$

As before, the fact that the description *the king of France* is anaphorically linked to john is made explicit by translating the whole derivation into a derivation in CTT.

(139)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[x:\mathsf{ind}]^1 \quad \overline{\mathsf{france}:\mathsf{ind}}}{\mathsf{king}(x,\mathsf{france}):\mathsf{prop}} \quad \overline{\mathsf{john}:\mathsf{ind}} \quad p:\mathsf{king}(\mathsf{john},\mathsf{france})}{(\mathsf{john},p):(\Sigma x:\mathsf{ind})\,\mathsf{king}(x,\mathsf{france})}\ \exists F,\ 1}{\pi_l(\mathsf{john},p):\mathsf{ind}}\ \varepsilon F}{\mathsf{bald}(\pi_l(\mathsf{john},p)):\mathsf{prop}}}{\neg\,\mathsf{bald}(\pi_l(\mathsf{john},p)):\mathsf{prop}}\ \neg F$$

Here $\pi_l(\mathsf{john},p)$ computes to john by left projection. Accordingly, we can obtain the following judgement in CTT.

(140) $\neg\,\mathsf{bald}(\mathsf{john}):\mathsf{prop}$

This means that the utterance of (135) in this context could communicate that John is not bald.

The second and third options involve accommodation. Consider a context in which the hearer does not know whether or not there is a king of France. In this context, the hearer might perform accommodation, which transforms the original derivation into the following:

(141)

$$\cfrac{\cfrac{\cfrac{[x:\mathsf{ind}]^1 \quad \overline{\mathsf{france}:\mathsf{ind}}}{\mathsf{king}(x,\mathsf{france}):\mathsf{prop}}\ \mathsf{king}F}{\exists x\,\mathsf{king}(x,\mathsf{france}):\mathsf{prop}}\ \exists F,\ 1 \qquad \cfrac{\cfrac{\cfrac{[\exists x\,\mathsf{king}(x,\mathsf{france})]^2}{\varepsilon x\,\mathsf{king}(x,\mathsf{france}):\mathsf{ind}}\ \varepsilon F}{\mathsf{bald}(\varepsilon x\,\mathsf{king}(x,\mathsf{france})):\mathsf{prop}}\ \mathsf{bald}F}{\neg\,\mathsf{bald}(\varepsilon x\,\mathsf{king}(x,\mathsf{france})):\mathsf{prop}}\ \neg F}{\exists x\,\mathsf{king}(x,\mathsf{france})\wedge\neg\,\mathsf{bald}(\varepsilon x\,\mathsf{king}(x,\mathsf{france})):\mathsf{prop}}\ \wedge F,\ 2$$

The translation into CTT yields a derivation of the following judgement.

(142) $(\Sigma y:(\Sigma x:\mathsf{ind})\,\mathsf{king}(x,\mathsf{france}))\,\neg\,\mathsf{bald}\,(\pi_l(y)):\mathsf{prop}$

This corresponds to the global accommodation reading in Heim's (1983) original analysis as seen in Section 3.2.3.

The third option is this. Consider a context in which the hearer knows that there is no king of France. Given this context, to avoid a contradiction, the hearer must perform accommodation at one stage above in the original derivation. This yields the following result:

(143)

$$\cfrac{\cfrac{\cfrac{\cfrac{[x : \mathsf{ind}]^1 \quad \overline{\mathsf{france} : \mathsf{ind}}}{\mathsf{king}(x, \mathsf{france}) : \mathsf{prop}}\;\mathsf{king}F}{\exists x\, \mathsf{king}(x, \mathsf{france}) : \mathsf{prop}}\;\exists F,\ 1 \qquad \cfrac{\cfrac{[\exists x\, \mathsf{king}(x, \mathsf{france})]^2}{\varepsilon x\, \mathsf{king}(x, \mathsf{france}) : \mathsf{ind}}\;\varepsilon F}{\mathsf{bald}(\varepsilon x\, \mathsf{king}(x, \mathsf{france})) : \mathsf{prop}}\;\mathsf{bald}F}{\exists x\, \mathsf{king}(x, \mathsf{france}) \wedge \mathsf{bald}(\varepsilon x\, \mathsf{king}(x, \mathsf{france})) : \mathsf{prop}}\;\wedge F,\ 2}{\neg(\exists x\, \mathsf{king}(x, \mathsf{france}) \wedge \mathsf{bald}(\varepsilon x\, \mathsf{king}(x, \mathsf{france}))) : \mathsf{prop}}\;\neg F$$

It is easily verified that the translation in CTT gives rise to a derivation of the judgement:

(144) $\neg(\Sigma y : (\Sigma x : \mathsf{ind})\, \mathsf{king}(x, \mathsf{france}))\, \mathsf{bald}\, (\pi_l(y)) : \mathsf{prop}.$

This corresponds to the local accommodation reading in Heim's analysis.

5.5 Quantified sentences

In Section 3.2.4, we examined the presuppositions of quantified sentences. Let us see what predictions our account makes for such quantificational cases. We will concentrate on the four representative cases discussed in Section 3.2.4, and see that our account gives desired results for these cases.

Example 5.6 (Existential sentences) We start by examining an existential construction that has a similar structure to Heim's example in (57) on page 189.

(145) Some man loves his wife.

This sentence can be represented in ILε as

(146) $\exists x(\mathsf{man}(x) \wedge \mathsf{love}(x, \varepsilon y\, \mathsf{wife}(y, x))),$

where we assume, for simplicity, that the antecedent of the embedded pronoun *his* is already resolved: it is bound to the subject NP *some man.* Then a derivation of the judgement (146) : prop looks as follows.

(147)

$$
\dfrac{\dfrac{\dfrac{[x:\mathsf{ind}]^1}{\mathsf{man}(x):\mathsf{prop}} \qquad \dfrac{[x:\mathsf{ind}]^1 \quad \dfrac{\begin{array}{c}[x:\mathsf{ind}]^1\ [\mathsf{man}(x)]^2\\ \vdots \\ \exists y\,\mathsf{wife}(y,x)\end{array}}{\varepsilon y\,\mathsf{wife}(y,x):\mathsf{ind}}\,\varepsilon F}{\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)):\mathsf{prop}}}{\mathsf{man}(x)\wedge\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)):\mathsf{prop}}\,\wedge F,\ 2}{\exists x(\mathsf{man}(x)\wedge\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)))}\,\exists F,\ 1
$$

Thus, our theory predicts that (145) presupposes that there is a derivation from the judgements $x:\mathsf{ind}$ and $\mathsf{man}(x)$ to the judgement $\exists y\,\mathsf{wife}(x,y)$. This prediction is similar to the universal presuppositions that Heim's (1983) dynamic semantics predicts for existential constructions such as (57). Indeed, the required derivation can be obtained if we have a judgement

$$\forall x(\mathsf{man}(x)\to\exists y\,\mathsf{wife}(y,x))$$

as an assumption.[9] Note that by way of accommodation we may also obtain the following proposition:

(148) $\exists x\,(\mathsf{man}(x)\wedge\exists y\,(\mathsf{wife}(y,x)\wedge\mathsf{love}(x,y)))$.

Again, the results are similar to what Heim's dynamic semantics predicts for such a case with the help of local accommodation.

We may also introduce an operation analogous to intermediate accommodation, proposed by van der Sandt (1992). It allows us the following transformation.

$$
\dfrac{A:\mathsf{prop}\qquad \begin{array}{c}B\\ \vdots\\ C:\mathsf{prop}\end{array}}{A\to C:\mathsf{prop}}\,{\to}F
\quad\Longrightarrow\quad
\dfrac{\dfrac{A:\mathsf{prop}\quad B:\mathsf{prop}}{A\wedge B:\mathsf{prop}}\,\wedge F \qquad \begin{array}{c}\dfrac{[A\wedge B]}{B}\,\wedge E\\ \vdots\\ C:\mathsf{prop}\end{array}}{(A\wedge B)\to C:\mathsf{prop}}\,{\to}F
$$

[9] If we take it that the antecedent of the embedded pronoun *his* is not yet resolved, we start with a formula like $\exists x(\mathsf{man}(x)\wedge\mathsf{love}(x,\varepsilon y(\exists x(\mathsf{male}(x)\wedge\mathsf{wife}(y,x)))))$. It is then easily verified that the intended reading is captured by the derivation of $\exists x(\mathsf{man}(x)\wedge \mathsf{love}(x,\varepsilon y(\exists z(\mathsf{male}(z)\wedge\mathsf{wife}(y,z))))):\mathsf{prop}$ from $\forall x(\mathsf{man}(x)\to\exists y\,\mathsf{wife}(y,x))$ in which the witness for the existential quantifier $\exists z$ associated $\mathsf{male}(z)$ is given by the existential quantifier $\exists x$ associated with $\mathsf{man}(x)$.

This transformation can be regarded as a special case of the general procedure in (134).

Example 5.7 (The nuclear scope of universal quantification) Next we consider the case in which a description occurs in the nuclear scope of a universal quantifier.

(149) Every man loves his wife.

This sentence is formalized as

(150) $\forall x\,(\mathsf{man}(x) \rightarrow \mathsf{love}(x, \varepsilon y\,\mathsf{wife}(y, x)))$,

where we assume that the pronoun *his* is bound to the subject NP *every man* in a similar way to (146). Then by the following derivation, which is similar to the one in (147), we can predict the universal presupposition for (149). This is the reading in which the description *his wife* is bound using the assumption that every man has a wife.

(151)

$$\dfrac{\dfrac{\dfrac{[x : \mathsf{ind}]^1}{\mathsf{man}(x) : \mathsf{prop}} \qquad \dfrac{[x : \mathsf{ind}]^1 \qquad \dfrac{\begin{matrix}[x : \mathsf{ind}]^1\ [\mathsf{man}(x)]^2 \\ \vdots \\ \exists y\,\mathsf{wife}(y, x)\end{matrix}}{\varepsilon y\,\mathsf{wife}(y, x) : \mathsf{ind}}}{\mathsf{love}(x, \varepsilon y\,\mathsf{wife}(y, x)) : \mathsf{prop}}}{\mathsf{man}(x) \rightarrow \mathsf{love}(x, \varepsilon y\,\mathsf{wife}(y, x)) : \mathsf{prop}}\ {\rightarrow}F,\ 2}{\forall x(\mathsf{man}(x) \rightarrow \mathsf{love}(x, \varepsilon y\,\mathsf{wife}(y, x))) : \mathsf{prop}}\ \forall F,\ 1$$

Other interpretations are also possible. Starting with the derivation in (151) where the assumption $\exists y\,\mathsf{wife}(y, x)$ remains open, we have four possibilities for accommodation.

First, the intermediate accommodation we have just introduced yields the following derivation.

(152)

$$
\dfrac{\dfrac{\dfrac{[x:\mathsf{ind}]^1}{\mathsf{man}(x):\mathsf{prop}} \quad \dfrac{\dfrac{[y:\mathsf{ind}]^2 \quad [x:\mathsf{ind}]^1}{\mathsf{wife}(y,x):\mathsf{prop}}}{\exists y\,\mathsf{wife}(y,x):\mathsf{prop}}\,2}{\mathsf{man}(x)\wedge\exists y\,\mathsf{wife}(y,x):\mathsf{prop}} \quad \dfrac{[x:\mathsf{ind}]^1 \quad \dfrac{\dfrac{[\mathsf{man}(x)\wedge\exists y\,\mathsf{wife}(y,x)]^3}{\exists y\,\mathsf{wife}(y,x)}}{\varepsilon y\,\mathsf{wife}(y,x):\mathsf{ind}}}{\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)):\mathsf{prop}}}{\dfrac{\mathsf{man}(x)\wedge\exists y\,\mathsf{wife}(y,x)\rightarrow\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)):\mathsf{prop}}{\forall x(\mathsf{man}(x)\wedge\exists y\,\mathsf{wife}(y,x)\rightarrow\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x))):\mathsf{prop}}\ \forall F,\ 1}\ {\rightarrow}F,\ 3
$$

The resulting reading can be glossed as "Every man who has a wife loves it." Next, the local accommodation yields the following derivation.

(153)

$$
\dfrac{\dfrac{\dfrac{[x:\mathsf{ind}]^1}{\mathsf{man}(x):\mathsf{prop}} \quad \dfrac{\dfrac{\dfrac{[y:\mathsf{ind}]^2 \quad [x:\mathsf{ind}]^1}{\mathsf{wife}(y,x):\mathsf{prop}}}{\exists y\,\mathsf{wife}(y,x):\mathsf{prop}}\,2 \quad \dfrac{[x:\mathsf{ind}]^1 \quad \dfrac{[\exists y\,\mathsf{wife}(y,x)]^3}{\varepsilon y\,\mathsf{wife}(y,x):\mathsf{ind}}}{\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)):\mathsf{prop}}}{\exists y\,\mathsf{wife}(y,x)\wedge\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)):\mathsf{prop}}\ \wedge F,\ 3}{\mathsf{man}(x)\rightarrow\exists y\,\mathsf{wife}(y,x)\wedge\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)):\mathsf{prop}}\ {\rightarrow}F}{\forall x(\mathsf{man}(x)\rightarrow\exists y\,\mathsf{wife}(y,x)\wedge\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x))):\mathsf{prop}}\ \forall F,\ 1
$$

The result can be glossed as "Every man has a wife and loves it." Third, by way of global accommodation, we obtain a derivation ending with the following judgement.

(154) $\exists y\,\mathsf{wife}(y,x)\wedge\forall x(\mathsf{man}(x)\wedge\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x))):\mathsf{prop}$

This option is ruled out, since the formula contains a free variable. These three options give the same results as van der Sandt's (1992) analysis discussed in Section 3.3.2

Finally, we can perform accommodation at one stage before the application of $\forall F$ in the derivation (151). This yields a derivation ending with the following judgement.

(155) $\forall x(\exists y\,\mathsf{wife}(y,x)\wedge(\mathsf{man}(x)\supset\mathsf{love}(x,\varepsilon y\,\mathsf{wife}(y,x)))):\mathsf{prop}$

Here the information that every individual in the domain of discourse has a wife is accommodated.

Example 5.8 (The nuclear scope of *no*) It is not difficult to see that the case in which ε-terms appear in the nuclear scope of *no* is handled in the same way. Thus, the sentence in (156a) is represented as (156b), where we concentrate on the definite description *its king* and do not take into account how the pronoun *its* is bound to the subject NP *no nation*.

(156) a. No nation cherishes its king. [= (55)]

b. $\neg\exists x(\mathsf{nation}(x) \wedge \mathsf{cherish}(x, \varepsilon y\, \mathsf{king}(y, x)))$

Then, setting aside the possibilities of accommodation, we predict that for (156b) to express a proposition, there needs to be a derivation of $\exists y\, \mathsf{king}(y, x)$ from $x : \mathsf{ind}$ and nation, which amounts to saying that every nation has a king. Hence, we obtain the same prediction as Heim's dynamic semantics (cf. Example 3.25).

Finally, let us look at an example in which a description appears in the restrictor of a universal quantifier.

Example 5.9 (The restrictor of a universal quantifier) We consider the example discussed in Section 3.2.4.

(157) a. Every man who serves his king will be rewarded. [=(60)]

b. $\forall x(\mathsf{man}(x) \wedge \mathsf{serve}(x, \varepsilon y\, \mathsf{king}(y, x)) \to \mathsf{rewarded}(x))$

Again, it is easily seen that if we set aside the option of accommodation, we obtain the universal presupposition as in Heim's dynamic semantics (cf. Example 3.27).

5.6 Summary and prospects

We have shown how the proof-theoretic framework based on intuitionistic ε-calculus $\mathsf{IL}\varepsilon$ accounts for the existential presuppositions triggered by definite descriptions in natural language. In particular, we have shown how processes of presupposition resolution, more specifically, processes of presupposition projection and accommodation, are formally represented as processes of constructing and transforming derivations in a proof system. The underlying motivation for our approach is that presupposition resolution is best viewed as *inference*, the sort of inference that can be reconstructed as a formal derivation in a suitable proof system. This idea is not inconsistent with the standard conception of presuppositions in dynamic theories (dynamic semantics and DRT), according to which presuppositions are requirements placed on background contexts. But the important difference is that in our conception, what the hearers are required to do is to perform a certain inference to determine whether the conditions induced by a presupposition trigger are satisfied in a given context; and the internal structure of such an inference, which is formally represented as a proof structure, is relevant to the determination of the overall interpretation of an utterance. Note that in our conception, contexts are essentially *structured* entities in a similar way to the DRSs adapted in DRT. By contrast, in dynamic semantics, a context is identified as a set of possible worlds as in DS (or alternatively, a set of assignments as in DS_q), and thus, it is regarded as *unstructured*. As we discussed in Section 3.4, this makes it difficult for dynamic semantics to discriminate between the relevant and irrelevant parts of the antecedently given information in determining the required presupposition, and hence, to answer to the proviso problem.

We have seen that our proof-theoretic approach satisfies the requirements for a proper formal theory of presuppositions that were summarized at Section 3.4: (i) it is flexible enough to account for the fact that presuppositions may or may not depend upon the antecedently given information; (ii) it provides correct predictions for a range of quantified sentences; and (iii) it can handle the interaction between presupposed information and reasoning about implicit assumptions. Moreover, by assigning ε-terms to expressions giving rise to existential presuppositions such as descriptions and pronouns in a uniform way, we can integrate a method of internal coindexing into our proof-theoretic framework.

Finally, let us mention one possible direction to extend our approach to handle presuppositional triggers other than those that we have discussed so far. In $\mathsf{IL}\varepsilon$, presuppositions are only generated by ε-terms; more specifically, among the formation rules of $\mathsf{IL}\varepsilon$, only the rule εF is responsible for making the judgements of the form $A : \mathsf{prop}$ dependent upon truth-judgements of the form $A\ \mathsf{true}$. It is interesting to extend the system so that formation rules for expressions other than terms can generate such a dependency. For instance, if we try to incorporate into our framework the binary operator $\setminus\!\!\setminus$ as we used in formalizing dynamic semantics, we will need a formation rule for $\setminus\!\!\setminus$ in which the judgement of the form $A \setminus\!\!\setminus B : \mathsf{prop}$ depends upon the judgement of the form $A\ \mathsf{true}$. Alternatively, and more directly, we can adopt a system with *partial predicates* by making formation rules for predicates dependent on truth-judgements. This would be a simple method to handle presuppositions induced by aspectual verbs such as *stop*. Thus, an application of the formation rule for *stop* would look as follows.

(158)

$$\frac{x : \mathsf{ind} \quad \mathsf{used\text{-}to\text{-}do}(x, \mathsf{smoking})}{\mathsf{stop}(x, \mathsf{smoking}) : \mathsf{prop}}\ \mathsf{stop}F$$

Needless to say, to incorporate such partial predicates into our framework and make correct predictions for a wide range of data, we need to work out the semantics of verbs, tense, and aspect, among others.[10] This is left for future work. Applications to a wider range of presupposition triggers as we mentioned in Section 2.2, including factive verbs, additive particles like *too*, cleft constructions, and iterative expressions like *again*, are also left for future investigation.

[10]For an extensive study of these issues within the framework of DRT, see Kamp and Reyle (1993); Kamp, van Genabith, and Reyle (2011).

Chapter 3

Contextualism and Propositions Expressed

1. Introduction to Chapter 3

Recent studies in the semantics and pragmatics of natural language have shown that there is a considerable gap between the linguistic meaning of a sentence and the proposition expressed by an utterance of that sentence. This raises the question: What kinds of pragmatic tasks are involved in the determination of the proposition expressed by an utterance? There are two influential approaches to this question, which we call "Indexicalism" and "Contextualism." Contextualism holds that purely pragmatic processes called "free enrichment" are involved in the derivation of the proposition expressed. Indexicalism, on the other hand, denies the existence of such processes, and maintains that no pragmatic processes are allowed to affect the proposition expressed by an utterance unless the linguistic meaning of the sentence itself so demands. Thus, according to Indexicalism, all elements in the proposition expressed by an utterance (i.e., the truth-conditional content of an utterance) are linguistically *controlled* in the sense that they result from fixing the values of indexical elements in the logical form (cf. Stanley 2000).

We agree with the Contextualists that not only processes of fixing values of indexical elements in the logical form (i.e., what is usually called "saturation") but also pragmatic processes of free enrichment are involved in determining the proposition expressed by an utterance. However, in our view, the standard conception of Contextualism, as defended by relevance theorists (Sperber and Wilson 1986/95; Carston 2002a), is in fact very radical in that it holds that there are almost no semantic factors or constraints involved in the way the process of free enrichment works. Thus we will argue that both Indexicalism and the standard version of Contextualism are mistaken in their conception of the way linguistic semantics is related to the pragmatic processes involved in the determination of the proposition expressed. Based on a close analysis of predicational copular sentences, we will show that there is an interesting constraint on the applicability of free enrichment, and argue that the existence of such a constraint poses serious problems to both Indexicalism and the standard version of Contextualism. More specifically, we argue that free enrichment is blocked for property concepts, i.e., those concepts that are most typically expressed by predicate nominals. Then we will propose a new version of Contextualism that is compatible with the claim that there is a semantic constraint on free enrichment. The theory proposed here is an elaboration of ideas presented in Nishiyama and Mineshima (2005, 2006a, 2006b, 2007a, 2007b, 2010).

The structure of Chapter 3

This chapter is structured as follows.

In Section 2, we will briefly review the controversy between Contextualism and Indexicalism, focusing on the issue whether a purely pragmatic process of free enrichment is involved in the derivation of a proposition expressed. We then present a refinement of the classification of pragmatic processes involved in the determination of a proposition expressed proposed in Carston (2004). Our classification of pragmatic processes rests on a novel characterization of the distinction between free enrichment and the so-called *ad hoc* concept construction. Based on this classification, we clarify in what sense free enrichment can be regarded as a purely pragmatic process.

In Section 3, we present the main claim that free enrichment is blocked for property concepts. Several arguments motivating this claim will be presented, and they are extended to cases of adjectives.

Our claim is further elaborated in Section 4. We will take up possible counter-arguments to our claim, and argue that they are all defective. Furthermore, we attempt to answer the question why free enrichment could never intrude into property concepts.

In Section 5, we will argue that the so-called the “over-generation” problem pointed out by Stanley (2002, 2005) against Contextualism can be avoided in our Contextualist framework. We also discuss a recent proposal by Hall (2008) that tries to account for the over-generation problem solely in terms of pragmatic principles.

In Section 6, we will draw some consequences on the controversy between Contextualism and Indexicalism, and discuss how to understand the Underdeterminacy Thesis, one of the central claims of Contextualism.

2. Background on Indexicalism and Contextualism

We start with reviewing some background on the debate between Indexicalism and Contextualism over basic issues in the semantics-pragmatics interface.

2.1 Three levels of meaning

Although there are many subtle differences among various authors, it is generally accepted that three notions that play a role in utterance interpretations can be distinguished: (i) the linguistic meaning of a sentence S, (ii) the proposition P expressed by uttering S in a given context, and (iii) what is implicated by saying P in that context. A simple illustration is provided by the following example.

(1) I am a philosopher.

This sentence has a certain conventional meaning that is constant across contexts. Such a context-independent level of meaning is what we call the *linguistic meaning* of a sentence. It is now widely accepted that there is such a level of meaning; it is called "linguistically encoded meaning (LEM)" by relevance theorists (e.g. Carston 2002b, 2008), and "standing linguistic meaning" by proponents of Indexicalism such as Stanley (2005) and King and Stanley (2005). Roughly, this level of meaning is determined by the syntactic structure of a sentence and the meanings of the individual words constituting that sentence. For ease of exposition, we assume in the following discussion that there are lexically and structurally disambiguated semantic (conceptual) representations that correspond to this level of meaning. Following the standard terminology in relevance theory (Sperber and Wilson 1986/95), we call such semantic representations *logical forms*, and the process of arriving at a logical form of sentence S the process of *linguistic decoding*.

While the linguistic meaning of (1) remains the same across contexts, the sentence (1) is used to express different propositions on different contexts of utterance. Thus, if someone, say Lisa, utters (1), it would express the proposition that Lisa is a philosopher. With a different speaker or a different time, the proposition expressed by (1) would be different accordingly. This second level of meaning is what we call "the proposition expressed by an utterance of a sentence" or simply "the proposition expressed."

Important issues here are what role the linguistic meaning of a sentence plays in determining the proposition expressed and to what extent the proposition expressed is dependent upon pragmatic inferences. There are various answers to these questions; accordingly this second level of meaning is characterized variously as "what is said" (Grice 1989; Recanati 1989, 2004), "explicature" (Sperber and Wilson 1989/1995; Carston 2000, 2004), "semantic content" (Stanley 2005; King and Stanley 2005), "impliciture" (Bach 1994), and so on. Nonetheless, there is a general agreement that several roles are assigned to the proposition expressed by an utterance. Most importantly, it serves to capture the intuitive truth-conditional content of what the speaker says by uttering a sentence on a given occasion. Hence it must be a propositional, truth-evaluable entity, rather than an incomplete, non-propositional one. Also, it is this level of propositions, not the first level of meaning, that plays a role in deriving contextual implications from the speaker's utterance. The proposition expressed by an utterance serves as a premise in the inference process of deriving further implications.

The third level is the so-called "what is implicated" or "implicature," a notion that originated from the work of Paul Grice (Grice 1989). For example, an utterance of (1) in response to the question "Can you read Latin?" can communicate that the speaker, Lisa, can read Latin. What is implicated by uttering a sentence on a particular context is determined by a certain specific intention the speaker has in mind, namely, what is called "communicative intention" in relevance theory (cf. Sperber and Wilson 1986/95). On the basis of the proposition expressed by an utterance and some general pragmatic principles governing its use, the hearer has to infer the implicatures by specifying the speaker's intention.

Our main interest is in the relationship between the first and the second levels of meaning, that is, between linguistic meaning and proposition expressed. How is the linguistic meaning of a sentence S related to the proposition expressed by an utterance of S? One naive approach to this question is what we call *Literalism*, following the terminology of Recanati (2004).

(2) **Literalism**. The proposition expressed by the utterance of a sentence S is solely determined by the linguistic meaning associated with S. No pragmatic inferences are required in order to determine the proposition expressed by an utterance.

According to Literalism, the proposition expressed is fixed by the linguistic rules independently of any pragmatic consideration. It should be noticed that this does not necessarily mean that Literalists deny the existence of context-sensitive expressions in natural languages. Rather, they hold that such contextual expressions are confined to what Kaplan (1989) calls "pure indexicals," i.e., words like *I*, *today*, and *yesterday* whose denotations are automatically determined if objective features of the context of utterance, such as the speaker and the date, are given.[1] Kaplan emphasizes this point when he says about pure indexicals;

> The linguistic rules which govern their use fully determine the referent for each context. No supplementary actions or intentions are needed. (Kaplan 1989: 491)

The Literalist conception in (2) is compatible with the existence of this kind of indexical expression. The point is that the Literalist view given in (2) can allow contributions of context-sensitive elements to the proposition

[1] Here and henceforth, linguistic expressions are indicated in italics.

expressed, only if the denotation of that context-sensitive expression is fixed by linguistic rules alone, independently of considerations of the speaker's beliefs and intentions.

It is difficult to maintain Literalism, however, since context-sensitive expressions in natural languages are not restricted to indexicals in this narrow sense. It is now widely accepted that the class of context-sensitive expressions or constructions whose interpretations contribute to propositions expressed is larger than is traditionally assumed. We will examine some typical examples in the next section. Indeed, as some authors have noted, even indexicals such as *now* and *here* are not "pure" indexicals in the strict sense above, since the speaker's intentions can play a crucial role in determining the temporal or spatial scopes of the denotations of *now* and *here*.[2] For example, the possible denotations of the indexical expression *here* uttered on a particular context can range from the very small section of space occupied by the speaker to more inclusive areas such as the room, building, town, or country in which the utterance occurs. Thus, given the fact that almost all cases of reference assignment to indexical expressions require some pragmatic considerations of inferring speaker's intention, we conclude that Literalism, as construed above, must be rejected.

Recently, there are new attempts to defend some version of Literalism, called "Semantic Minimalism," which are proposed by Borg (2004, 2007) and Cappelen and Lepore (2005). They claim that the semantic content of an utterance departs only minimally from the linguistic meaning of a sentence. More specifically, they argue that it is not the job of "minimal" propositions to capture our intuitive judgement of what speakers say when they utter sentences. A principal objection to Minimalism is that such a level of minimal proposition plays no relevant role in utterance comprehension; see Recanati (2004: chap.4) and Carston (2008) for detail criticisms from the Contextualist point of view. Although there is now a considerable literature on the debate between Contexualism and Minimalism (see, e.g. Preyer and

[2]See Perry (2001: 61) and Carston (2002a: 218, note 48). Perry (2001) only classifies *I* and *today* as what he calls "automatic indexicals," i.e. those indexical expression whose denotation is fixed independently of speaker's intention.

Peter 2007; Bezuidenhout 2006; Cappelen and Lepore 2006), to follow up this matter would take us beyond the scope of this chapter.

2.2 Two approaches to propositions expressed

To deny the Literalist conception of propositions expressed commits ourselves to the following claim, which we call *the Underdeterminacy Thesis* following Carston (2002a: 19–20).[3]

(3) **The Underdeterminacy Thesis**. The linguistic meaning of a sentence used underdetermines the proposition expressed by the utterance.

The Underdeterminacy Thesis claims that there are considerable pragmatic tasks involved in arriving at the proposition expressed. Thus, anyone who accepts the Underdeterminacy Thesis commits herself to the rejection of Literalism. However, there is an ambiguity about the word "pragmatic processes" or "pragmatic tasks" here. Accordingly, there are several ways of interpreting the Underdeterminacy Thesis, depending upon how to understand the extent of the required pragmatic processes.[4] Among them, we focus on two dominant views, which we call *Indexicalism* and *Contextualim*, following the terminology of Recanati (2004).

2.2.1 Indexicalism

According to Indexicalism, there are only two kinds of pragmatic tasks involved in the determination of the proposition expressed, namely, disambiguation and saturation.[5] Disambiguation is a process of selecting one

[3] The Underdeterminacy Thesis (or sometimes called the *Underdetermination Thesis*) is particularly emphasized by proponents of Contextualism, including Bezuidenhout (2002), Carston (2000, 2002a, 2004), Neale (1990, p.114; 2004, p.88), Recanati (2004), Sperber and Wilson (1986/95), and Wilson and Sperber (2002, 2005), among others.

[4] We put aside the question whether all sentences in natural language require pragmatic tasks, in other words, the question whether there is a sentence which does not require any pragmatic task to understand. For a discussion, see Carston (2002a).

[5] For advocates of Indexicalism, see Stanley (2000, 2002, 2005), Stanley and Szabó (2000), King and Stanley (2005), Taylor (2001), and Martí (2006). Indexicalism has been

among a number of possible interpretations provided by the linguistic system itself. The interpretation of the bracketed element in (4) is due to disambiguation.

(4) John wrote a letter. [A LETTER OF THE ALPHABET]

Saturation is a process of supplying contextual values, not only for explicit indexical expressions such as pronouns and demonstratives as in (5a), but also for "hidden" indexicals involved in the logical form of a sentence, as exemplified in (5b).

(5) a. He fell down yesterday.
    b. John's paper is too long. [THE PAPER IN RELATION R TO JOHN] [TOO LONG FOR X]

Indexicalism holds that all elements in the proposition expressed by an utterance are linguistically controlled in the sense that they result from fixing the values of elements in the logical form (cf. Stanley 2000, 2005; Stanley and Szabó 2000). Thus, according to Indexicalism, the proposition expressed by an utterance departs from the linguistic meaning of the sentence uttered only when the linguistic meaning itself demands that some contextual value be assigned. On this account, any context sensitivity that affects the proposition expressed is traceable to some indexical element in the logical form.

Now consider the following examples. In appropriate contexts, the utterances of (6a), (7a) and (8a) could express the propositions shown in (6b), (7b) and (8b) respectively.

(6) a. The painter disappeared.
    b. THE PAINTER [LIVING IN THIS VILLAGE] DISAPPEARED.

(7) a. I met every linguist.
    b. I MET EVERY LINGUIST [ATTENDING THE PARTY].

(8) a. It's snowing.
    b. IT'S SNOWING [IN TOKYO] [AT TIME t_k].

partly motivated by analyses of context-dependent expressions within formal semantics, for example, the analysis of quantifier domain restriction by von Fintel (1994).

Indexicalists take these as instances of saturation.[6] They maintain that the logical form of each sentence contains some hidden variables—the domain variables in (6) and (7) and the time and location variables in (8)—that must be filled for each utterance to express a truth-evaluable proposition.

As particularly emphasized by Stanley (2002, 2005), one motivation for Indexicalism is to preserve the idea of the compositionality of truth-conditional contents. According to it, the intuitive truth-conditional content of a sentence uttered on a given occasion must be compositionally derivable, i.e., it must be obtained by assigning values to the elements of the logical form of a sentence and combining them in accord with its structure. Indexicalists respond to apparent counter-examples to this claim, such as ones in (6)–(8), by positing some covert syntactic variables in logical forms.

A general worry about Indexicalism is that the postulation of such covert variables tends to be non-explanatory; sometimes it may be possible to explain *why* the construction in question can have such context-sensitive interpretations without complicating the syntax and semantics of that construction.[7] These considerations provide us with initial motivations to explore an alternative position, namely Contextualism. Contextualists hold that interpretations like (6b), (7b), and (8b) can be accounted for in terms of pragmatic principles, without stipulating the existence of hidden variables in logical forms. Such a pragmatic approach to analyzing propositions expressed would be generally preferable, because the mechanisms used are independently needed for other pragmatic phenomena such as implicature

[6]For a treatment of nominal expressions as in (6) and (7) within the framework of Indexicalism, see Stanley and Szabó (2000) and Stanley (2002). For a discussion of time and location as in (8), see Stanley (2000).

[7]Stanley (2000, 2002) attempts to provide some linguistic evidence for certain covert variables (the so-called "binding argument"). There is a considerable literature on Stanley's argument. For critical discussions, see Bach (2000), Neale (2000, 2004), Recanati (2004: chap.7), and Elbourne (2008). In addition, Indexicalism faces a number of objections based on theoretical and empirical grounds. For arguments against Stanley's version of Indexicalism, see Carston (2000), Perry (2001), Wilson and Sperber (2002, 2005), and Hall (2008). In Section 3.4, we will give yet another argument against Stanley's Indexicalism on the basis of the claim that predicate nominals do not allow contextual restrictions.

derivations. The main worry about the Contextualist view is that pragmatic inferences are so powerful and unconstrained that they generate interpretations that do not actually occur (cf. Stanley 2002, 2005). This is the so-called the "overgeneration" problem and will be discussed in Section 6.

2.2.2 Contextualism

Let us review the Contextualist view in more detail. Contextualists agree that there are pragmatic processes of disambiguation as in (4) and saturation as in (5). As mentioned above, however, they deny that (6), (7), and (8) are instances of saturation.[8] As Recanati (1989, 2004) observes, the pragmatic processes involved in (5) are *mandatory* in the sense that they must be carried out in any context in which the linguistic expressions at issue are used; without such a process, the utterance cannot communicate a complete proposition. In contrast, the pragmatic processes involved in (6), (7), and (8) are *optional*: the utterances could communicate complete propositions without those processes. In this respect, the pragmatic processes involved in (5) are essentially different from those involved in (6), (7), and (8). Following the terminology of relevance theory (see, in particular, Carston 2004) we call the type of pragmatic process involved in (6), (7), and (8) "free enrichment."

Free enrichment is the process of adding further conceptual materials to the logical form without any linguistic mandate. Such additional materials are often called "unarticulated constituents" (cf. Perry 1986). The bracketed elements in (6b), (7b), and (8b) are typical examples of unarticulated constituents. As further examples, consider sentences (9a), (10a), and (11a), which are often discussed by relevance theorists.[9]

(9) a. They got married and had many children.
b. They got married and [then] had many children.

(10) a. She insulted him and he left the room.
b. She insulted him and [as a result] he left the room.

[8] See Footnote 3 for proponents of Contextualism.

[9] An earlier discussion is found in Carston (1988). Further examples and discussion can be found in Carston (2000, 2002a, 2004).

(11) a. Mary took out her key and opened the door.
b. MARY TOOK OUT HER KEY AND [THEN] OPENED THE DOOR [WITH THE KEY].

The utterances of (9a), (10a), and (11a) could express the propositions as enriched in (9b), (10b), and (11b) respectively. Note that each utterance could be understood literally without the bracketed elements. According to relevance theory, such literal interpretations are often not worth processing, hence optional inferential processes automatically take place in order that the resulting overall interpretation would be relevant enough to hearers. This means that the motivation for these processes taking place is not linguistic as in the case of saturation, but pragmatic through and through. The same remark applies to the examples in (6), (7), and (8) discussed above.

We assume here that the process of free enrichment takes place not for a proposition as a whole but for a particular constituent of a proposition; in other words, free enrichment is a *local* process (cf. Carston 2000a; Recanati 2004; Hall 2008). Notice that, in a typical case, the process of free enrichment is triggered by the fact that the (literal) proposition would be not relevant enough to attract the hearer's attention. However, this does not imply that the target of free enrichment is the proposition as a whole. Rather, the fact is that the proposition might be made more relevant through enriching some constituent of it. Thus, in dealing with examples of free enrichment, it is important to ask which constituent of the proposition is enriched.

Before moving on, it will be useful to distinguish several types of free enrichment. In examples like (6) and (7), what is enriched is a concept encoded by a nominal expression. For example, in (6), the concept [PAINTER] is enriched to the concept [PAINTER LIVING IN THIS VILLAGE]. The process underlying such cases may be called the *nominal* enrichment. On the other hand, in examples like (8), (9), (10), and (11), the concepts encoded by verbal expressions are enriched; thus, in (8), the concept [SNOWING] is enriched to the concept [SNOWING IN TOKYO AT TIME t]. This process may be called the *verbal* enrichment. In discussing examples involving conjunction as in (9), (10), and (11), people often talk as if the literal interpretation of AND is modified so that various temporal or causal interpretations of conjunction, such as [and then] and [and as a result], are obtained. However, as shown in (12), the same type of enrichment occurs when two sentences are simply conjoined.

(12) a. She insulted him. [As a result] He left the room.

b. Mary took out her key. [Then] She opened the door.

In these cases, the enriched material such as [then] or [as a result] modifies the concept expressed by a verbal phrase in the second sentence. Thus, the examples like (9), (10), and (11) can also be classified as verbal enrichment, in a similar way to the example in (8).

Another kind of example that is often taken up by relevance theorists as an instance of free enrichment is the so-called "subsentential" or "incomplete" utterances. Typical examples are the following:

(13) a. [Context: Holding up a bottle of wine.] From France.

b. This wine is from france

(14) a. Nice shirt.

b. You are wearing a nice shirt

It is widely agreed that the expressions in (a) can be used to express the propositions in (b). There is an ongoing debate about the nature of such subsentential utterances. Contextualists, including Recanati (2004), Carston (1988, 2002a, 2004), Sperber and Wilson (1986/95), and Hall (2008), claim that examples like (13) and (14) are genuine examples of free enrichment. On the other hand, indexicalists, including Stanley (2000, 2002, 2005) and King and Stanley (2005), argue that these examples are best accounted for by positing hidden sentential structures. Although this issue is important for a comprehensive understanding of pragmatic processes involved in the recovery of a proposition expressed, we will put it aside here and concentrate on the enrichment of fully sentential examples.

2.3 The classification of pragmatic processes

We have already mentioned three types of pragmatic processes, namely, disambiguation, saturation, and free enrichment. There is another type of pragmatic process which Contextualists admit. Following relevance theorists (Sperber and Wilson 1986/95, 1998; Carston 1996, 2002a, 2004; Wilson and Sperber 2002), we call this type of processes "*ad hoc* concept construction."[10]

Ad hoc concept construction is the process of replacing an encoded lexical concept appearing in the logical form with a contextually adjusted one, so that the concept interpreted as communicated by the particular lexical item is different from the lexically encoded concept. The following are examples from Carston (1996).

(15) a. I want to meet some bachelors. [BACHELORS*]
b. This steak is raw. [RAW*]
c. Our boss is a bulldozer. [BULLDOZER*]

Suppose (15a) is uttered by a young lady who has decided to go to a party to find a potential husband. What she communicates by *bachelor* is not the encoded lexical concept but rather something much more specific, say, [BACHELORS ELIGIBLE FOR MARRIAGE], which is indicated by [BACHELORS*].[11] Here, a *narrowing* of the encoded lexical concept is involved. Now suppose (15b) is uttered by a customer at a restaurant. What he communicates by the predicate *raw* is not the encoded lexical concept [UNCOOKED] but a *loosening* of the concept, namely, [UNDERCOOKED] which is indicated by [RAW*]. Similarly, in (15c), what the speaker communicates by *bulldozer* is

[10] The same type of pragmatic process is called "specifization" in Bach (1994) and Recanati (2004: chap.2), and "modulation" in Recanati (2004: chap.9). Recanati (2004: 24) mentions the following pair of examples involving a mass term *rabbit* as a case of "specifization."

(i) He wears rabbit. [RABBIT FUR]
(ii) He eats rabbit. [RABBIT MEAT]

In our terminology, these examples are a typical case of *ad hoc* concept construction.

[11] Following the standard practice in relevance theory, we use capitals like "BACHELOR" to denote entities at the level of conceptual representation (i.e., concepts and propositions).

not the encoded lexical concept, but a looser concept like [being powerful and aggressive].[12]

As in the case of free enrichment, *ad hoc* concept construction is also an optional process in the sense that the utterance of a sentence could deliver a complete proposition without such a process. For example, the utterance of (15b) could be understood literally by taking *raw* as expressing the encoded concept [uncooked], though it would be false in usual situations. These literal interpretations are often not worth processing, so that optional inferential processes automatically take place in order that the resulting overall interpretation would be relevant enough.

Carston (2004: 643) suggests that "[*ad hoc* concept construction], like free enrichment, takes us well away from encoded linguistic meaning and has no linguistic mandate." It should be noted, however, that the range of possible interpretation provided by *ad hoc* concept construction is strictly constrained by the hearer's lexical and encyclopedic knowledge concerning the original encoded concept. For example, [bachelor*] in (15a) is derived by combining the original encoded concept [bachelor] with the encyclopedically related concept [being eligible for marriage]. In the case of [raw*] in (15b), the range of possible interpretation is restricted to some class of related concepts, varying from [uncooked] to various degrees of [undercooked]. Similarly, [bulldozer*] in (15c) is derived from the original encoded concept by deleting one of its defining properties such as [being a machine used for knocking down buildings], and leaving only the encyclopedically associated concept [being powerful and aggressive].

As these examples show, the process of *ad hoc* concept construction should be characterized as the process of making pragmatic adjustment within the restricted range of interpretation licensed by a hearer's general lexical and encyclopedic knowledge concerning the original encoded concept. The process of free enrichment, on the other hand, is not bounded in this way. For example, in the interpretation of (6a) discussed above, the additional concept [living in this village] cannot be derived from the encoded

[12] For a more discussion on the distinction between narrowing and loosening, see Carston (1996).

concept [PAINTER] through some general lexical-encyclopedic knowledge. In this respect, there is an essential difference between free enrichment and *ad hoc* concept construction.

Free enrichment and *ad hoc* concept construction are not always carefully distinguished in the literature.[13] Thus, Carston (2004) takes the following examples as typical cases of free enrichment.

(16) a. She has a brain. [A HIGH-FUNCTIONING BRAIN]
b. It's going to take time for these wounds to heal. [CONSIDERABLE TIME]

In the light of the discussion above, this analysis is questionable. In (16a), the lexically encoded concept [BRAIN] is pragmatically adjusted so that the utterance would express some interesting fact rather than an obvious truth. Although the process is generated for purely pragmatic reasons, the possible range of interpretation would surely be bounded by the lexical concept [BRAIN]. (16b) is analogous to (15b) in that the pragmatic process gives rise to the adjusted concept with a suitable degree of application. In the case of (16b), the encoded concept [TAKING TIME] is pragmatically adjusted. Thus, these cases should be taken as instances of *ad hoc* concept construction.

As a further illustration of the distinction between free enrichment and *ad hoc* concept construction, consider again the paradigm example of free enrichment in (11), repeated here.

(11) a. Mary took out her key and opened the door.
b. Mary took out her key and [THEN] opened the door [WITH THE KEY].

Recanati (2004: 25) suggests that some cases involving free enrichment may be construed as instance of *ad hoc* concept construction (i.e., what he calls

[13]It should be emphasized that according to the taxonomy of pragmatic processes in explicature derivation proposed by Sperber and Wilson (1986/95) and Carston (2002a), free enrichment and *ad hoc* concept construction are classified into the same type of pragmatic process; both processes are labeled as "enrichment." What we adopt here is a more fine-grained taxonomy, which is basically the one presented in Carston (2004). This new taxonomy should be distinguished from the earlier proposals in relevance theory.

"specification" or "modulation"; see Footnote 10). Thus he says of the example in (11b) that the implicit instrument in the second conjunct could be construed either (i) as an unarticulated constituent corresponding to the implicit prepositional phrase *with the key*, or (ii) as an *ad hoc* concept [open with key], derived from the general concept [open] by a process of narrowing. However, as Carston (2007: 46) correctly points out, even if a hearer could derive the narrowed concept [open with key], she would still need to pragmatically infer that Mary opened the door *with the key mentioned in the first conjunct*, in order to obtain the intended meaning of (11a). In our terms, while the *ad hoc* concept [open with key] might be derived from the concept [open] using some lexical-encyclopedic knowledge, the concept involving reference to a *particular* object, i.e., [open with the key], cannot be derived in that way. In other words, it is not predictable from our general knowledge about the concept [open] that what the speaker intends to communicate is something about a particular key Mary took out (at a particular place at a particular time). Thus in our view, the concept [with the key] in the example (11b) should be construed as a result of free enrichment, rather than *ad hoc* concept construction.

We have so far classified the hearer's task in constructing a proposition expressed into four types:

(17) a. **Disambiguation**: Selecting one of the candidate conceptual representations provided by the linguistic system.
b. **Saturation**: Supplying contextual values to variables in the logical form.
c. **Ad hoc concept construction**: Replacing an encoded concept appearing in the logical form with a contextually adjusted one.
d. **Free enrichment**: Adding further conceptual constituents to the decoded logical form.

When Carston (2004: 639) introduces the process of free enrichment, she says "it is 'free' in that it is not under linguistic control." However, the notion of "linguistic control" here could be construed in two ways. A pragmatic process is said to be under linguistic control if it satisfies (18a) or (18b).

(18) a. The process is *mandated* in the sense that it must be carried out in any context in which the linguistic expression at issue is used.

b. The process is *bounded* in the sense that the range of possible interpretations is provided by the linguistic system itself or by hearer's lexical/encyclopedic knowledge concerning the original encoded concept.

Then, the four types of processes in (17) can be classified as follows.

	Mandated?	Bounded?
Disambiguation	Yes	Yes
Saturation	Yes	No
Ad hoc concept construction	No	Yes
Free enrichment	No	No

Among the four types of processes, free enrichment is special in that it is neither mandated nor bounded. This is why free enrichment is considered to be a "purely pragmatic" process and thus to play a distinctive role in the determination of a proposition expressed.

2.4 The standard conception of free enrichment

According to Contextualism, there is a gap between the linguistic meaning of a sentence and the proposition expressed that is to be filled by pragmatic processes, in particular, by free enrichment. Now the question arises: what role does the linguistic meaning of a sentence play in deriving a proposition expressed via a process of free enrichment? In particular, what kind of constraint is imposed on processes of free enrichment? In this section, we make explicit the standard answer to these questions within the framework of Contextualism. Roughly speaking, according to the standard Contextualist view, the linguistic meaning of a sentence is just an *evidence* to recover a proposition expressed, in a similar way as the proposition expressed is an evidence to derive a conversational implicature of the utterance. Thus, on this view, what plays a crucial role in constraining free enrichment is a *pragmatic* consideration, such as the relevance and informativeness of the resulting propositions. By contrast, we will propose an alternative conception, according to which there is a *semantic* constraint on a process of free enrichment. The main aim of the next section is to argue for this alternative conception.

Before going on, let us first explain what we mean by "semantics" here. Throughout our discussion, we assume the relevance-theoretic framework (Sperber and Wilson 1986/95; Carston 2002a), which is one of the most developed theories among currently available Contextualist positions. By "semantics" we mean what is often labeled by relevance theorists as "linguistic semantics" or "translational semantics." Its role is to map one representation, i.e., a natural language sentence, into another representation, i.e., a semantic or conceptual representation called a *logical form*.[14] The sort of semantics which explicates the relation between propositions expressed and what they represent may be called "denotational semantics," in distinction from linguistic or translational semantics. Thus, on this view, "semantic" interpretations take place in two stages. At the first stage, linguistic semantics provides an algorithm that assigns a semantic representation, i.e., a logical form, to a natural language sentence (or a set of semantic representations to an ambiguous sentence). Such representations are often not fully propositional but only schemas to be further elaborated. Thus they are pragmatically selected, completed, enriched, and modulated in various ways as described in the last section, so as to yield a fully propositional form expressed by the utterance. At the second stage of semantic interpretation, denotational semantics gives truth-conditions to the fully propositional representations.[15]

[14]See Sperber and Wilson (1986/95: 257–8) and Carston (2002a: 58). The view outlined here is shared by some authors outside relevance theory as well. See, for example, Neale (2004: 82).

[15]It should be mentioned that the so-called "structured propositions" view, as defended by Soames (1987) and King (2007), among others, also shares such a two-stage view of semantic interpretation. What is special about the conception of relevance theory is the appreciation of the roles played by pragmatic inferences that bridge the output of the first

A variant of the relevance-theoretic view is discussed in Neale (1990, 2004), who calls it the "explicit" approach to pragmatic enrichment. According to Neale (2004: 121), enrichment is a process whereby a natural language expression that is actually uttered (e.g., *the painter* in example (6)) is mapped into another natural language expression, namely, an expression that the speaker *could have* used if she was asked to be more explicit (e.g., *the painter living in this village*). In other words, the sentence actually uttered is a shorthand for a richer phrase the speaker might have used.[16] (Neale carefully distinguishes such a process of "contextual recovery" from a process involved in syntactic ellipsis such as VP ellipsis as discussed in syntactic theory.) For concreteness, we focus on the relevance-theoretic view on free enrichment and regard it as a representative position in Contextualism; but it should be noted that our argument in following sections is neutral with respect to the choice between the relevance-theoretic view and the explicit view, and hence that it can be applied to the explicit view on enrichment as well.

With this picture in mind, let us now return to the main issue. The question is: what kind of constraint is imposed on free enrichment? And what role does the linguistic meaning of a sentence play when a process of free enrichment takes place?

The standard answer to this question within Contextualism is two-fold. First of all, free enrichment is *pragmatically* constrained. Let us state the pragmatic constraint on free enrichment in the following way:

(19) **Pragmatic Constraint on Free Enrichment (PC)**
Free enrichment can take place only when the resulting proposition is consistent with pragmatic considerations such as relevance and informativeness.

More specifically, according to relevance theory, one general pragmatic principle, namely, the principle of relevance, is at work in the determination

phase (i.e., a linguistic meaning or logical form) with the input of the second phase (i.e., a proposition expressed or explicature).

[16] A related view is found in Bach (1994, 2000). See also Elbourne (2008) for a discussion on the difference between the relevance-theoretic view and the explicit view.

of a proposition expressed as well as in the derivation of an implicature. Roughly, at each stage in disambiguation, saturation, free enrichment and *ad hoc* concept construction (as well as implicature derivations), the hearer would choose the interpretation (hypothesis) that requires the least effort and abandon it when it fails to be consistent with the principle of relevance. Carston calls such a general procedure "the least effort strategy," and describe it in the following way:

(20) Consider interpretations (disambiguations, saturations, enrichments, implicatures, etc) in order of accessibility (i.e., follow a path of least effort in computing cognitive effects); stop when the expectation of relevance is achieved. (Carston 2002b: 139)

For some illustrations of how this strategy works, see Wilson and Sperber (2002: 606–619), Carston (2002a: 142–145), and Carston (2002b: 139–140). On this view, the linguistic meaning of a sentence uttered (i.e., the logical form of a sentence) is regarded as having just an evidential role in the identification of the proposition expressed by an utterance.[17]

However, to say that the linguistic meaning of a sentence is just an evidence in determining the proposition expressed does not mean that the role of linguistic meaning in this process is exactly the same as the one a proposition expressed plays in the derivation of the implicatures of an utterance. The crucial fact is that the proposition expressed is built out of the semantic representation (i.e., the logical form) associated with the linguistic expression used. This point can be clearly seen in the following remark by Recanati (2004: 6) on the difference between what is said (i.e., propositions expressed in our terms) and what is implicated.

> The difference between 'what is said' and 'what is implicated' is that the former is constrained by sentence meaning in a way in which the implicatures aren't. What is said results from fleshing out the meaning of the sentence (which is like a semantic 'skeleton') so as to make it propositional. The propositions one can arrive at through this process of contextual enrichment or

[17] See Carston (2002b: 130).

> 'fleshing out' are constrained by the skeleton which serves as input to the process. Thus 'I am French' can express an indefinite number of propositions, but the proposition in question all have to be compatible with the semantic potential of the sentence; this is why the English sentence 'I am French' cannot express the proposition that kangaroos have tails. There is no such constraint on the propositions which an utterance of the sentence can communicate through the mechanism of implicature. Given enough background, an utterance of 'I am French' might implicate that kangaroos have tails. What is implicated is implicated by virtue of an inference, and the inference chain can (in principle) be as long and involve as many background assumptions as one wishes. (Recanati 2004: 6)

To use the terminology of relevance theory, the proposition expressed (i.e., explicature) must be a *development* of the logical form of a sentence used; in other words, it must be composed of all the elements of the logical form together with some additional elements.[18] Furthermore, it can be reasonably assumed that free enrichment must preserve the *semantic type* of an element in a logical form; accordingly, free enrichment can never give rise to meaningless propositions.[19] For instance, in (21a), the concept [LINGUIST] and the enriched concept [LINGUIST ATTENDING THE PARTY] are of the same semantic type; both are semantically monadic predicates. Similarly, in (21b), the concept [OPENED THE DOOR] and the enriched concept [OPENED THE DOOR WITH THE KEY] are of the same semantic type.

(21) a. I MET EVERY LINGUIST [ATTENDING THE PARTY]

[18] See Sperber and Wilson (1986/95: 182) and Carston (2002a: 116–125).

[19] Such a constraint seems to be implicitly assumed by contextualists in discussing examples of enrichment (i.e., what we called nominal and verbal enrichment) as we saw in Section 2.2. Recanati (2011: 11, footnote 9) briefly mentioned this kind of constraint. We should note that Recanati (2011) does not distinguish between what we called free enrichment and *ad hoc* concept construction. We will remain neutral about the question whether a process of *ad hoc* concept construction in our sense always preserves the semantic type of a concept to which it applies.

b. JOHN OPENED THE DOOR [WITH THE KEY]

Let us say that contextualists are committed to the *Minimal Linguistic Constraints on Free Enrichement*, abbreviated as MLC.

(22) **Minimal Linguistic Constraint on Free Enrichment (MLC)**

a. A proposition expressed by an utterance of a sentence S is obtained via free enrichment only if it is a *development* of the logical form of S.

b. Free enrichment must preserve the *semantic type* of an element in a logical form; that is, if concept α^+ is obtained by enriching concept α in a proposition, then α and α^+ must be of the same semantic type.

The standard Contextualist conception of how free enrichment works in deriving propositions expressed can be summarized as follows.

(23) **The standard conception of free enrichment**
A process of free enrichment can take place if it observes the pragmatic constraint (PC) and the minimal linguistic constraint (MLC).

That is, according to the standard conception, the only constraints imposed on free enrichment are PC and MLC. In fact, it turns out that this conception imposes a weak constraint on how free enrichment works; the standard conception says that, except for MLC, there is no linguistic factor or constraint involved in the way the process of free enrichment works. More specifically, the standard conception is committed to the view that free enrichment could apply to *any* concept in the logical form of a sentence, as far as PC and MLC are observed. Thus, Sperber and Wilson (1986/95: 176) claim:

> The semantic representations recovered by decoding are useful only as a source of hypotheses and evidence for the second communication process, the inferential one. Inferential communication involves the application, not of special purpose decoding rules, but of general-purpose inference rules, which apply to any conceptually represented information.

We agree with the standard Contextualism that it is not saturation but free enrichment that is involved in the derivations of the propositions expressed in such examples as (6)–(11). Thus, we admit the existence of free enrichment, and reject the Indexicalist view that such examples are construed as instance of saturation. However, we will argue that the standard Contextualist claim in (23) is false. In the next section, we will show that the applicability of free enrichment is far more restricted than is assumed within the current framework of Contextualism as we reviewed so far, and in particular, that there is a certain semantic restriction on how free enrichment works.

3. Semantic constraint on free enrichment

The goal of this section is to argue that there is a certain restriction on how free enrichment can work in deriving the proposition expressed by an utterance. Specifically, we will first focus on predicational sentences like (24), and argue that in no context can the process of free enrichment apply to the concept encoded by a predicate nominal like *a painter* in (24).

(24) John is a painter.

This is to say that we cannot imagine any context in which the concept encoded by *a painter* in (24) might be enriched so that (24) would express the proposition in (25).

(25) JOHN IS A PAINTER LIVING IN THIS VILLAGE.

Since the data bearing on this matter are often not easy to assess, we will present several arguments to defend our claim. Before going on, we provide some preliminary background on the semantic interpretation of predicate nominals.

3.1 Predicate nominals

A copular sentence of the form "NP_1 is NP_2" has several readings.[1] Among them we pay special attention to the *predicational* reading. Typical examples are the following.

(26) a. Mary is a linguist.

[1]For discussions of the classification of copular sentences, see Higgins (1973), Declerck (1988), Nishiyama (2003), and Mikkelsen (2005). For a useful survey, see Mikkelsen (2011).

b. John is a fool.

Noun phrases occurring in a predicate position, such as *a linguist* and *a fool* in (26), are called "predicate nominals." Predicate nominals are non-referential in that they stand for a property ascribed to the referent of the subject noun phrase. Thus, the predicate nominal *a linguist* in (26a) does not denote any individual but stands for a property ascribed to Mary. Similarly, in (26b), the property of being a fool is ascribed to John.

This characteristic of predicate nominals is more manifest when we consider examples involving negation like (27):

(27) John is not a linguist.

Intuitively, an utterance of (27) does not establish an individual referent corresponding to the noun phrase *a linguist*. Rather, (27) is interpreted as expressing that John lack a certain property. These peculiarities of predicate nominals are already pointed out by Geach (1980: 36):

> [...] if I use the term "man" in the context "... is a man" or "... isn't a man", it is mere nonsense to ask which man or men would be referred to, or whether every man or just some man would be meant. If I said "Tibbles isn't a dog" and some non-philosopher asked me with apparent seriousness "Which dog?", I should be quite bewildered—I might conjecture that he was a foreigner who took "isn't" to be the past tense of a transitive verb.

There are several arguments which show that predicate nominals are semantically different from "referential" noun phrases. First, as noted by Kuno (1970), Doron (1988), and Mikkelsen (2005), among others, three is a close connection between the referentiality of a noun phrase and the choice of a pronominal expression. Consider the following examples, taken from Kuno (1970: 365).

(28) a. *A doctor*$_i$ came to see me. I could trust {the doctor$_i$/him$_i$}.

b. My brother is *a doctor*. I cannot trust {*the doctor/him}.

In (28a), the noun phrase *a doctor* appears in an argument position of the sentence and can be the antecedent of the definite noun phrase *the doctor* as well as the personal pronoun *him*. In (28b), the noun phrase *a doctor* appears as a predicate nominal. In this case, *a doctor* cannot be interpreted as the antecedent of the definite noun phrase *the doctor*. The personal pronoun *him* is acceptable, but it is anaphoric on the noun phrase *my brother* in the subject position, not on the predicate nominal *a doctor*. As Kuno (1970: 365) observes, the pronoun that is anaphoric on the predicate nominal must be *it*, rather than personal pronouns such as *he* or *she*.

(29) He is *a fool*, although he doesn't look {it, *him}.

This contrast can be explained by assuming that noun phrases in argument position can establish an individual referent, whereas predicate nominals denote a property and hence do not establish such an individual referent.

This explanation is further supported by the following example.[2]

(30) He is *a gentleman*, {which/*who} his brother is not.

(30) shows that the relative pronoun formed on the predicate nominal *a gentleman* must be *which*, rather than *who*. Note that *which* is also used for an adjective such as *foolish*, which inherently denotes a property.

(31) John is foolish, which you are not.

Additionally, the interrogative pronoun *what*, but not *who*, is used for asking a property ascribed to the referent of the subject noun phrase.

(32) *What* is he? — He is {tall/a doctor}.

It should be noted here that predicate nominals do also appear in the environments other than the post-copula position of a predicational sentence.

(33) a. Socrates became *a philosopher*. (Geach 1962: 35–36; Higgins 1973: 224–225; Williams 1983)
b. John used to be *a philosopher*. (Higgins 1973: 225)

[2]See Kuno (1970: 365) and Mikkelsen (2005: 95–99) for more detailed discussions.

c. John considers Mary *a genius.* (small clause complements: Rothstein 2006: 57)

d. John, as *a judge*, earns $50,000 a year. (*as*-pharses: Szabó 2003)

e. Being *a non-smoker*, Mary approves the new law. (free adjuncts: Jäger 2003)

f. He was born *a Republican* and he died *a Republican.* (secondary predication: Rothstein 2006: 223)

Each construction can be understood as involving a predication at some level. Thus, (33c) involves the predication that Mary is a genius, and (33d) involves the predication that John is a judge.

Let us introduce some terminology. We are concerned with noun phrases that have a structure as shown in Figure 3.1.

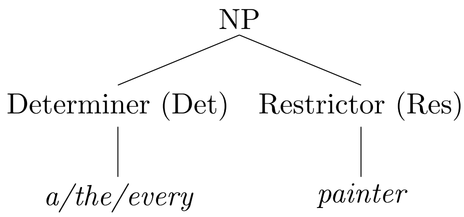


Fig. 3.1 The structure of noun phrases

When a noun phrase of the form $[[_{\text{Det}}\,\alpha][_{\text{Res}}\,\beta]]$ appears in an argument position of a sentence, where α is a determiner and β is its restrictor, we say that the concept expressed by β is an *object-directed concept.* The semantic function of an object-directed concept is to indicate who or what the entire concept corresponding to the noun phrase in question refers to or quantifies over.[3] On the other hand, when a noun phrase of the form $[[_{\text{Det}}\,\alpha][_{\text{Res}}\,\beta]]$

[3] As is often observed, quantificational phrases such as *every F* or *most F* cannot be used as a predicate nominal; see Williams (1983: 425–6), Higginbotham (1987), Doron (1988: 297–9), and Fara (2001: 17–8) for detailed discussions.

(i) *John is every linguist from New York.

We can say that the restrictor *F* of a quantificational phrase of the form *every F*, *most F*, and the like inherently expresses an object-directed concept. Note also that such quantificational noun phrases can introduce individuals that are referred back to by a appears as a predicate nominal, we call the concept expressed by β a *non-object-directed concept* or, simply, a *property concept*. The semantic function of a property concept is to indicate a property that is to be ascribed to an object (or objects) introduced in an argument position of a sentence.

As an illustration, consider the following:

(34) a. A linguist disappeared.
b. [[A LINGUIST] DISAPPEARED]

(35) a. John is a linguist.
b. [JOHN IS [A LINGUIST]]

The sentences in (34a) and (35a) express the propositions in (34b) and (35b) respectively. We say that the concept LINGUIST in (34b) is an object-directed concept, while the concept LINGUIST in (35b) is a property concept.

Note that the concept LINGUIST itself is a *monadic* concept, that is, a semantically one-place predicate.[4] This means that whether a concept F is object-directed or not is not determined by the semantic type of F that can be specified independently of how it occurs in a proposition. Rather it is determined by the position that the concept F occupies in a proposition. To say that the concept F is an object-directed concept or a property concept is a claim about a particular role played by F in a proposition.

pronominal phrase (cf. Evans 1980).

(ii) Every student$_i$ arrived. They$_i$ registered.

In these respects, the semantic function of quantificational noun phrases is crucially distinguished from that of predicate nominals.

[4]We are assuming that each concept is associated with an adicity that determines how it can be combined with other concepts to form a proposition. There is a debate about what semantic type is to be assigned to a predicate nominal as a whole (e.g., *a linguist* in *John is a linguit*), as well as to the copular verb *be*; the classical reference is Partee (1986); see also Fara (2001), Neale (2005), Kripke (2005), and Heim (2011) for discussion. We are not committed to a particular view on this issue. We will also remain neutral about the question whether definite and indefinite descriptions appearing in argument position should be treated as a quantificational expression, i.e., an expression of type $((e,t),t)$, or as a referring expression, i.e., an expression of type e. Our argument below will be independent of these issues. (See Chapter 2 of this dissertation for a defense of the analysis of definite descriptions as referring expressions.)

In the next section, we will see how free enrichment can take place for object-directed concepts. Then, in Section 3.3, we will turn to the case of property concepts.

3.2 Free enrichment and object-directed concepts

The process of free enrichment can take place for an object-directed concept, i.e., a concept expressed by a noun phrase appearing in an argument position. Typical examples are the following.[5]

(36) a. Every linguist disappeared. [EVERY LINGUIST ATTENDING THE PARTY]

b. A painter died. [A PAINTER LIVING IN THIS VILLAGE]

c. She gave presents to some children but not to others. [SOME CHILDREN AT THE PARTY]

For each sentence in (36), when uttered in the right situation, we naturally interpret it with the enriched material indicated in brackets. Without these enriched materials, the utterances would communicate irrelevant propositions. In the case of (36a), the propositions literally expressed would be obviously false in usual situations. In the case of the examples in (36b) and (36c), the literal propositions might be true but not informative enough to attract the hearer's attention. This means that each example satisfies the pragmatic constraint on free enrichment (PC) in (19). Note also that each enriched proposition preserves the logical form of an original sentence; hence the minimal linguistic constraint on free enrichment (MLC) in (22) is observed as well. On the standard relevance-theoretic account, then, the process of free enrichment is automatically invoked in these cases, and those enriched propositions which satisfy the hearer's expectation of relevance are derived.

There are several points to note about the examples in (36). First, free enrichment can take place not only for the concepts expressed by definite noun phrases such as *every painter* and *the painter*, but also for the concepts

[5] The example (36c) is indebted to Robyn Carston (personal communication).

expressed by indefinite noun phrases such as *a painter* in (36b) and *some children* in (36c).[6] Second, free enrichment can take place for the concept expressed by the subject noun phrase, as in (36a) and (36b), as well as for the concept expressed by the object noun phrase, as in (36c). Third, each example in (36) involves a *narrowing* of the object-directed concepts. Thus in the case of (36a), the concept [LINGUIST] is made more specific by the enriched material [ATTENDING THE PARTY], resulting in a more restricted extension. Similarly for the cases of (36b) and (36c).

3.3 Free enrichment and predicate nominals

With the case of object-directed concepts in mind, let us turn to the case of property concepts, i.e., concepts expressed by predicate nominals. We claim that free enrichment is blocked for property concepts.

Consider the following conversation. Suppose that, in a certain village meeting, the village chief Lisa finds an unfamiliar guy. She wonders whether the guy is an inhabitant of her village or not. John is aware of her worry and utters (37).

(37) He is a painter.

Here, (37) is a predicational sentence, in which the noun phrase *a painter* functions as a predicate nominal. In this context, it seems impossible to interpret John's utterance of (37) as expressing a proposition like (38b) by enriching the concept expressed by *a painter*. Rather, it should be interpreted literally as in (38a).

(38) a. BILL IS A PAINTER
b. BILL IS A PAINTER [FROM THIS VILLAGE]

Indeed, we cannot imagine any context in which the concept encoded by *a painter* in (37) might be enriched. Note that, in this context, while the proposition in (38a) is not relevant enough, the proposition in (38b) would

[6] For discussion on whether indefinites could exhibit such pragmatic interpretations as free enrichment, see Bach (1994); Stanley and Szabó (2000).

most likely contribute to the satisfaction of Lisa's expectation of relevance, since she wants to know whether the guy is living in her village or not. That is, the pragmatic constraint on free enrichment (PC) in (19) seems to be satisfied in this case. In this respect, the situation is analogous to the examples in (36b) and (36c). In both cases, the literal proposition is not informative enough to meet the hearer's expectation of relevance. Additionally, it seems clear that the minimal linguistic constraint on free enrichment (MLC) in (22) is satisfied as well, because the proposition in (38b) is a development of (38a) and the enriched concept [A PAINTER [FROM THIS VILLAGE]] is of the same semantic type as the original concept [A PAINTER]; both are semantically monadic predicates.

Hence, the contextualist account predicts that the literal proposition should be enriched to a point where the hearer's expectation of relevance would be satisfied. However, this is not possible in the case of (37), in contrast to the case of (36b) and (36c). This means that the standard conception of free enrichment as summarized in (23) gives a wrong prediction in the case of concepts expressed by predicate nominals.

It might be objected here that in the above example, since John's utterance to Lisa gives her no grounds for supposing that the guy is from their village rather than from somewhere else, she could not enrich the proposition one way or the other even if free enrichment were a technical possibility.[7] The objection might go further as follows. Consider a context in which Lisa knows that all the people coming to the meeting are inhabitant of her village but doesn't know who the particular guy in front of her is. John is aware of her concern and utters (37). Then, it might be claimed that John's utterance would express the enriched proposition in (38b). This means that if it were assumed that the domain at issue was just people from the village, the property concept encoded by the predicate nominal *a painter* could be enriched as in (38b).

To this objection, we can reply as follows. It is true that John's utterance of (37) in this context can be understood as communicating (38b). In this case, however, (38b) can be taken as a contextual implication of the

[7] This possible objection is pointed out by Robyn Carston (personal communication).

utterance, rather than the proposition expressed by the utterance. In fact, (38b) follows deductively from (38a) together with the assumption that Bill is from the village. And the latter assumption follows from the premise that the domain at issue consists of just people from the village, together with the premise that Bill, the guy John points out, belongs to that domain.

This alternative explanation in terms of contextual implication is preferable because it is consistent with the case in which a quantificational noun phrase appears in the subject position of a predicational sentence. To see this, consider the utterance of a quantified predicational sentence in (39).

(39) Everyone is a painter.

(40) a. EVERYONE [FROM THE VILLAGE] IS A PAINTER.
b. EVERYONE IS A PAINTER [FROM THE VILLAGE].

Suppose that, as before, the domain at issue consists of just people from the village. In this context, the utterance would be interpreted as (40a) rather than as (40b). Given the fact that the noun phrase *a painter* in (39) has the same semantic status as the one in (37), that is, both are predicate nominals in predicational sentences, it is natural to conclude that, in the case of (37) too, the additional element [FROM THE VILLAGE] does not apply to the property concept [A PAINTER]. Thus we conclude that if the assumption that the speaker is talking about people from the village is contextually available, the proposition in (38b) derives as a contextual implication of the utterance of (37). The objection mistakenly construes it as a proposition expressed by the utterance.

Argument from negation

In examples like (36a), the existence of free enrichment is easily detectable because it changes the truth value of a proposition; the literal proposition is false but the enriched proposition is true. In such cases, the enriched materials appear in the so-called *downward entailing contexts*, such as the scope of negation or the restrictor of a universal quantifier.[8] To

[8]Let C and C' be concepts such that the extension of C' is a subset of the extension of C. If a proposition $\varphi(C)$ entails a proposition $\varphi(C')$, C is said to appear in a downward entailing context in $\varphi(C)$.

test our claim, then, consider a case in which a predicate nominal appears in a downward entailing context. Suppose that Lisa and John go to a dinner party, where some painters are gathering. They are looking for a painter living in their village. Although Lisa happens to find a painter, she realizes that he is not an inhabitant of her village. Suppose, then, Lisa utters (41) to John.

(41) That guy is not a painter.

It is clear that John will regard her utterance as false, because the guy she pointed to is a painter. Note that if it were possible to interpret her utterance as communicating the proposition enriched as in (42), her utterance would be true.

(42) That guy is not a painter living in her village

However, John cannot interpret (41) in this way, even if he knows that Lisa is looking for a painter living in her village: it is quite impossible for John to regard her utterance as true. Note that the situation here is analogous to the case of (36a). In both cases, the literal proposition is obviously false. Thus the standard contextualist account predicts that the process of free enrichment should be automatically invoked so as to give a true proposition. However, this is not possible in the case of (41). The concept encoded by *a painter* in (41) cannot be enriched.

Argument from definite descriptions

It is well known that not only an indefinite description such as *a painter* but also a definite description such as *the king of France* and *Mary's husband* can appear as a predicate nominal in a predicational sentence.[9]

(43) a. John is not the king of France.
b. John became the king of France.
c. John is Mary's husband.

For example, the noun phrase *the king of France* in (43a) does not refer to any individual, but expresses a property. Note that a definite description

[9] See Fara (2001) and references given there.

appearing in a post-copula position can have a referential interpretation as well. Thus, by taking the noun phrase *the king of France* as referential, (43a) can be interpreted as expressing that John is not identical to the individual referred to by that noun phrase. This type of reading for (43a) is called an *identity reading.*[10]

Now consider the following sentence.

(44) John is the painter. [*PREDICATIONAL READING]

In usual contexts, while it is possible to interpret (44) as an identity sentence by taking the noun phrase *the painter* as referential, it is impossible to take (44) as a predicational sentence. This can be explained on the basis of the following hypothesis:

(45) A copular sentence of the form "*a* is the *F*" is acceptable as a predicational sentence if it is commonly assumed by the speaker and hearers that there is exactly one *F* in the world.

We usually assume that there are a lot of painters in the world. However, the use of a definite description *the painter* implies that there exists exactly one painter in the world, and this contradicts the common assumption.

Now consider the following sentence.

(46) John is the painter from Lisa's village. [okPREDICATIONAL READING]

[10]Given a copular sentence containing a definite description in the post-copula position, we can check whether it has a predicational reading or an identity reading by the following tests. If a sentence of the form "a is the F" has an identity reading, it is possible to use a pronominal phrase to refer back to the object introduced by the noun phrase *the F*. Thus, if we take the first sentence in (i) as an identity sentence, it is possible to use a pronoun *they* as in the second sentence.

(i) John is not the chairman of the committee. They are not the same person.

On the other hand, if we take the first sentence as a predicational one, it is impossible to use the pronoun *they* to refer back to both John and the chairman of the committee. It should be further noticed that (43a) has another reading, a *specificational reading*. On this reading, (43a) says the answer to the question "Which person is the chairman of the committee ?" is not John. We will discuss the specificational reading in the next section.

Suppose that there is exactly one painter from Lisa's village, and that this fact is commonly assumed by the speaker and hearers. Then, (46) is clearly acceptable as a predicational sentence, because the condition in (45) is satisfied. Note that even in the same setting, it is impossible to interpret (44) as a predicational sentence. If it were possible to enrich (44) as in (46), (44) would be acceptable as a predicational sentence, since the condition in (45) would be satisfied. However, the predicational reading of (44) is quite impossible. This counts as evidence that the property concept expressed by *the painter* in (44) blocks free enrichment.

From the above arguments, we claim the following.

(47) **The Semantic Constraint on Free Enrichment (SC).**
Free enrichment is blocked for property concepts, whereas it is allowed for object-directed concepts.

This claim diverges from the standard conception of Contextualism, according to which the process of free enrichment is directed by pragmatic considerations (relevance-seeking heuristics in the case of relevance theory), in accordance with the minimal linguistic constraints (MLC). Under this conception, we should be able to put additional material into any position in the logical form of a given utterance. Our arguments show that this is not the case. Thus, we claim that the Contextualist claim in (23) does not hold.

(23) **The standard conception of free enrichment.**
A process of free enrichment can take place if it observes the pragmatic constraint (PC) and the minimal linguistic constraint (MLC).

Note that we accept PC and MLC; thus our claim is that in addition to PC and MLC, the semantic constraint SC is at work for the process of free enrichment.

3.4 An argument against Indexicalism

We have argued that the process of free enrichment is blocked for concepts expressed by predicate nominals. It should be noted here that the same point does not hold for the other three types of pragmatic processes discussed earlier, namely, disambiguation, saturation, and *ad hoc* concept construction. It is easy to see that these processes can be applied to concepts expressed by predicate nominals, as shown in the following examples:

(48) a. This is a bat. [A FLITTERMOUSE] (Disambiguation)
b. John is the chairman. [THE CHAIRMAN OF THIS COMMITTEE] (Saturation)
c. He is an enemy. [AN ENEMY OF OUR GROUP] (Saturation)
d. Jane is a bulldozer. [BULLDOZER*] (*Ad hoc* concept construction)

Interpretation of utterances of each sentence would, in an appropriate context, include the bracketed element by using the respective pragmatic processes.[11] It should be noted that in contrast to the case of free enrichment, each process can apply to a property concept under the scope of negation.

(49) a. This is not a bat. [A FLITTERMOUSE] (Disambiguation)
b. John is not the chairman. [THE CHAIRMAN OF THIS COMMITTEE] (Saturation)
c. He is not an enemy. [AN ENEMY OF OUR GROUP] (Saturation)
d. Jane is not a bulldozer. [BULLDOZER*] (*Ad hoc* concept construction)

These facts show that free enrichment is crucially distinguished from the other three types of processes in that it is blocked for concepts in a predicative position.

From this, we can draw an argument against the indexicalist view that free enrichment is reducible to saturation. According to Stanley and Szabó

[11] In discussing whether indefinite descriptions can have pragmatic enrichment, Bach (2000) mentions the following example:

(i) Dr. Atkins is not [what I would describe as] a physician but a quack.

In our criterion, this additional bracketed element is not externally supplied to the concept [A PHYSICIAN], hence not caused by free enrichment. Rather, it is due to the adjustment of the concept [A PHYSICIAN] and hence is best classified as an instance of *ad hoc* concept construction.

(2000) and Stanley (2002), any nominal expression, whether it appears in an argument position or in a predicative position, contains a hidden domain variable, which restricts the extension of the nominal expression. However, what our argument shows is that predicate nominals never allow contextually restricted interpretations. The trouble for Indexicalism is that, if a predicate nominal contains a hidden domain variable, it is difficult to explain why contextual restriction as in the case of a noun phrase in argument position is not allowed for predicate nominals. For example, Indexicalism predicts that the predicate nominal *a painter* in (50a) has a domain variable D as in (50b), which restricts the extension of the nominal expression *painter*.[12]

(50) a. John is a painter.
b. John is a painter$_D$.
c. John is a painter [A PAINTER FROM THIS VILLAGE].

Thus, it wrongly predicts that the contextual restriction in (50c) is possible. The point is that Indexicalists like Stanley and Szabó (2000) and Stanley (2002) fail to notice the fact that the applicability of contextual restriction (i.e., free enrichment in our terms) is sensitive to the semantic function an expression plays in a proposition. In this respect, Stanley and Szabó's version of Indexicalism has a serious problem.[13]

3.5 Verbs and adjectives

Verb phrases can be regarded as expressing object-directed concepts, because they are naturally analyzed as expressions involving reference to events

[12]Indeed, Stanley and Szabó (2000) and Stanley (2002) posit a more complex form of hidden variable in a nominal expression in order to capture what Stanley and Szabó (2000: 242) called "quantified contexts." However, it does not matter here whether we take such a complicated form or the simple form as in (50b), since both predict the same reading in (50c) for (50a).

[13]The same difficulty applies to those who deny the existence of free enrichment by assuming that all cases of free enrichment could be reanalyzed as cases of *ad hoc* concept construction. Carston (2000) suggests this position without endorsing it. Recanati (2004: 25) also suggests this possibility.

or states.[14] This view is supported by the fact that an anaphoric pronoun can refer back to a verb phrase that represents an event or state. Thus, the pronoun *it* can refer to an event as in (51a) or a state as in (51b).

(51) a. It's snowing. It will continue until tomorrow.
b. They had many children. It was five years ago.

Given the hypothesis that the class of concepts that allow free enrichment coincides with the class of object-direct concepts in our sense, we predict that free enrichment can take place for concepts expressed by verb phrases. This prediction is borne out by the examples discussed in Section 2; see examples in (8), (9), (10), and (11). Thus, the verb phrase *is snowing* in (8a) and *had many children* in (9a), repeated below, can be regarded as expressing object-directed concepts that involve reference to an event of snowing and a state of having many children, respectively.

(8) a. It's snowing.
b. IT'S SNOWING [IN TOKYO] [AT TIME t].

(9) a. They got married and had many children.
b. THEY GOT MARRIED AND [THEN] HAD MANY CHILDREN.

Here, the enriched materials make these object-directed concepts more specific by means of temporal and locative modification.

Next, let us consider the following predicative constructions involving adjectives.

(52) a. This steak is raw.
b. John is tall.

It is well known that these constructions show a variety of linguistic underdeterminacy effects as in the following examples.

(53) a. This steak is raw [TO SUCH AND SUCH DEGREE].

[14]As is well known, Davidson (1967) argues that action verbs are best analyzed as involving existential quantification over events. Higginbotham (1985) and Parsons (1990), among others, generalize Davidson's approach to verbs in general, including state verbs.

b. John is tall [compared to his classmates/for a football player].

It should be noted that the processes at issue here are not instances of free enrichment. The process of modifying the degree of the application of a predicate as in (53a) is an instance of *ad hoc* concept construction, as discussed in Section 2.3. The process of providing the criterion of the application of a predicate, as in (53b), is an instance of saturation: without the criterion, the utterance of a sentence containing the adjectives would not deliver a truth-evaluable proposition. These processes are linguistically mandated or bounded in the sense discussed before, hence not purely pragmatic in our sense.

We assume that not only predicate nominals but also adjectives express property concepts. This accounts for why there is no context in which (52b) could be freely enriched as follows.

(54) John is [tall and clever]

If it were possible to interpret (52b) as in (54), it would be a genuine instance of free enrichment. In fact, (54) is a development of the logical form of (52b). However, the interpretation in (54) is not possible, even though we can easily manipulate the context in such a way that the literal interpretation of (52b) is not relevant enough so that there is good reason to derive the proposition in (54). The standard Contextualist account fails here just as in the case of predicate nominals.

One might argue that the following sentence is a counter-example to our claim.

(55) My husband is a gentleman [in my parent's home].

The interpretation of an utterance of (55), in an appropriate context, may include the bracketed element which seems to be provided on pragmatic grounds alone. Indeed, this is an instance of free enrichment. However, it should be noted that (55) could be paraphrased as in (56).

(56) In my parent's home, my husband is a gentleman.

It is doubtful that such an occurrence of *in my parent's home* is a genuine modifier of the nominal expression *gentleman*. This means that the free enrichment does not take place for the property concept GENTLEMAN. One plausible explanation is that the enriched element [IN MY PARENT'S HOME] restricts the concept associated with the entire verb phrase, namely, [IS A GENTLEMAN], which may be analyzed as involving reference to a certain state.[15]

The next question is why free enrichment is blocked for property concepts. Before tackling this important question, however, we first discuss some possible counter-examples to our claim. This may help clarify our position.

[15] Maienborn (2001) calls modifiers like *In my parent's home* in (56) "frame-setting" modifiers and semantically distinguishes them from standard adverbial modifiers.

4. Some challenges and the nature of free enrichment

In this section, we take up some apparent counter-examples to the claim that free enrichment does not take place for property concepts. We argue that they are based upon misunderstandings of the relevant notions and hence are not genuine counter-examples to our claim.

4.1 Specificational sentences

First, consider the following example.

(57) a. John is the bank robber.

b. JOHN IS THE BANK ROBBER [WHO WAS ARRESTED AT LEICESTER SQUARE YESTERDAY]

An utterance of (57a) in a proper context can be interpreted as expressing the proposition in (57b). This case might look like a counter-example to our thesis. However, (57) is not a predicational sentence but what is called a *specificational* sentence; the noun phrase *the bank robber* in (57a) works differently from the case of predicate nominals in predicational sentences. To see this, let us first explain the difference between predicational and specificational sentences. Roughly, a copular sentence of the form "NP_1 is NP_2" has a specificational reading if NP_2 stands for a predicate and NP_1 specifies the value which satisfies it.[1] Some examples are given in (58).

[1]See Higgins (1973), Declerck (1988), Nishiyama (1997, 2003) and Mikkelsen (2005) for more discussion on specificational constructions. Higgins (1973) calles the NP_2 in a specificational sentence a *superscriptional* noun phrase. Nishiyama (1997, 2003) calls it a

(58) a. John's article was the cause of the riot.
b. John's tie is what I don't like about him.

At a certain level of semantic representation, the noun phrase in the post copular position of a specificational sentence is associated with a WH-question. For instance, the noun phrase *the cause of the riot* in (58a) is closely associated with the WH-question, (59).

(59) What was the cause of the riot?

The whole sentence (58a) says the answer to this WH-question is John's article. In contrast, the predicate nominal in a predicational sentence is not associated with such a specificational WH-question. For instance, as noted in Section 3.1, the predicational sentence in (60) should not be regarded as an answer to the questions in (61).

(60) John is a linguist. (PREDICATIONAL READING)

(61) a. Who is a linguist?
b. Which person is a linguist?

Rather (60) can be regarded as an answer to the following kind of question:

(62) What is John?

Another characteristic which distinguishes specificational sentences from predicational ones is that unlike predicational sentences, the order of the two noun phrases in specificational sentences is reversible. Thus, the sentences in (58) can be paraphrased as follows.

(63) a. The cause of the riot was John's article.
b. What I don't like about John is his tie.

On the other hand, the order of the two noun phrases in (60) is not reversible:

(64) *A linguist is John.

In these two respects, there is a crucial difference between predicational and specificational sentences.[2]

noun phrase involving a variable (NPIV).

[2] A further example may help clarify the distinction between the two readings:

Having the distinction between the two types of copular sentences in mind, let us go back to the example in (57). Our claim that (57a) is not predicational but specificational is based on the following observations. To begin with, (57a) is not associated with the predicational question as in (65a), but with the specificational question as in (65b).

(65) a. What is John?
 b. Who is the bank robber?

The point is further confirmed by the fact that (57a) can be paraphrased as (66).

(66) The bank robber is John.

We can conclude that *the bank robber* in (57a) is not a predicate nominal expressing a property concept, and hence, that (57) is not a counter-example to our claim.

A possible confusion comes from the following kind of example: (67a) can easily be interpreted as (67b).

(67) a. John is a painter from a village.
 b. John is a painter from a village [NEAR HERE].

From this, it might be concluded that the concept expressed by a predicate nominal could be enriched. In this example, however, the process of free enrichment applies to the object-directed concept expressed by the noun phrase *a village*, which is a part of the property concept expressed by the

(i) What I don't eat is food for the dog. (Declerck 1988: 69)

(i) is ambiguous between a predicational reading and a specificational reading. On the predicational reading, (i) is a comment about a particular food identified as "something which I don't eat." (i) says of that food that it has the property expressed by *food for the dog*. Notice that here *food for the dog* is not a referring expression. On the specificational reading, (i) can be interpreted as (ii).

(ii) It is food for the dog that I don't eat.

Under this interpretation, *what I don't* eat in (i) is not referential. It contains a variable, which is supposed to be assigned a value. *Food for the dog* specifies that value.

whole noun phrase *a painter from a village*. Thus, this is not a counter-example to our claim either.

Another potential argument against our claim concerns the following sentence.[3]

(68) This is every student. [every student in my class]

The interpretation of an utterance of (68) may include the bracketed element which is provided by free enrichment. It should, however, be noted that *every student* in (68) does not express a property of the referent of *this*; indeed, it can be argued that (68) is not a predicational sentence but a certain type of identity sentence.[4] Therefore, we conclude that this is not a genuine counter-example too.

4.2 Hall's alleged counter-example

Hall (2008) presents some arguments against the claim that free enrichment is blocked for predicate nominals.[5] In this subsection, we discuss Hall's (2008) alleged counter-example and reject it by showing that her argument depends on some misconceptions of the relevant semantic properties of predicate nominals.

Before examining Hall's example, it should be pointed out that Hall seems to misunderstand the thesis at issue. Nishiyama and Mineshima (2006b) presented an example such as (69) as evidence to support the thesis in (70).

(69) a. That guy is not a painter.
b. That guy is not a painter [from our village].

(70) Free enrichment cannot take place for the concepts expressed by predicate nominals.

[3]This example is pointed out by Chris Tancredi (personal communication).

[4]See Doron (1988) for this view and further discussion on quantifiers in predicate position.

[5]Hall (2008) criticizes the view put forward by Nishiyama and Mineshima (2006b), which is an earlier version of the view we present here.

However, Hall (2008: 431) takes (69) as an example to support the claim in (71).

(71) Free enrichment cannot take place for the concepts expressed by indefinite descriptions.

It is clear, from the discussion so far, that the hypothesis defended by taking up an example like (69) is not (71) but (70). That is, the question is whether free enrichment can take place for predicate nominals, not for indefinite descriptions. Note that whether a noun phrase is an indefinite description or not solely depends on its form, i.e., whether it is of the form *an F*. By contrast, whether a noun phrase is a predicate nominal or not depends on the position it occupies in a sentence, more specifically, whether it appears in the post copular position of a predicational sentence.

Indeed, there are counter-examples to (71), which are already discussed in (36) at Section 3.2. The relevant examples are repeated here:

(72) a. A painter died. [A PAINTER LIVING IN THIS VILLAGE]

b. She gave presents to some children but not to others. [SOME CHILDREN AT THE PARTY]

In these examples, indefinite noun phrases appear in argument position and hence express object-directed concepts. In this case, it would be easy to imagine a context in which an utterance of each sentence is relevant when it is interpreted as including the bracketed element. Thus, a process of free enrichment *can* take place for the indefinite description appearing in argument position. This fact is compatible with the claim in (70) but constitutes a clear counter-example to the claim in (71).

Contrary to what she claims, however, Hall's argument (as we will discuss below) could be viewed as an attempt to show the thesis (70) does not hold for some case. Setting aside the above problem, then, let us move on to examine her argument.

Hall (2008: 431) considers the following example:[6]

[6] Hall (2008) ascribes this example to Richard Breheny.

(73) **Context**: At a departmental party for professors and students. The professors all attend evening classes at a different college, so they are students too, and this is (mutually) known to the interlocutors, who are trying to tell if people at the party are students in the department, or professors. The speaker points at a professor and utters (74).

(74) He is not a student.

(75) a. He is not a student.
b. He is not a student [in this department].

Hall claims that the utterance of (74) in this context would be judged to be *true*, even though all professors attend evening classes and the speaker and the hearer know it. According to Hall, this data can naturally be explained if we assume that the free enrichment to the concept a student takes place so that the utterance of (74) is interpreted as (75b) rather than (75a). Hall's argument here can be summarized as follows.

1. In the context (73), the utterance of (74) can be judged as true.[7]
2. If the utterance of (74) expresses the proposition in (75a), it would be judged *false*, given the fact that all the professors attend evening classes and the speaker and the hearer share this fact.

[7]Strictly speaking, Hall says that an utterance of (74) is "uniformly judged true"(p.432). Surely, it is natural to take an utterance of (74) to be true in the context (73). However, it is possible to judge the utterance to be false; for example, the hearer who knows that the professor in question studies at evening classes can respond to the speaker of (74) in the following way.

(i) No, he is a student. He studies at evening classes.

To defend Hall's position, it might be replied that the interpretation which takes (74) to be false is due to the fact that (74) is interpreted as in (ii):

(ii) He is not a student [in evening classes].

However, whether this reply is true or not, Hall's argument should be based on the claim that the utterance of (74) *can* be true, rather than "uniformly judged true." Such a modification seems not to affect the main point of her argument.

3. Hence, the utterance of (74) does not express the proposition expressed as in (75a).

4. On the other hand, if the utterance of (74) expresses the proposition in (75b), it would be judged as *true*, on the basis of the same assumption.

5. Therefore, it is reasonable to take the utterance of (74) to express the proposition in (75b).

The problem of this argument is that there are several ways to interpret the proposition in (75b); indeed, the notation in (75b) allows several different interpretations. To begin with, the concept STUDENT has two possible interpretations: (i) it may be interpreted as an *unsaturated* concept, i.e., a concept like ENEMY or CHAIRMAN which requires saturation, or (ii) it may be interpreted as a saturated concept, i.e., a concept like PAINTER which does not require saturation. In the former case, the proposition in (75b) would be derived from the logical form as shown in (76) in terms of saturation, rather than free enrichment.

(76) HE IS NOT A STUDENT IN *e*.

Thus, if Hall's example is concerned with *free enrichment*, the concept STUDENT needs to be interpreted as a saturated concept. This interpretation, however, leads to some inconsistency in Hall's argument. The thesis that the proposition in (75b) is taken to be true crucially depends on the interpretation of the negation in (75b). To see the point, consider first the following example, in which a saturated noun *painter* appears.

(77) He is not a painter from this village.

Depending upon how to interpret the focus of negation, (77) has at least three readings.

(78) a. HE IS NOT [A PAINTER FROM THIS VILLAGE]$_F$.
b. HE IS NOT A PAINTER [FROM THIS VILLAGE]$_F$.
c. HE IS NOT [A PAINTER]$_F$ FROM THIS VILLAGE.

Here the focused part is indicated by [...]$_F$. Now consider the following conversation:

(79) A: He is a painter from our village.

B: No, he is not a painter

In this case, it is natural to interpret B's utterance as (78c), which can be paraphrased in the following way.

(80) He is from this village but is not a painter.

This means that B's utterance in (79), if enriched, implies (81).

(81) He is not a painter.

The same thing can be said to Hall's example. If the concept student is a saturated concept, the proposition expressed by (74) in that context has the focus structure as indicated in (82a), and can be paraphrased as (82b). Thus, it entails the proposition in (82c).

(82) a. He is not [a student]$_{F}$ in this department

b. He is in this department, but he is not a student.

c. He is not a student. [=(75a)]

However, if the utterance of (74) implies (82c), Hall's argument is committed to the claim that the utterance of (74) would be judged to be *false* after all. In short, if the concept student is taken to be a saturated concept and the focus of negation is on this concept, then the reading in which (74) is construed to be true cannot be salvaged by appealing to free enrichment.

In our view, nouns such as *student* and *professor* are semantically ambiguous: they may be interpreted as expressing saturated concepts or unsaturated concepts. Accordingly, there are two possible logical forms associated with (74).

(83) a. He is not a student

b. He is not a student in e

Under the interpretation in (83a), the utterance of (74) is false.[8] By contrast, under the construal in (83b), there are two possible ways of supplying a value to e, as in (84a) and (84b), respectively.

[8]This interpretation is one that the hearer who utters (i) of footnote 7 would typically have in mind.

(84) a. He is not a student in [this department]
b. He is not a student in [evening classes]

The natural reading in this context is the one in (84a); the utterance of (74) would then be true. Thus, in our view, the reading in which the utterance of (74) is true is captured in terms of saturation, rather than free enrichment. To sum up, the two readings, namely, the reading in which the utterance of (74) is true and the reading in which it is false, are compatible with our position: in the former case, free enrichment does not take place since the concept expressed by *a student* is a saturated concept. In the latter case, a pragmatic process can take place but it is not an instance of free enrichment but saturation. Hence we conclude that Hall's example is not a genuine counter-example to our thesis.

Finally, let us see Hall's positive proposal on this matter. Based on the above argument, Hall (2008) suggests the following hypothesis.[9]

(85) **Hall's pragmatic constraint (I)**
Free enrichment can take place for given/backgrounded contents, while it is blocked for at-issue contents.

The distinction between given and at-issue contents could be interpreted in several ways, but Hall does not settle one interpretation. One likely interpretation is that a given content corresponds to a question under discussion, whereas an at-issue content corresponds to an answer to it. Under this construal, however, there are clear counter-examples to Hall's hypothesis. Consider:

(86) To be a painter is fun.

It seems to be plausible to regard *a painter* to be part of the given content of this sentence, since it is part of the subject noun phrase of a predicational sentence. However, if our argument in the previous section is correct, it is a predicate nominal and hence cannot be subject to free enrichment. On the other direction, consider Mary's utterance of (87b) given the question under discussion indicated in (87a)

[9] Hall (2008) presents and defends some additional pragmatic constraints, which we will discuss in Section 6.

(87) a. **Question under discussion**: How did you open the door?

b. Mary: I took out the key and opened the door [BY USING THE KEY].

In (87b), the enriched element as indicated in the brackets answers the question under discussion. This shows that free enrichment can take place for for at-issue contents (construed as an answer to a question). Thus, we conclude that the Hall's hypothesis (85) fails to account for the distribution of free enrichment.

4.3 The nature of free enrichment

We have argued so far that free enrichment is blocked for property concepts, and that the standard contextualist view, including the current framework of relevance theory, fails to explain this fact. As we saw in Section 2.2.1, one major advantage of the standard contextualist position over the indexicalist position is that it dispenses with covert variables and hence can have explanatory power; it sets out to explain why free enrichment can take place in terms of pragmatic considerations, without semantic resources such as covert variables. The situation is analogous to one in which conversational implicatures are derived from propositions expressed using some pragmatic principles; in this case it is generally accepted that it is desirable to handle conversational implicatures without complicating semantic properties of the linguistic expressions in question.

Now the claim that there is a *semantic* constraint on free enrichment may face a similar challenge; it would lack explanatory force if the semantic constraint in question was simply stipulated. To meet this challenge, we will argue, in this subsection, that the applicability of free enrichment is explained in terms of the difference in semantic function between object-directed concepts and property concepts. That is, we will not only argue that there is a semantic constraint on free enrichment but also attempt to answer *why* such a constraint plays a role in deriving the proposition expressed by an utterance. More specifically, we will try to answer why free enrichment is applicable to object-directed concepts, while it is blocked for property concepts. In what follows, we will outline the answer to this question.

To begin with, it is important to recognize the difference in function between object-directed concepts and property concepts. Recall typical constructions that involve expressions associated with object-directed concepts.

(88) {A/Every/Some} painter disappeared.

Here the noun phrase *a/every/some painter* occurs in argument position of the sentence and hence the concept PAINTER expressed by the nominal expression *painter* is an object-directed concept. As noted in Section 3.1, such concepts are typically associated with objects in the world; thus, in the case of (88), it is meaningful to ask which painter or painters the speaker is talking about by using the noun phrase *a/every/some painter*. Note also that, as discussed in Section 3.1, the occurrence of an object-directed expression allows us to use anaphoric expressions to refer back to the entity or entities introduced by that expression.

As for object-directed concepts, then, we can argue in the following way.

(R1) The function of object-directed concepts is to pick out an object or a set of objects that a speaker intends to talk about.
(R2) The expression the speaker uses to indicate an object-directed concept is just a hint or a clue that might help the hearer to pick out the object or the set of objects.
(R3) Hence, the hearer has to undertake the process of adding conceptual material that would provide information sufficient to pick out the intended object. This is nothing but the process of free enrichment.
(R4) For the purpose of picking out objects, a more specific concept with a restricted range of application must be provided by the process of adding some conceptual material; to use the terminology of relevance theory, the process in question is a *narrowing* of the encoded concept.

It is useful to restate the claim in (R4) as an additional constraint on free enrichment.

(89) The process of free enrichment must be a *narrowing* process.

On the standard Contextualist view, free enrichment takes place in order to make sense of *what proposition the speaker intends to express*, that is, to make sense of the overall proposition expressed by the speaker's utterance or, more generally, the speaker's communicative intention. In our view, the role of free enrichment is more specific: it takes place in order to specify *what object or objects the speaker intends to talk about*; in other words, the role of free enrichment is to make sense of the speaker's *referential intention*.

Now let us turn to the case of property concepts. Consider a typical instance of predicational sentences:

(90) Mary is a painter.

In this sentence, the noun phrase *a painter* is a predicate nominal and the concept PAINTER expressed by the nominal expression *painter* is a property concept (i.e., non-object-directed concept). In this case, it is meaningless to ask which painter the speaker intends to talk about by using the predicate nominal *a painter*. Furthermore, as noted in Section 3.1, a personal pronoun such as *he* or *she* cannot be anaphoric on this predicate nominal (cf. (29) discussed at page 291).

In the case of property concepts, then, we can argue as follows.

(P1) The function of property concepts is not to pick out objects but to indicate a property ascribed to an object.

(P2) Accordingly, it does not require the hearer to supply some additional material that would provide information for picking out objects.

(P3) In order to interpret a property expression, the only thing the hearer has to do is determine what concept it expresses in a given context.

(P4) In order to do this, the hearer has to decode the linguistic expression used, and if necessary, to undertake the other three types of pragmatic processes, namely, disambiguation, saturation, and *ad hoc* concept construction.

(P1) says that the function of a property concept in a proposition is to *predicate* something of an object introduced by some other part of the proposition. Object-directed concepts, by contrast, play a different function: the concepts themselves point to objects in the world. Free enrichment is special in that it is sensitive to this difference in the semantic function between object-directed and non-object-directed concepts. In short, a process of free enrichment can only apply to a concept that points to objects, and its effect is to narrow down that concept. Recall here that as shown by the examples in (48) at page 301, the other three types of pragmatic processes, namely, disambiguation, saturation, and *ad hoc* concept construction, operate on a concept independently of its semantic function in a proposition.

Our conception of free enrichment may be clarified in terms of a model of how the various pragmatic processes work in the course of the derivation of a proposition expressed. According to this model, disambiguation, saturation, and *ad hoc* concept construction, on the one hand, and free enrichment and reference assignment, on the other, work at different stages in the cognitive process of deriving the proposition expressed by an utterance. More specifically, we hypothesize that there are two different phases in the process of deriving the proposition expressed. The first phase consists of the process of what we call *concept-determination*, namely, the process of determining the concept expressed by a linguistic expression; such a process takes place independently of the semantic function a concept plays in a proposition. Pragmatic processes performed in the phase of content-determination include disambiguation, saturation, and *ad hoc* concept construction. The overall process of determining a concept expressed is shown in Figure 4.1.

The second phase consists of the process of what we call *reference-determination*, namely, the process of determining the reference of an object-directed concept provided as an output by a concept-determination process. The pragmatic processes pertaining to reference-determination include free enrichment and reference assignment, as pictured in Figure 4.2.

It is worth noting that under the standard conception of relevance theory (cf. Carston 2004), both (91) and (92) are regarded as cases of saturation.

(91) a. I like John's book. [THE BOOK WRITTEN BY JOHN]
b. This paper is too long. [TOO LONG FOR PUBLICATION IN THIS JOURNAL]

(92) a. John/He/That guy is young. [JOHN$_x$ / HE$_x$ / THAT GUY$_x$]
b. The murderer is insane. [THE MURDERER$_y$] (referentially used

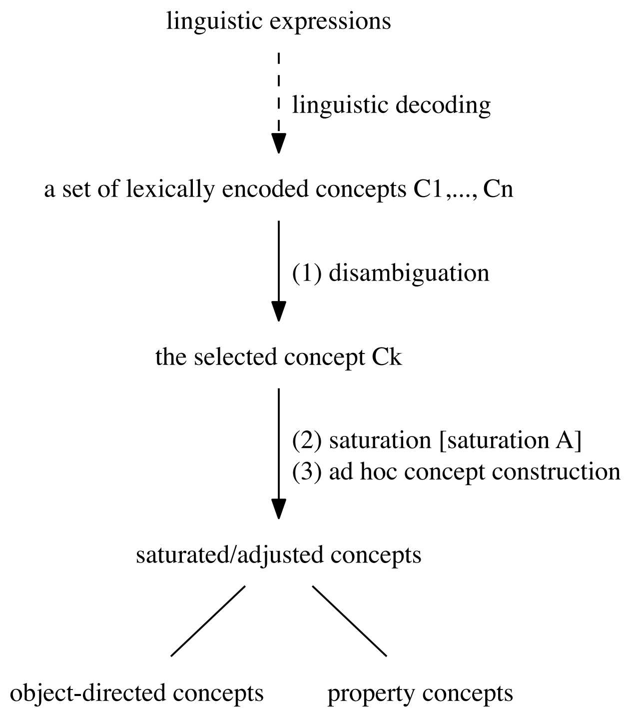


Fig. 4.1 A model of concept-determination processes.

descriptions)

In our view, there is an important difference between the examples in (91) and those in (92). In (91), some concepts are contextually provided in order to determine complete propositions. Thus, in the case of (91a), some relation R that holds between THE BOOK and JOHN is contextually provided. In the case of (91b), a concept that fills the gap X in [TOO LONG FOR X] is supplied. We can say that "saturation" in (91) is an instance of the process of concept-determination. On the other hand, the examples in (92) are instances of reference assignment, where indices like x, which point to particular objects in the world, are assigned to concepts expressed

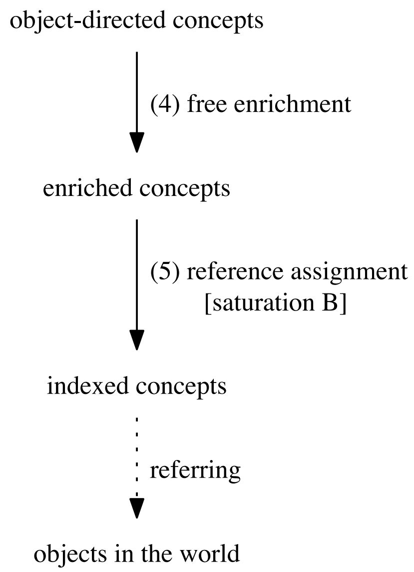


Fig. 4.2 A model of reference-determination processes.

by referential expressions. To distinguish these two types of "saturation" processes, we call the processes operating in (91) "saturation" (Saturation A), and those in (92) "reference assignment" (Saturation B).

It should be noted that saturation in our sense (i.e., Saturation A) can be applied to property concepts, as shown in (93).

(93) a. This is John's book. [A BOOK WRITTEN BY JOHN] (predicational reading)
b. He is an enemy. [AN ENEMY OF OUR GROUP] [= (48c)]

In contrast, reference assignment (i.e., Saturation B) cannot be applied to property concepts. Relevant examples are the following.

(94) a. John is that guy. [THAT GUY_x]. (identity reading)
b. He is John Smith. [(i) JOHN SMITH_x (identity reading) / (ii) A MAN NAMED "JOHN SMITH" (predicational reading)]

The only possible reading of (94a) is the identity reading, where the noun phrase *that guy* functions as a referential noun phrase. The example in (94b), where a proper name *John Smith* occurs in a predicative position, can be interpreted either as having the identity reading or the predicational reading. In the case of the predicational reading, the noun phrase *John Smith* does not need to be indexed, since it does not purport to refer to an individual at all. Indeed, it can be paraphrased as *a man named "John Smith,"* which stands for a property ascribed to the individual referred to by the subject noun phrase.[10] This contrast would suggest that saturation and reference assignment are different kinds of pragmatic processes which work at different stages of derivation: like disambiguation and *ad hoc* concept construction, saturation works in the phase of concept-determination, whereas like free enrichment, reference assignment works in the phase of reference-determination.

[10]For a discussion on proper names in a predicative position, see Nishiyama (2003).

5. Free enrichment and the overgeneration problem

5.1 A problem of "overgeneration"?

Stanley (2005) presents the following argument against those who admit optional pragmatic processes like free enrichment. An utterance of (95a) can communicate the proposition expressed by (95b), but never the proposition expressed by (95c).

(95) a. Every Frenchman is seated.
b. Every Frenchman in the classroom is seated.
c. Every Frenchman or Dutchman is seated.

Stanley argues that if the process of free enrichment were constrained only by general pragmatic principles, it would be a mystery why an interpretation like (95c) is blocked, since in certain contexts there might be a good pragmatic reason to derive (95c) from the utterance of (95a). According to Stanley, if we admit the process of free enrichment, there would be numerous sentences that would be assigned interpretations that they do not actually have. This is what he calls the problem of "over-generation" for the free enrichment view. Thus Stanley (2005: 243, footnote 13) says,

> Indeed, in the thousands of pages that have been written over the last decade arguing for pragmatic (non-semantic) accounts of a wide range of apparently semantic phenomena, I am not aware of a single attempt to provide a response to the threat of over-generation to pragmatic theories. Indeed, I am not even

> aware, aside from passing footnote references, [of] a discussion of the over-generation threat facing such theories.

On the position of Indexicalism defended by Stanley, the problem of "over-generation" does not arise, simply because it is assumed that there is no process of "free enrichment," and hence, that all elements in the truth-conditional content of an utterance are the result of assigning values to the elements in the logical form of a sentence uttered. As we have seen so far, Stanley's indexicalist view nonetheless faces the challenge from Contextualism: appealing to covert variables tends to be non-explanatory and hence less attractive than the contextualist view in which how free enrichment works could be explained in terms of pragmatic notions.

Now we agree that the standard contextualist view has difficulty in accounting for the problem of "over-generation", since in the standard view there is almost no linguistic constraint upon the applicability of free enrichment. However, this does not mean that we must deny the existence of free enrichment. Indeed, given our semantic constraint on free enrichment, we can avoid the problem of "over-generation," and defend the existence of free enrichment. Moreover, as we argued in Section 4.3, the existence of a semantic constraint on free enrichment is not stipulated but can be explained in terms of differences in function between object-directed and non-object-directed concepts.

Indeed, Stanley's example above is explained in terms of the constraint we already observed. Recall that we proposed the following constraint on free enrichment in Section 4.3.

(89) The process of free enrichment must be a narrowing process.

To see how this constraint works to avoid the problem of "over-generation," look at the examples in (95) above. In relevance-theoretic terms, the process of modifying the concept [FRENCHMAN] in (95b) is an instance of narrowing, namely a process of restricting the extension of a concept, whereas the process in (95c) is an instance of loosening, namely a process of broadening the extension of a concept. In fact, the interpretation in (95c) does not contribute to the determination of what the speaker intends to talk about.

Thus, the constraint in (89) correctly predicts that the reading in (95b) is available, while the reading in (95c) is not.

5.2 Hall's pragmatic account

The semantic constraint on free enrichment we propose here is not assumed within the standard framework of relevance theory. Thus, Carston (2002a: 40) claims:

> [A]s well as not uniquely determining the objects they can be used to refer to, natural language expressions seem to be intrinsically underdetermining of the properties and relations they may be used to predicate of an object.

This means that standard relevance theory does not rely on the semantic constraints on free enrichment to avoid Stanley's over-generation problem. Recently, however, Hall (2008) has defended standard relevance theory against Stanley's objection, arguing that an interpretation like (95c) can be blocked by general pragmatic considerations alone. Thus, Hall claims:

> In principle, any local enrichments may be possible, but since enrichment, like any pragmatic process, will take place only as far as it has some worthwhile effects, it should be possible to predict, for any given utterance, how much enrichment will take place [...]." (Hall 2008: 452–453)

In this section, we argue that Hall's pragmatic account has some serious problems. In Section 5.2.1, we take up cases of the over-generation of disjunctive elements, and in Section 5.2.2, we consider cases of conjunctive elements.

5.2.1 Disjunction

Hall's pragmatic solution for the over-generation problem of disjunctive elements as in (95c) is based on the following claim:

(96) **Hall's pragmatic constraint (II)**
Any pragmatic inference that derives a less informative proposition than its premise(s) is blocked since it lacks a motivation for expending extra effort.

Her argument runs as follows (Hall 2008: 451–452):

A1. A pragmatic inference from (95a) to (95c) requires *or*-introduction.

A2. *or*-introduction always has a trivial result, since the output proposition, schematically represented as P *or* Q, is considerably less informative than the input proposition P.

A3. Pragmatic inference involves the expenditure of processing effort, so it requires some motivation (e.g., the lack of expected informativeness or relevance); in the absence of any motivation for expending extra effort, pragmatic inference won't get off the ground.

A4. Any inference that derives a less informative proposition than its premise(s) lacks a motivation for expending extra effort.

A5. Therefore, the inference from (95a) to (95c) is blocked.

This account has several problems. The first one is concerned with the notion of "*or*-introduction" in A1. It should be noted that the rule of *or*-introduction in the standard sense is a logically valid inference rule, which takes an arbitrary proposition as premise, and derives its disjunction with any other arbitrary proposition as conclusion (cf. Sperber and Wilson 1986/95: 96).

(97) *or*-introduction
Input: P
Output: P *or* Q

However, note that (95c) cannot be paraphrased as (98).

(98) Every Frenchman is seated or every Dutchman is seated.

Thus, the pragmatic inference from (95a) to (95c) is not a case of *or*-introduction in the sense of (97). One might argue that "*or*-introduction" in A1 should be understood as "*or*-enrichment" in the following sense.

(99) *or*-enrichment is a general pragmatic process that takes an arbitrary proposition as input and derives the one enriched with a disjunctive element somewhere in the proposition.

Accordingly, A1 and A2 should be replaced by A1′ and A2′, respectively.

A1′. A pragmatic inference from (95a) to (95c) requires *or*-enrichment.

A2′. *or*-enrichment always has a trivial result, since the output proposition is considerably less informative than the input proposition.

However, there are counter-examples to A2′. Recall Stanley's examples in (95a) and (95c).

(95) a. Every Frenchman is seated.
c. Every Frenchman or Dutchman is seated.

Here, (95a) does not entail (95c); rather, (95c) entails (95a). Note that according to the standard definition of informativeness, proposition P is more informative than proposition Q if and only if P entails Q but Q does not entail P, i.e., P asymmetrically entails Q. If we adopt this definition of informativeness, then it follows that the output proposition (95c) is more informative than the input proposition (95a). More generally, adding a disjunctive element to a concept appearing in a downward entailment context such as the restrictor of a universal quantifier and the antecedent of a conditional gives rise to a more informative proposition. Some other counter-examples to A2′ are given in (100) and (101).

(100) a. If John met a painter, I will be surprised.
b. If John met a painter or a writer, I will be surprised.

(101) a. Everyone who meets a painter will be disappointed.
b. Everyone who meets a painter or a writer will be disappointed.

Here the propositions in (b) are more informative than the propositions in (a); in these cases, *or*-enrichment does not weaken but strengthen the proposition expressed. We conclude that even if we replace A1 and A2 with

A1′ and A2′, Hall's pragmatic account still cannot block the over-generation caused by disjunctive elements.

The second problem with Hall's pragmatic account is that the claim in A4, repeated here, is simply false.

A4. Any inference that derives a less informative proposition than its premise(s) lacks a motivation for expending extra effort.

There are two cases in which a pragmatic inference derives a less informative proposition but has a motivation for expending effort. The first one is concerned with free enrichment. Consider the following examples.

(102) a. Every window is open.
b. Every window in Mary's room is open.

(102a) can express the proposition in (102b) via free enrichment. But (102b) is less informative than (102a), since (102a) entails (102b) while (102b) does not entail (102a).

The second type of counter-examples to A4 is concerned with *ad hoc* concept construction. Consider (103):

(103) a. I met a Frenchman.
b. I met a Frenchman or Belgian. (cf. Hall 2008: 451)
c. I MET A FRENCHMAN*
d. [FRENCHMAN*] = [FRENCH-SPEAKER]

(103a) can be loosely interpreted as (103b), when French-speakers are relevant in the context. This interpretation is an instance of *ad hoc* concept construction, since the similarity between the concept FRENCHMAN and BELGIAN (i.e., the property of being a French-speaker in this case) is crucial for deriving the loosened interpretation. Note that (103b) is less informative than (103a), because (103a) asymmetrically entails (103b). Since A4 is claimed to hold for any kind of pragmatic process, it fails to provide the reason why the disjunctive interpretation is possible in the case of *ad hoc* concept construction, whereas it is impossible in the case of free enrichment.

This would suggest that the difference between these two cases cannot be explained by general pragmatic considerations alone.

In contrast to Hall's pragmatic constraints, our semantic constraints on free enrichment, repeated below, can account for the phenomena discussed so far.

(104) **Semantic Constraints on Free enrichment:**

a. Free enrichment is applicable to object-directed concepts, but it is blocked for property concepts in general.

b. If free enrichment is applicable, it must be a narrowing process.

Free enrichment is allowed in (102b), since the nominal *window* in *every window* expresses an object-directed concept and the process in question is a narrowing process. On the other hand, the interpretation in (103b) is an instance of *ad hoc* concept construction, hence the constraint in (104b) does not apply to this case.

5.2.2 Conjunction

Stanley also presents an example involving conjunction as a case of overgeneration by free enrichment (Stanley 2002: 165–6). Suppose that Bill utters (105b) in the context shown in (105a).

(105) a. Context: Everyone who likes Sally likes his mother.

b. Bill: Everyone likes Sally. (cf. Stanley 2002: 165)

Even in the context of (105a), Bill's utterance in (105b) could not communicate the proposition in (106).

(106) Everyone likes sally and his mother.

According to Stanley, however, if there were a process of free enrichment, then it would be possible to utter (105b) and thereby successfully deliver (106) as an explicature. But this is not possible. Thus, he concludes that the hypothesis that there are such pragmatic processes should be rejected. Hall (2008) takes up this example and argues that the alleged case of overgeneration can be avoided using general pragmatic constraints.

Hall's account is based on the following constraint.

(107) **Hall's pragmatic constraint on free enrichment (III)**
If an assumption (developed from the logical form) is needed as a premise in the derivation of further intended aspects of meaning, then it cannot be developed any further at the level of proposition expressed. (Hall 2008: 447)

Then Hall's account goes as follows.

1. The proposition everyone likes Sally is needed as a premise to an inference process together with the context given in (105a) to derive the proposition everyone likes his mother.
2. Thus, the constraint in (107) predicts that the proposition everyone likes Sally cannot be developed any further at the level of proposition expressed.
3. Hence, Bill's utterance in (105b) cannot express the proposition in (106).

In this case, Hall's pragmatic constraint in (107) successfully explains why adding a conjunctive element as in (106) is blocked.

However, there is a more difficult case, as pointed by Hall herself. Consider Bill's utterance of (108c) in the context of (108a) and (108b).

(108) a. Context: If John likes Sally and his mother, then he likes dominant women. (*If P and Q then R*)
b. Independently motivated assumption: John likes his mother. (Q)
c. Bill: John likes Sally. (P)

(109) #Explicature: John likes sally and his mother. (P *and* Q)

Even in the context of (108a), Bill's utterance could not express the proposition in (109). But the constraint in (107) predicts that it could, since the proposition in (109) is needed as an input to an inference process together with the context in (108a) to derive the proposition John likes dominant women (R).

To solve this problem, Hall considers two possible ways of deriving the same implication (Hall 2008: 448ff.), which we call "Derivation A" and "Derivation B."

- Derivation A starts with the logical form P and the independently motivated assumption Q. Then the explicature, P *and* Q, is derived via free enrichment. Since the context contains the assumption *If* P *and* Q *then* R, the contextual implication R is derived by an application of *modus ponens*.

- Derivation B starts with the logical form P, and the explicature is the same proposition P. Since the context contains the assumption *If* P *and* Q *then* R, the contextual implication *If* Q *then* R can be derived by an application of *conjunctive modus ponens* as shown in (110) below. Then, given an independently motivated assumption Q, one can obtain the desired contextual implication R via modus ponens.

(110) Conjunctive modus ponens (cf. Sperber and Wilson 1986/95: 99)

Input: (i) If P and Q, then R
(ii) P

Output: If Q then R

It has been argued within relevance theory that conjunctive modus ponens is psychologically plausible, while introduction rules, including the rule of *and*-introduction, are not; that is to say, introduction rules play no part in the spontaneous deductive processing of information (cf. Sperber and Wilson 1986/95: 95–100). Based on these claims, Hall (2008) argues that Derivation B is psychologically more plausible than Derivation A.

Hall's argument here can be summarized in the following way.

1. Given the context in (108a) (= If P and Q, then R), the proposition John likes Sally (= P) is needed as a premise in the derivation via conjunctive modus ponens to yield the proposition If John likes his mother, then he likes dominant women (= If Q then R).

2. Then, the pragmatic constraint in (107) implies that the proposition John likes Sally cannot be developed any further at the level of proposition expressed.

3. Hence, Bill's utterance in (108c) cannot communicate (109) as a proposition expressed (i.e., as an explicature).

Now we can see that the general pragmatic constraint in (107) implies the following constraint as a corollary.

(111) In any context, a proposition P cannot be developed into a proposition that is logically equivalent to P *and* Q at the level of proposition expressed.

However, there are two problems with Hall's pragmatic account. Firstly, consider Bill's utterance of (112c) in the context of (112a) and (112b).

(112) a. Context: If someone likes Sally and his mother, then people will be surprised.
b. Independently motivated assumption: Everyone likes his mother.
c. Bill: Someone likes Sally.

(113) #Explicature: Someone likes Sally and his mother.

Even in the context of (112a), Bill's utterance in (112c) could not express the proposition in (113). But the pragmatic constraint in (107) predicts that it could, since the proposition in (113) is needed as an input to an inference process together with the context in (112a) to derive the proposition PEOPLE WILL BE SURPRISED. Note here that Hall's derivation using conjunctive modus ponens cannot apply to this case, since (113) cannot be paraphrased as a propositional conjunction as shown in (114).

(114) Someone likes Sally and someone likes his mother.

Thus, Hall's revised solution is not enough to avoid the over-generation problem for conjunctive elements.

Secondly, if our claims in the previous sections are correct, free enrichment can never take place for property concepts. However, Hall's pragmatic constraint in (107) fails to account for this type of restriction. If there was such a process of enriching a property concept, it would be a process operating on a constituent of a proposition, not a process going beyond the level of full proposition. Thus, if there was such a process of enriching the concept PAINTER in (115a) so as to obtain the proposition in (115b), it would be a process that *develops* a given logical form.

(115) a. JOHN IS A PAINTER.
b. JOHN IS A PAINTER [LIVING IN THIS VILLAGE].

This case is structurally similar to the standard case of nominal restriction, i.e., the case of enriching the concept PAINTER in (116a) to derive the proposition in (116b).

(116) a. {A/EVENRY} PAINTER DISAPPEARED.
b. {A/EVENRY} PAINTER [LIVING IN THIS VILLAGE] DISAPPERED.

To use Hall's terminology, the incorporation of an additional material in both cases arises from *local* development, as opposed to *global* inference. If our argument in Section 3 is correct, however, there is an important contrast between these two cases: enriching an object-directed concept as in (116) is possible, whereas enriching a property concept as in (115) is not. If the constraint in (107) does not allow free enrichment to apply to property concepts, then it would lead to the undesirable consequence that free enrichment does not take place for nominal concepts at all, whether they appear in argument position or in predicative position. We can conclude that the type of constraint on free enrichment as exemplified in (115) is not accounted for in terms of purely pragmatic principles such as the one in (107).

Note that our constraints correctly block the interpretations in (106), (109), and (113), since in these cases, the additional elements do not narrow down object-directed concepts.

(106) #Everyone likes Sally and his mother.

(109) #John likes Sally and his mother.

(113) #Someone likes Sally and his mother.

We thus conclude that Hall's pragmatic approach to the "overgeneration" problem faces serious troubles, and that our semantic account is preferable both on conceptual and empirical grounds.

6. Summary and Conclusion

In the final section, we will discuss how Contextualism and the Underdeterminacy Thesis can be properly understood given the existence of the semantic constraints on free enrichment.

As we argued in Section 2.4, the semantic constraints on free enrichment we proposed are not assumed in the standard framework of Contextualism, including relevance theory. However, to claim this is not to commit ourselves to Indexicalism. Unlike indexicalists, we do not deny the existence of free enrichment. Thus, we propose an alternative version of Contextualism, which is compatible with the claim that free enrichment can never intrude into the position occupied by a property concept. To clarify our position, it is worth reconsidering how the Underdeterminacy Thesis, one of the most important claims in relevance theory, should be understood given our previous argument. As noted in Section 2, the Underdeterminacy Thesis is characterized as follows (cf. Carston 2002a: 19–20):

> **The Underdeterminacy Thesis (UT)**: The linguistically encoded meaning of a sentence used underdetermines the proposition expressed by the utterance.

UT claims that there are considerable pragmatic tasks involved in arriving at the proposition the speaker intends to express. Thus, anyone who accepts **UT** is committed to the following claim.[1]

[1] In Section 2, the position rejecting **UT** is called *Literalism*, following the terminology of Recanati (2004). According to Literalism, the proposition expressed by an utterance is fixed by the linguistic rules independently of any pragmatic consideration. As noted in Section 2, it is difficult to maintain Literalism, because even the tasks of disambiguation and saturation require some pragmatic consideration.

(117) **Non-literalism**: The proposition expressed by the utterance of a sentence S is not solely determined by the linguistic rules associated with S. Some pragmatic processes are required in order to determine the proposition expressed by an utterance.

However, there is a vagueness about the word "pragmatic processes." Accordingly, there are at least three ways of interpreting **UT**, depending on how to understand the extent of the required pragmatic processes. The first reading is what we called "Indexicalism" in Section 2:

(118) **Indexicalism**. The pragmatic processes involved in the determination of the proposition expressed by an utterance are exhausted by linguistically mandated ones, namely, disambiguation and saturation.

As discussed in Section 2, indexicalists like Stanley (2000) deny the existence of optional pragmatic processes like free enrichment, and claim that all apparent cases of free enrichment should be analyzed as instances of saturation.[2]

The second reading of **UT** is as follows:

(119) **Radical Contextualism**:

a. The pragmatic processes involved in the determination of the proposition expressed by an utterance are not exhausted by linguistically mandated ones, namely, disambiguation and saturation. That is, some optional pragmatic processes are required.

b. Optional processes involved in the determination of the proposition expressed by an utterance are solely directed by pragmatic considerations in the sense that, except for the minimal linguistic constraint (MLC), there is no linguistic factor or constraint involved in the way the processes work.

[2]It is noted that Stanley (2005), against relevance theorists, argues that the alleged cases of *ad hoc* concept construction do not contribute to the truth-conditional content of an utterance. See Wilson and Sperber (2002, 2005) for a defense of the relevance-theoretic view.

We call this position "Radical Contextualism." Like Radical Contextualists, we do not deny the existence of optional pragmatic processes, in particular, the existence of free enrichment. Thus, we accept the claim in (119a). In this respect, we side with the contextualist positions. What distinguishes our position from Radical Contextualism is rejection of the claim in (119b). According to the claim in (119b), the conceptual schema delivered by the encoded logical form of an utterance provides the mere "skeleton" on which a proposition expressed is developed, and except for the case of disambiguation and saturation, the gap between the "skeleton" and the proposition expressed is bridged by pragmatic considerations without semantic constraints. We reject this view. As discussed in Section 2, we take there to be two types of optional pragmatic processes, namely free enrichment and *ad hoc* concept construction. We claim that the conceptual schema delivered by the logical form of an utterance plays a more important role in deriving the proposition expressed through each type of optional pragmatic process, in the following way:

(120) a. *Ad hoc* concept construction is linguistically bounded in the sense that the range of possible interpretations is provided by the lexical-encyclopedic knowledge concerning the encoded concept.

b. The applicability of free enrichment is constrained by the semantic function an encoded concept plays in a proposition. In particular, free enrichment is blocked for property concepts.

Thus we propose the following interpretation of **UT**:

(121) **Our version of Contextualism**

a. The pragmatic processes involved in the determination of the proposition expressed by an utterance are not exhausted by linguistically mandated ones, namely, disambiguation and saturation. [= (119a)]

b. Optional processes involved in the determination of the proposition expressed by an utterance are semantically constrained as in (120).

The claim in (121b) implies that the "skeleton" provided by the encoded logical form of an utterance imposes a strong semantic constraint on the way in which a possible proposition could be developed by optional pragmatic inferences. In particular, how free enrichment works is sensitive to whether the target concept is object-directed or not. In this respect, our position is essentially different from the radical contextualist position in (119). The positions we considered so far may be summarized as follows:

Does the linguistic meaning fully determine
the proposition expressed by an utterance?

Yes
Literalism

No
Non-literalism
Are optional pragmatic processes involved
in the determination of a proposition expressed?

Yes
Contextualism
Are there semantic constraints imposed
on optional pragmatic processes?

Yes
Our position

No (except for MLC)
Radical Contextualism

No
Indexicalism

To sum up, we have shown that free enrichment can never intrude into the position occupied by a property concept in a logical form. To interpret property expressions such as predicate nominals and adjectives, hearers cannot use free enrichment at all. Based on the existence of this semantic constraint, we then argued that both the standard version of Contextualism (Radical Contexualism), including current relevance theory, and Indexicalism are mistaken in their conception of the way that linguistic semantics is related to the pragmatic processes involved in the determination of the proposition expressed. Radical Contextualism is mistaken in that it holds that a purely pragmatic process of free enrichment is not semantically constrained. Indexicalism, on the other hand, is mistaken in that it denies the existence of free enrichment and maintains that no pragmatic processes are allowed to enter into the proposition expressed unless the linguistic meaning of the sentence itself so demands. Furthermore, Indexicalism provides wrong predictions for predicational sentences. By contrast, our new version of Contextualism is sensitive to the semantic constraint of free enrichment, and the over-generation problem of Contextualism, pointed by Stanley (2002, 2005), can be appropriately avoided within this framework.

Summary of the Thesis

In this thesis, we have investigated various aspects of inference in natural language. Our main contributions can be summarized as follows.

In Chapter 1, we have introduced a syllogistic proof system for reasoning with inclusion and exclusion relations, aiming at a better understanding of inferences with basic categorical (quantified) sentences in natural languages. Compared to existing systems for syllogistic inferences, our proof system has the advantage of simplicity; in particular, it succeeds in formulating categorical syllogisms without appealing to axioms or inference rules concerned with sentential negation. It also contributes to the program of natural logic, one of whose aims is to study how significant parts of natural language inferences can be formalized in a more simple and efficient logical system than standard systems such as first-order logic. We proved the completeness and normalization theorems, and then provided a precise characterization of normal proofs in this proof system. The simplicity of the system enables us to characterize the notion of normal proofs in a perspicuous way, and thereby to make comparison with other logical systems easier.

We then showed that categorical syllogisms with and without existential import can be faithfully embedded into our proof system. To make a connection with standard logical systems, we also showed that our proof system can be faithfully embedded into an implicational fragment of a natural deduction system for minimal propositional logic. The proofs of these results relied on purely proof-theoretic methods, i.e., syntactic transformation between normal proofs in each system.

We also established a correspondence between our proof system and an inference system for Euler diagrams. This connection is theoretically important because both syllogistic and diagrammatic inferences have been of central importance in the cognitive study of human deductive reasoning, yet a precise relationship between linguistic and diagrammatic proof systems has not been provided in the previous literature.

As a step towards a more expressive system that is suitable for representing a wider range of inferences in natural languages, we extended our basic syllogistic system with intersection, and showed a completeness result with respect to its natural semantics. This kind of extensions will be a basis for a more realistic framework for linguistic analyses.

In Chapter 2, we studied presuppositions from a proof-theoretical point of view. We focused on the case of existential presuppositions triggered by definite descriptions, which has long been discussed in the philosophy and linguistics literature and hence serves a representative case for the study of presuppositions.

We started with a brief overview of the controversy between quantificational and referential analyses of descriptions; we provided some linguistic evidence against the quantificational (Russellian) analysis and then argued that the referential (presuppositional) analysis has an explanatory advantage in that it can explain the peculiar behavior of descriptions in terms of a general mechanism of presupposition projection.

We compared two influential formal approaches to presupposition projection: Dynamic Semantics and Discourse Representation Theory. We argued that both approaches have empirical and methodological problems and, accordingly, that it is worth exploring an alternative framework in which inferences with presuppositions play a central role in accounting for projection behavior.

We then proposed a proof-theoretical framework for handling existential presuppositions of descriptions, building on the study of ε-calculus and constructive type theory in logic and computer science. In this framework, processes of representing and reasoning about presuppositional contents are formalized as natural-deduction proofs. We showed that the difficulties confronting the previous approaches (Dynamic Semantics and Discourse Representation Theory) can naturally be avoided without complicating the semantic mechanisms handling presupposition projection. We also discussed how to extend our theory to handle various kinds of accommodation strategies (i.e., local, global, and intermediate accommodation). In future work, it will be interesting to see how our proof-theoretical framework can apply to various presupposition triggers other than the definite descriptions as discussed in the current linguistics literature.

In Chapter 3, we discussed the nature of pragmatic processes called *free enrichment* from a relevance-theoretic point of view. We started by classifying three levels of meaning involved in the overall process of utterance interpretation, namely (i) linguistic meaning, (ii) proposition expressed, and (iii) implicature. We concentrated on the distinction between (i) and (ii), and addressed the question of how a hearer can bridge the gap between the linguistic meaning of a sentence and the proposition expressed by the utterance of that sentence.

Two approaches to this issue were discussed: Indexicalism and Contextualism. Indexicalism postulates covert variables for various context-sensitive constructions, whereas Contextualism attempts to fill a gap between a linguistic meaning and the proposition expressed, solely by relying on pragmatic considerations, that is, without complicating the syntax and semantics of the constructions in question.

We classified four types of pragmatic processes pertaining to the derivation of a proposition expressed: disambiguation, saturation, free enrichment, and *ad hoc* concept construction. While the Contextualist position admits all four types of pragmatic processes, the Indexicalist position only admits disambiguation and saturation. We provided a novel characterization of the distinction between free enrichment and *ad hoc* concept construction, a distinction that has remained obscure in the literature. We pointed out that under the standard Contextualist conception, free enrichment is solely constrained by pragmatic considerations (except for the minimal linguistic constraint).

We then argued that this standard conception of free enrichment has a serious problem. The main empirical problem posed for the Contextualist position is that free enrichment is blocked for a concept expressed by a predicate nominal, i.e., what we called a *property concept*. In our view, free enrichment is only applicable to an *object-directed concept*, a concept whose semantic function consists in referring to or ranging over objects in the world. We argued that both Indexicalist and Contextualist views have difficulty in accounting for such a constraint as imposed by predicate nominals. More specifically, the Indexicalist position stipulates covert variables for nominal expressions, whether they appear in argument position or in predicative position; consequently, it predicts that contextual restriction on predicate nominals is possible in principle, independently of the semantic function it plays in a proposition. If our argument is correct, this assumption has to be rejected. Contextualism, on the other hand, fails to explain why the process of free enrichment is blocked for property concepts, since the only constraints (except for the minimal linguistic constraint) imposed on the way it works are pragmatic ones.

We discussed some challenges to our claim, and argued that they did not make genuine counterexamples. We then argued that the semantic constraint on free enrichment can be explained in terms of the differences in function between object-directed concepts and property concepts. Our conception of free enrichment was made clear by comparing it with the recent pragmatic approach proposed by Hall (2008), who defended the standard framework of relevance theory. We argued that Hall's argument against our claim is based on a certain misconception of the relevant semantic notions; more specifically, Hall's argument fails to appreciate the semantic difference between object-directed and property concepts.

Finally, we addressed the Indexicalist objection to those who admit free enrichment, as first pointed out by Jason Stanley in his series of papers (Stanley 2002, 2005). Hall (2008) addressed Stanley's objection from the standpoint of relevance theory. We argued that while Hall's pragmatic approach fails to handle some typical examples discussed by Stanley and others, the semantic constraints on free enrichment we proposed can naturally avoid Stanley's overgeneration problem without further stipulations.

Currently, various conceptions of semantics-pragmatics interface, other than those discussed in this chapter, have been proposed by linguists and philosophers. We left it for future research to explore the consequence of our view on these various conceptions of how linguistic meaning is related to the proposition expressed by an utterance or, more generally, how pragmatic inferences intervene in semantic representations.

References

Aczel, Peter. (1980). Frege structures and the notions of proposition, truth and set. J. Barwise, H. J. Keisler and K. Kunen (eds.), *The Kleene symposium*, 31–59. Amsterdam: North-Holland.

Asher, Nicholas. (2011). *Lexical Meaning in Context: A Web of Words.* Cambridge: Cambridge University Press.

Avigad, Jeremy, Edward Dean, and John Mumma. (2009). A formal system for Euclid's Elements. *The Review of Symbolic Logic* 2(4), 700–768.

Bach, Kent. (1994). Conversational impliciture, *Mind and Language* 9, 124–162.

Bach, Kent. (2000). Quantification, qualification and context: a reply to Stanley and Szabó. *Mind and Language* 15, 262–283.

Barker, Chris and Pauline Jacobson (eds). (2007). *Direct Compositionality.* Oxford: Oxford University Press.

Barwise, J. and R. Cooper. (1981). Generalized quantifiers and natural language. *Linguistics and Philosophy* 4, 159–219.

Barwise, Jon and John Etchemendy. (1996). Visual information and valid reasoning. In G. Allwein and J. Barwise (eds), *Logical Reasoning with Diagrams.* Oxford: Oxford University Press, 3–26.

Beaver, David. (1992). The kinematics of presupposition. In P. Dekker and M. Stokhof (eds.), *Proceedings of the Eighth Amsterdam Colloquium*, 17–36. Amsterdam: Institute of Logic, Language and Computation.

Beaver, David. (1994). When variables don't vary enough. In M. Harvey and L. Santelmann (eds.), *Proceedings of Semantics and Linguistic Theory IV*, 35–60. Ithaca, NY: Cornell University.

Beaver, David. (1997). Presupposition. In J. van Benthem and A. ter Meulen (eds.), *The Handbook of Logic and Language*, 939–1008. Amsterdam: Elsevier.

Beaver, David. (2001). *Presupposition and Assertion in Dynamic Semantics*. Stanford, CA: CSLI Publications.

Beaver, David and Emiel Krahmer. (2001). A partial account of presupposition projection. *Journal of Logic, Language and Information* 10, 147–182.

Beaver, David and Henk Zeevat. (2007). Accommodation. In G. Ramchand and C. Reiss (eds.), *Oxford Handbook of Linguistic Interfaces*, 503–538. Oxford: Oxford University Press.

Belnap, Nuel D. (1970). Conditional assertion and restricted quantification. *Noûs* 4(1), 1–12.

Bezuidenhout, Anne. (2002). Truth-conditional pragmatics. *Philosophical Perspectives* 16, 105–134.

Bezuidenhout, Anne. (2006). The Coherence of Contextualism. *Mind and Language* 21, 1–10.

Blackburn, Patrick and Johan Bos. (2005). *Representation and Inference for Natural Language: A First Course in Computational Semantics.* Stanford, CA: CSLI.

Blackburn, Patrick and Johan Bos. (1999). *Working with Discourse Representation Theory: An Advanced Course in Computational Semantics.* `http://homepages.inf.ed.ac.uk/jbos/comsem/book2.html`

Blackett, D. (1983). *Elementary Topology.* New York: Academic Press.

Blakemore, Diane. (2002). *Relevance and Linguistic Meaning: the Semantics and Pragmatics of Discourse Markers*. Cambridge: Cambridge University Press.

Blamey, Stephen. (1986/2002). Partial logic. In D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, Second edition, Volume 5*, 261–353. Dordrecht: Kluwer, 2002. [First edition in D. Gabbay and F. Guenthner (eds.), *The Handbook of Philosophical Logic, vol. IV*, 1–70, Dordrecht: Reidel, 1986.]

Borg, Emma. (2004). *Minimal Semantics*. Oxford: Oxford University Press.

Borg, Emma. (2007). Minimalism versus contextualism in semantics. In G. Preyer and G. Peter (eds.), *Context-Sensitivity and Semantic Minimalism: New Essays on Semantics and Pragmatics*. 339–359. Oxford: Oxford University Press.

Cappelen, H. and E. Lepore. (2005). *Insensitive Semantics*. Oxford: Blackwell.

Cappelen, H. and E. Lepore. (2006). Response. *Mind and Language* 21, 50–73.

Carlström, Jesper. (2005). Interpreting descriptions in intensional type theory. *Journal of Symbolic Logic* 70(2), 488–514.

Carston, Robyn. (1988). Implicature, explicature, and truth-theoretic semantics. In R. Kempson (ed.), *Mental Representations: The Interface between Language and Reality*. 155–181. Cambridge: Cambridge University Press.

Carston, Robyn. (1996). Enrichment and loosening: complementary processes in deriving the proposition expressed. *UCL Working Papers in Linguistics* 8, 610–88.

Carston, Robyn. (2000). Explicature and semantics. *UCL Working Papers in Linguistics* 12, 1–44.

Carston, Robyn. (2002a). *Thoughts and Utterances: The Pragmatics of Explicit Communication*. Oxford: Blackwell.

Carston, Robyn. (2002b). Linguistic meaning, communicated meaning and cognitive pragmatics. *Mind and Language* 17, 127–148.

Carston, Robyn. (2004). Relevance theory and the saying/implicating distinction. In L. Horn and G. Ward (eds.) *The Handbook of Pragmatics*. 633–656. Oxford: Blackwell.

Carston, Robyn. (2007). How many pragmatic systems are there? In M. Frápolli (ed), *Saying, Meaning and Referring: Essays on François Recanati's Philosophy of Language*. New York: Palgrave Macmillan, 18–48.

Carston, Robyn. (2008). Linguistic communication and the semantics/pragmatics distinction. *Synthese* 165, 321–345.

Carston, Robyn. (2009). The explicit/implicit distinction in pragmatics and the limits of explicit communication. *International Review of Pragmatics* 1, 35–62.

Cellucci, Carlo. (1995). On Quine's approach to natural deduction. In P. Leonardi and M. Santambrogio (eds.), *On Quine*, 314–335. Cambridge: Cambridge University Press.

Charlow, Simon. (2009). Strong "predicative" presuppositions. Ms. NYU.

Chemla, Emmanuel. (2009). *Presuppositions of quantified sentences: experimental data. Natural Language Semantics* 17(4), 299–340.

Chierchia, Gennaro and Sally McConnell-Ginet. (2000). *Meaning and grammar: an introduction to semantics, Second edition*. Cambridge, MA: MIT Press.

Clark, Herbert H. (1975). Bridging. In R. C. Schank and B. L. Nash-Webber (eds.), *Theoretical issues in natural language processing*, 169–174. New York: Association for Computing Machinery.

Cooper, Robin. (1983). *Quantification and Syntactic Theory.* Dordrecht: Reidel.

Cooper, Robin. (2005a). Austinian truth, attitudes and type theory. *Research on Language and Computation* 3(4), 333–362.

Cooper, Robin. (2005b). Records and record types in semantic theory. *Journal of Logic and Computation* 15(2), 99–112.

Corcoran, John. (1972). Completeness of an ancient logic. *Journal of Symbolic Logic* 37, 696–705.

Corcoran, John. (1974). Aristotle's natural deduction system. In J. Corcoran (ed.), *Ancient Logic and its Modern Interpretations.* Dordrecht: Reidel, 85–131.

Crabbé, Marcel. (2001). The formal theory of syllogisms. *The Review of Modern Logic* 9, 29–52.

Curry, Haskell B. and Robert Feys. (1958). *Combinatory Logic*, volume 1. Amsterdam: North-Holland.

Davidson, Donald. (1967). The logical form of action sentences. In N. Rescher (ed.), *The Logic of Decision and Action.* Pittsburgh: University of Pittsburgh Press. Reprinted in D. Davidson, *Essays on Actions and Events*, 105–122, Oxford: Clarendon Press, 1980.

Declerck, Renaat. (1988). *Studies on Copular Sentences, Clefts and Pseudoclefts.* Leuven: Leuven University Press.

Dekker, Paul. (1992). An update semantics for dynamic predicate logic. In P. Dekker and M. Stokhof (eds.), *Proceedings of the Eighth Amsterdam Colloquium*, 113–132. Amsterdam: Institute of Logic, Language and Computation.

Dekker, Paul. (1996). The values of variables in dynamic semantics. *Linguistics and Philosophy* 19, 211–257.

DeVidi, David. (2004). Choice principles and constructive logics. *Philosophia Mathematica* 3(12), 222–243.

Doron, Edit. (1988). The semantics of predicate nominals. *Linguistics* 26, 281–301.

Elbourne, Paul. (2005). *Situations and Individuals*. Cambridge, MA: MIT Press.

Elbourne, Paul. (2008). The argument from binding. *Philosophical Perspectives* 22. 89–110.

Elbourne, Paul. (2010). The existence entailments of definite descriptions. *Linguistics and Philosophy* 33, 1–10.

Englebretsen, George. (1981). *Three Logicians: Aristotle, Leibniz, and Sommers and the syllogistic*. Assen: Van Gorcum.

Euler, Leonhard. (1768). *Lettres à une Princesse d'Allemagne sur Divers Sujets de Physique et de Philosophie*. Saint-Pétersbourg, De l'Académie des Sciences. [English translation in D. Brewster, ed. 1853. *Letters of Euler on Different Subjects in Natural Philosophy: Addressed to a German Princess*. Vol. II, Letters 102–8. New York: Harper and Brothers Publishers.]

Evans, Gareth. (1980). Pronouns. *Linguistic Inquiry* 11, 337–362.

Fara, Delia Graff. (2001) Descriptions as predicates. *Philosophical Studies* 102, 1–42. Originally published under the name "Delia Graff".

Faris, J. A. (1955). The Gergonne relations. *Journal of Symbolic Logic* 20, 207–231.

Feferman, Solomon. (1995). Definedness. *Erkenntnis* 43, 295–320.

Fernando, Tim. (2001). Conservative generalized quantifiers and presupposition. *Proceedings of Semantics and Linguistic Theory XI*, 172–191. Ithaca, NY: Cornell University.

Fernando, Tim. (2009). Situations as indices and as denotations. *Linguistics and Philosophy* 32(2), 185–206.

von Fintel, Kai. (1994). *Restrictions on Quantifier Domains.* Ph.D thesis, University of Massachusetts, Amherst.

von Fintel, Kai. (2004). Would you believe it? The king of France is back! (Presuppositions and truth-value intuitions). In A. Bezuidenhout and M. Reimer (eds.), *Descriptions and Beyond.* 315–342. Oxford: Oxford University Press.

von Fintel, Kai. (2008). What is presupposition accommodation, again? *Philosophical Perspectives* 22(1). 137–170.

Flower, Jean and John Howse. (2002). Generating Euler diagrams. In M. Hegarty et al. (eds.), *Diagrammatic Representation and Inference: Proceedings of Diagrams 2002*, *Lecture Notes in Computer Science*, Volume 2317. 61–75.

Francez, Nissim, Roy Dyckhoff and Gilad Ben-Avi. (2010). Proof-theoretic semantics for subsentential phrases. *Studia Logica* 94, 381–401.

Frege, Gottlob. (1892). Über Sinn und Bedeutung. *Zeitschrift für Philosophie und philosophische Kritik* 100, 25–50. [English translation in P. Geach and M. Black (eds.), *Translations from the Philosophical Writings of Gottlob Frege*, 56–78. Oxford: Blackwell, 1960.]

Geach, Peter, T. (1950). Russell's theory of descriptions. *Analysis* 10(4), 84–88.

Geach, Peter, T. (1962/1980). *Reference and Generality.* Ithaca, NY: Cornell University Press. Second edition 1962; Third edition 1980.

Gentzen, Gerhard. (1934). Untersuchungen über das logische Schließen. *Mathematische Zeitschrift* 39, 176–210, 405–431. [M. Szabo, ed. and trans. Investigations into logical deduction. In *The Collected Papers of Gerhard Gentzen*, 1969, Amsterdam: North Holland.]

Gergonne, J. (1817). Essai de dialectique rationelle. *Annales de mathématiques pures et appliqées* 7, 189–228.

Geurts, Bart. (1996). Local satisfaction guaranteed. *Linguistics and Philosophy*, 19. 259–294.

Geurts, Bart. (1999). *Presuppositions and Pronouns*. Amsterdam: Elsevier.

Geurts, Bart. (2007). Existential import. In I. Comorovski and K. von Heusinger (eds.), *Existence: Semantics and Syntax*, Synthese language library, Volume 84. 253–271.

Glanzberg, Michael. (2007). Definite descriptions and quantifier scope: some Mates cases reconsidered. *European Journal of Analytic Philosophy* 3, 133–158.

Grice, Paul. (1989). *Studies in the Way of Words*. Cambridge, Mass.: Harvard University Press.

Groenendijk, Jeroen and Martin Stokhof. (1991). Dynamic predicate logic. *Linguistics and Philosophy* 14, 39–100.

Groenendijk, Jeroen, Martin Stokhof and Frank Veltman. (1996). Coreference and modality. In Shalom Lappin (ed.), *Handbook of Contemporary Semantic Theory*, 179–216. Oxford: Blackwell.

Groeneveld, Willem and Frank Veltman. (1994). Inference systems for update semantics. Institute for Logic Language and Computation, Department of Philosophy, University of Amsterdam.

Hall, Alison. (2008). Free enrichment or hidden indexicals? *Mind and Language* 23, 426–456.

Hammer, Eric. (1995). *Logic and Visual Information*. Stanford, CA: CSLI Publications, 1995.

Hammer, Eric and Sun-Joo Shin. (1998). Euler's visual logic. *History and Philosophy of Logic* 19, 1–29.

Heim, Irene. (1982). *The Semantics of Definite and Indefinite Noun Phrases*. Doctoral dissertation, University of Massachusetts, Amherst.

Heim, Irene. (1983). On the projection problem for presuppositions. In M. Barlow, D. Flickinger and M. Wescoat (eds.), *Proceedings of WCCFL 2*. 114–125. Stanford, CA: Stanford University.

Heim, Irene. (1991). Artikel und Definitheit. In Arnim von Stechow and Dieter Wunderlich (eds.), *Semantics: An International Handbook of Contemporary Research*, 487–535. Berlin: de Gruyter.

Heim, Irene. (1992). Presupposition projection and the semantics of attitude verbs. *Journal of Semantics* 9, 183–221.

Heim, Irene. (2011). Definiteness and indefiniteness. In K. von Heusinger, C. Maienborn and P. Portner (eds.), *Semantics* (HSK 33.2), Berlin: de Gruyter, 996–1025.

Heim, Irene and Angelika Kratzer. (1998). *Semantics in Generative Grammar*. Oxford: Blackwell.

Higginbotham, James. (1985). On semantics. *Linguistic Inquiry* 16, 547–593.

Higginbotham, James. (1987). Indefiniteness and predication. In E.J. Reuland and A. ter Meulen (eds.), *The Representation of (In)definiteness*. Cambridge, MA: MIT Press, 43–70.

Higgins, Roger. (1973). *The Pseudo-cleft Construction in English*. Ph.D thesis, Massachusetts Institute of Technology.

Hilbert, David and Paul Bernays. (1934) *Grundlagen der Mathematik*, Volume I. Berlin: Springer. [Second edition, 1968.]

Hintikka, Jaakko. (1959). Towards a theory of definite descriptions. *Analysis* 19, 79–85.

Howard, William A. (1980). The formulae-as-types notion of construction. In J. P. Seldin and J. R. Hindley (eds.), *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, 479–490. London: Academic Press.

Howse, John, Fernando Molina, Sun-Joo Shin, and John Taylor. (2002). On diagram tokens and types. In M. Hegarty et al. (eds.), *Diagrammatic Representation and Inference: Proceedings of Diagrams 2002. Lecture Notes in Computer Science*, Volume 2317, 146–160.

Howse, John, Fernando Molina, and John Taylor. (2000) SD2: A Sound and Complete Diagrammatic Reasoning System. *Proceedings of IEEE International Symposium on Visual Languages*. 127–134.

Howse, John, Gem Stapleton, and John Taylor. (2005). Spider diagrams. *LMS Journal of Computation and Mathematics* 8, 145-194.

Jäger, Gerhard. (2003). Towards an explanation of copula effects. *Linguistics and Philosophy* 26(5), 557–593.

Jäger, Gerhard. (2005). *Anaphora and Type Logical Grammar*. Springer.

Kadmon, Nirit. (2001). *Formal Pragmatics*. Oxford: Blackwell.

Kamp, Hans. (1981). A theory of truth and semantic representation. In J. Groenendijk, T. Janssen and M. Stokhof (eds.), *Formal Methods in the Study of Language*, Part 1, volume 135, 277–322. Amsterdam: Mathematical Centre Tracts.

Kamp, Hans and Uwe Reyle. (1993). *From Discourse to Logic*. Dordrecht: Kluwer.

Kamp, Hans, Josef van Genabith, and Uwe Reyle. (2011). Discourse representation theory. In D. Gabbay and F. Guenthner (eds.), *Handbook of Philosophical Logic, Second edition, Volume 15*, 125–394. Berlin: Springer.

Kaplan, David. (1989). Demonstratives. In J. Almog, J.Perry, and H.Wettstein (eds.), *Themes from Kaplan*, Oxford: Oxford University Press. 481–566.

Karttunen, Lauri. (1973). Presuppositions of compound sentences. *Linguistic Inquiry* 4, 167–193.

Karttunen, Lauri. (1974). Presupposition and linguistic context. *Theoretical Linguistics* 1, 181–193.

Karttunen, Lauri and Stanley Peters. (1979). Conventional implicatures. in C. Oh and D. Dineen (eds.), *Syntax and Semantics 11: Presupposition*, 1–56. New York: Academic Press.

King, Jeffrey. (2001). *Complex Demonstratives.* Cambridge, MA: MIT Press.

King, Jeffrey. (2007) *The Nature and Structure of Content.* New York: Oxford University Press.

King, Jeffrey and Jason Stanley. (2005). Semantics, pragmatics, and the role of semantic content. In Z. Szabó (ed.), *Semantics versus Pragmatics.* New York: Oxford University Press, 111–164.

Klinedinst, Nathan and Daniel Rothschild. (2012). Connectives without truth tables. *Natural Language Semantics* 20(2), 137–175.

Kneale, William and Martha Kneale. (1962). *The Development of Logic.* Oxford: Oxford University Press.

Krahmer, Emiel. (1998). *Presupposition and Anaphora.* Stanford, CA: CSLI Publications.

Krahmer, Emiel and Paul Piwek. (1999). Presupposition projection as proof construction. In H. Bunt and R. Muskens (eds.), *Computing Meaning*, 281–300. Dordrecht: Kluwer.

Kripke, Saul. (2005). Russell's notion of scope. *Mind* 114, 1005–1037.

Kripke, Saul. (2009). Presupposition and anaphora: remarks on the formulation of the projection problem. *Linguistic Inquiry* 40(3), 367–386.

Kuno, Susumu. (1970). Some properties of non-referential noun phrases. In R. Jakobson and S. Kawamoto (eds.), *Studies in General and Oriental Linguistics. Presented to S. Hattori on Occasion of his Sixtieth Birthday*, Tokyo: TEC, 348–373.

Lambek, Joachim. (1958). The mathematics of sentence structure. *The American Mathematical Monthly* 65(3), 154–170.

Lambert, Karel. (1991). *Philosophical Applications of Free Logic*. Oxford: Oxford University Press.

Lambert, Karel. (2003). The Hilbert-Bernays theory of definite descriptions. In Karel Lambert, *Free Logic: Selected Essays*, Cambridge: Cambridge University Press, 44–68.

Langendoen, Terence and Harris Savin. (1971). The projection problem for presuppositions. In Charles Fillmore and D. Terence Langendoen (eds.), *Studies in Linguistic Semantics*, New York: Holt, Reinhardt and Winston, 373–388.

Lasersohn, Peter. (1993). Existence presuppositions and background knowledge. *Journal of Semantics* 10(2), 113–122.

Levinson, Stephen. (1983). *Pragmatics*. Cambridge: Cambridge University Press.

Lewis, David. (1970). General semantics. *Synthese* 22, 18–67.

Lewis, David. (1979). Scorekeeping in a language game. *Journal of Philosophical Logic* 8, 339–359.

Ludlow, Peter and Stephen Neale. (1991). Indefinite descriptions: in defense of Russell. *Linguistics and Philosophy* 14, 171–202.

Łukasiewicz, Jan. (1957). *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*. Second edition, Oxford: Clarendon Press.

MacCartney, Bill and Christopher D. Manning. (2008). Modeling semantic containment and exclusion in natural language inference. *The 22nd International Conference on Computational Linguistics (Coling-08)*. 140–156.

Maehara, Shôji. (1955). The predicate calculus with ε-symbol. *Journal of the Mathematical Society of Japan* 7(4), 323–344.

Maienborn, Claudia. (2001). On the position and interpretation of locative modifiers. *Natural Language Semantics* 9, 191–240.

Martí, Luisa. (2006). Unarticulated constituents revisited. *Linguistics and Philosophy* 29, 135–166.

Martin, John. (1997). Aristotle's natural deduction reconsidered. *History and Philosophy of Logic* 18, 1–15.

Martin-Löf, Per. (1975). An intuitionistic theory of types: predicative part. In H. E. Rose and J. C. Sheperdson (eds.), *Logic Colloquium 1973*, 73–118. Amsterdam: North-Holland.

Martin-Löf, Per. (1982). Constructive mathematics and computer programming. In L. J. Cohen, J. Łoś, H. Pfeiffer, and K.-P. Podewski (eds.), *Logic, methodology, and philosophy of science VI*, 153–175. Amsterdam: North-Holland.

Martin-Löf, Per. (1983/1996). On the meanings of the logical constants and the justifications of the logical laws. *Nordic Journal of Philosophical Logic* 1(1), 11–60, 1996. [Originally presented at the meeting *Teoria della Dimostrazione e Filosofia della Logica*, Siena 6–9 April, 1983.]

Martin-Löf, Per. (1984). *Intuitionistic Type Theory.* Notes by G. Sambin. Naples: Bibliopolis.

Mates, Benson. (1973). Descriptions and reference. *Foundations of Language* 10, 409–418.

McAllester, David and Robert Givan. (1992). Natural language syntax and first-order inference. *Artificial Intelligence* 56, 1–20, 1992.

McCawley, James. (1979). Presupposition and discourse structure. In Choon- Kyu Oh and David Dineen (eds.), *Syntax and Semantics 11: Presupposition.* 371–388, Academic Press, New York.

Mercer, Robert. (1992). Default logic: towards a common logical semantics for presupposition and entailment. *Journal of Semantics* 9, 223–250.

Mikkelsen, Line. (2005). *Copular Clauses. Specification, Predication and Equation.* Amsterdam: Benjamins.

Mikkelsen, Line. (2011). Copular clauses. In K. von Heusinger, C. Maienborn and P. Portner (eds.), *Semantics* (HSK 33.2), Berlin: de Gruyter, 1805–1829.

Mineshima, Koji. (2008). A presuppositional analysis of definite descriptions in proof theory. In Ken Satoh, Akihiro Inokuchi, Katashi Nagao, and Takahiro Kawamura (eds.) *New Frontiers in Artificial Intelligence*, *Lecture Notes in Artificial Intelligence* Vol. 4914. 214–227. Springer-Verlag.

Mineshima, Koji, Mitsuhiro Okada, Yuri Sato, and Ryo Takemura. (2008). Diagrammatic reasoning system with Euler circles: theory and experiment design. In G. Stapleton, J. Howse, and J. Lee (eds.), *Diagrammatic Representation and Inference: Proceedings of Diagrams 2008*, *Lecture Notes in Computer Science*, Volume 5223. 188–205. Springer-Verlag.

Mineshima, Koji, Mitsuhiro Okada, and Ryo Takemura. (2009). Conservativity for a hierarchy of Euler and Venn reasoning systems. *Proceedings of Workshop on Visual Languages and Logic 2009.* CEUR Series, Volume 510. 37–61.

Mineshima, Koji, Mitsuhiro Okada, and Ryo Takemura. (2010). Two types of diagrammatic inference system: natural deduction style and resolution style. In A.K. Goel et al. (eds.), *Diagrammatic Representation and*

Inference: Proceedings of Diagrams 2010, *Lecture Notes in Computer Science*, Volume 6170, Springer-Verlag. 99–114.

Mineshima, Koji, Mitsuhiro Okada, and Ryo Takemura. (2012a). A diagrammatic inference system with Euler circles. *Journal of Logic, Language and Information* 21, 365–391.

Mineshima, Koji, Mitsuhiro Okada, and Ryo Takemura. (2012b). A generalized syllogistic inference system based on inclusion and exclusion relations. *Studia Logica* 100. Special issue, Logic and Natural Language, edited by Nissim Francez and Ian Pratt-Hartmann, 753–785.

Mints, Grigori. (1977). Heyting predicate calculus with epsilon symbol. *Journal of Soviet Mathematics* 8, 317–323.

Molina, Fernando. (2001). Reasoning with extended Venn-Peirce diagrammatic systems. Ph.D Thesis, University of Brighton.

Montague, Richard. (1970). English as a formal language. Reprinted in R. Thomason (ed.), *Formal Philosophy: Selected Papers of Richard Montague*, 188–221. New Haven: Yale University Press, 1974.

Montague, Richard. (1973). The proper treatment of quantification in ordinary English. In J. Hintikka, J. M. E. Moravcsik and P. Suppes (eds.), *Approaches to Natural Language*, 189–224. Dordrecht: Reidel. Reprinted in R. Thomason (ed.), *Formal Philosophy: Selected Papers of Richard Montague*, 247–270. New Haven: Yale University Press, 1974.

Montague, R. (1974). *Formal Philosophy: Selected Papers of Richard Montague*, R. Thomason (ed.), New Haven: Yale University Press.

Morgan, Jerry. (1969). On the treatment of presupposition in transformational grammar. In *Papers from the Fifth Regional Meeting, Chicago Linguistic Society*, 167–77, University of Chicago.

Moss, Lawrence S. (2008). Completeness theorems for syllogistic fragments. In F. Hamm and S. Kepser (eds.), *Logics for Linguistic Structures*. Berlin : Mouton de Gruyter, 143–173.

Moss, Lawrence S. (2010a). Syllogistic logics with verbs. *Journal of Logic and Computation* 20. 947–967.

Moss, Lawrence S. (2010b). Natural logic and semantics. In M. Aloni et al. (eds.), *Proceedings of 17th Amsterdam Colloquium*, LNAI 6042, 84–93.

Moss, Lawrence S. (2010c). Syllogistic logic with complements. In J. van Benthem, A. Gupta, and E. Pacuit (eds.), *Games, Norms and Reasons: Logic at the Crossroads*, Springer, 185–203.

Muskens, Reinhard. (1996). Combining Montague semantics and discourse representation. *Linguistics and Philosophy* 19(2), 143–186.

Muskens, Reinhard. (2011). A squib on anaphora and coindexing. *Linguistics and Philosophy* 34(1), 85–89.

Neale, Stephen. (1990). *Descriptions*. Cambridge, MA: MIT Press.

Neale, Stephen. (2000). On being explicit. *Mind and Language* 15, 284–294.

Neale, Stephen. (2004). This, that, and the other. In A Bezuidenhout and M. Reimer (eds.), *Descriptions and Beyond*. Oxford: Oxford University Press. 68–182.

Neale, Stephen. (2005). A century later. *Mind* 114, 809–871.

Nishihara, N. and K. Morita. (1988). An extended syllogistic system with conjunctive, disjunctive and complementary terms, and its completeness proof (in Japanese). *Trans. IEICE Japan*, Vol. J71-D, No.4. 693–704, 1988.

Nishihara, N., K. Morita, and S. Iwata. (1990). An extended syllogistic system with verbs and proper nouns, and its completeness proof. *Systems Computer Japan* 21, 96–111.

Nishiyama, Yuji. (1983). *Imiron* (Semantics). In *Eigogaku Taikei* (A Series of Studies on English), Volume 5. Tokyo: Taisyukan Syoten.

Nishiyama, Yuji. (1997). Attributive use and non-referential NPs. In M. Ukaji, T. Nakao, M. Kajita, and S. Chiba (eds.), *Studies in English Linguistics: A Festschrift for Akira Ota on the Occasion of His Eightieth Birthday*. Tokyo: Taishukan Publishing Company.

Nishiyama, Yuji. (2003). *Nihongo Meishiku no Imiron to Goyoron: Shijiteki Meishiku to Hi-shijiteki Meishiku [Semantics and Pragmatics of Noun Phrases in Japanese: Referential Noun Phrases and Non-referential Noun Phrases]*. Tokyo: Hituzi Shobo.

Nishiyama, Yuji and Koji Mineshima. (2005). Semantic constraints on free enrichment. Paper presented at the 9th International Pragmatics Conference, Riva del Garda, Italy.

Nishiyama, Yuji and Koji Mineshima. (2006a). Jojutsu Meishiku to Goyoronteki Kaishaku. [Predicate Nominals and Pragmatic Interpretation]. In Takashi Iida (ed.), *Seiyoseishin-shi ni okeru Gengo to Gengo-kan [Language, Culture and European Tradition]*, 21–50. The Institute of Cultural and Linguistic Studies, Keio University.

Nishiyama, Yuji and Koji Mineshima. (2006b). Property expressions and the semantics-pragmatics interface. Paper presented at the 3rd Łod Symposium, Łodz, Poland.

Nishiyama, Yuji and Koji Mineshima. (2007a). Contextualism and Indexicalism: the pragmatics of predicate nominals. Paper presented at 10th International Pragmatics Conference, Göteborg, Sweden.

Nishiyama, Yuji and Koji Mineshima. (2007b). Property expressions and the semantics-pragmatics interface. In Piotr Cap and Joanna Nijakowska (eds.) *Current Trends in Pragmatics*. 130–151. New Castle: Cambridge Scholars Publishing.

Nishiyama, Yuji and Koji Mineshima. (2010). Free enrichment and the over-generation problem. In Ewa Walaszewska et al. (eds.), *In the Mind* *and Across Minds: A Relevance-Theoretic Perspective on Communication and Translation*, 22–42. New Castle: Cambridge Scholars Publishing.

Nordström, Bengt, Kent Petersson, and Jan M. Smith. (1990). *Programming in Martin-Löf's Type Theory.* Oxford: Oxford University Press.

Okada, Mitsuhiro and Koji Mineshima (2009). Logic and dynamic state transition. (In Japanese). In S. Ishizaki et al. (eds.), *Gengo Shori-Gaku Jiten* (*Encyclopedia of Language Processing*). Tokyo: Kyoritu Syuppan.

Parsons, Terence. (1990). *Events in the Semantics of English.* Cambridge, MA: MIT Press.

Parsons, Terence. (2008). The development of supposition theory in the later 12th through 14th centuries. In D. Gabbay and J. Woods (eds.), *Handbook of the History of Logic. Volume 2: Mediaeval and Renaissance Logic*, Elsevier. 157–280.

Partee, Barbara H. (1986). Noun phrase interpretation and type-shifting principles. In J. Groenendijk, D. de Jongh and M. Stokhof (eds.), *Studies in Discourse Representation Theory and the Theory of Generalized Quantifiers*, 115–143. Dordrecht: Foris.

Perry, John. (1986). Thought without representation. *Proceedings of the Aristotelian Society, Supplementary Volume* LX, 137–151.

Perry, John. (2001). *Reference and Reflexivity.* Stanford, CA: CSLI Publications.

Pratt-Hartmann, Ian. (2003). A two-variable fragment of English. *Journal of Logic, Language and Information* 12, 13–45.

Pratt-Hartmann, Ian. (2004). Fragments of language. *Journal of Logic, Language and Information* 13, 207–223.

Pratt-Hartmann, Ian, and Allan Third. (2006). More fragments of language. *Notre Dame Journal of Formal Logic* 47, 151–177.

Pratt-Hartmann, Ian. (2009). No syllogisms for the numerical syllogistic. In O. Grumberg, M, Kaminski, S. Katz, S. Wintner (eds.), *Languages: From Formal to Natural*, *LNCS 5533*, Berlin, Heidelberg: Springer, 192–203.

Pratt-Hartmann, Ian, and Lawrence S. Moss. (2009). Logics for the relational syllogistic. *Review of Symbolic Logic* 2, 647–683.

Prawitz, Dag. (1965). *Natural Deduction.* Stockholm: Almqvist & Wiksell. Dover Edition, 2006.

Prawitz, Dag. (1971). Ideas and results in proof theory. In J.E. Fenstad (ed.), *Studies in Logic and the Foundations of Mathematics, Volume 63, Proceedings of the Second Scandinavian Logic Symposium.* Elsevier. 235–307.

Preyer, Gerhard and Georg Peter. (2007). *Context-Sensitivity and Semantic Minimalism: New Essays on Semantics and Pragmatics.* Oxford: Oxford University Press.

Ranta, Aarne. (1994). *Type-Theoretical Grammar.* Oxford: Oxford University Press, Oxford.

Recanati, François. (1989). The pragmatics of what is said. *Mind and Language* 4, 295–329.

Recanati, François. (2004). *Literal Meaning.* Cambridge: Cambridge University Press.

Recanati, François. (2011). *Truth-Conditional Pragmatics.* Oxford: Oxford University Press.

Rothschild, Daniel. (2007). Presuppositions and scope. *Journal of Philosophy* 104, 71–106.

Rothstein, Susan. (2006). Secondary predication. In M. Everaert and H. van Riemsdijk (eds.), *The Blackwell Companion to Syntax*, Volume IV, 209–233. Oxford: Blackwell.

Russell, Bertrand. (1905). On denoting. *Mind* 14. 479–493.

Russell, Bertrand. (1957). Mr. Strawson on referring. *Mind* 66. 385–389. Reprinted in B. Russell, *Essays in Analysis*, edited by D. Lackey. 120–126. London: George Allen & Unwin.

Russell, Bertrand and A. N. Whitehead. (1910). *Principia Mathematica*, Cambridge: Cambridge University Press. Second Edition: 1927.

Sánchez, Valencia, Víctor. (1991). *Studies on Natural Logic and Categorial Grammar*, Ph.D thesis, University of Amsterdam.

Sato, Yuri, Koji Mineshima, and Ryo Takemura. (2010a). The efficacy of Euler and Venn diagrams in deductive reasoning: Empirical findings. In A.K. Goel, M. Jamnik, and N.H. Narayanan (eds.), *Proceedings of Diagrams 2010*, *Lecture Notes in Computer Science*, Volume 6170, Springer-Verlag. 6–22.

Sato, Yuri, Koji Mineshima, and Ryo Takemura. (2010b). Constructing internal diagrammatic proofs from external logic diagrams. *Proceedings of 32nd Annual Conference of the Cognitive Science Society*. 2668–2673.

Saurer, Werner. (1993). A natural deduction system for discourse representation theory. *Journal of Philosophical Logic* 22, 249–302.

Schlenker, Philippe. (2008). Be Articulate: a pragmatic theory of presupposition projection. *Theoretical Linguistics* 34(3), 157–212.

Schlenker, Philippe. (2009). Local contexts. *Semantics and Pragmatics* 2(3), 1–78.

Schlenker, Philippe. (2011a). DRT with local contexts. *Natural Language Semantics* 19(4). 373–392.

Schlenker, Philippe. (2011b). The Proviso Problem: a note. *Natural Language Semantics* 19(4). 395–422.

Schock, Rolf. (1968). *Logics without Existence Assumptions.* Stockholm: Almqvist and Wiskell.

Schoubye, Anders J. (2009). Descriptions, truth value intuitions, and questions. *Linguistics and Philosophy* 32, 583–617.

Scott, Dana S. (1967) Existence and denotation in formal logic. In R. Schoenman (ed.), *Bertrand Russell — Philosopher of the Century*, 181–200. London: Allen and Unwin.

Scott, Dana S. (1979). Identity and existence in intuitionistic logic. In M. P. Fourman, C. J. Mulvey. and D. S. Scott (eds.), *Applications of sheaves. Lecture Notes in Mathematics*, vol. 753. Berlin: Springer, 660–696.

Shimojima, Atsushi. (1996). *On the Efficacy of Representation.* Ph.D. thesis, Indiana University.

Shin, Sun-Joo. (1994). *The Logical Status of Diagrams.* Cambridge: Cambridge University Press.

Shirai, Kokio. (1971). Intuitionistic predicate calculus with ε-symbol. *Annals of the Japan Association for Philosophy of Science* 4, 49–67.

Singh, Raj. (2007). Formal alternatives as a solution to the proviso problem. In T. Friedman and M. Gibson (eds.), *Proceedings of SALT 17*, 264–281.

Smiley, Timothy. (1960). Sense without denotation. *Analysis* 20, 125–135.

Smiley, Timothy. (1973) What is a syllogism? *Journal of Philosophical Logic* 1, 136–154.

Soames, Scott. (1987). Direct reference, propositional attitudes, and semantic content. *Philosophical Topics*, Volume XV, No. 1, 47–87.

Soames, Scott. (1989). Presupposition. In D. Gabbay and F. Guenthner (eds.), *Handbook of philosophical logic*, vol. 4, Dordrecht: Reidel. 553–616.

Sommers, Fred. (1982). *The Logic of Natural Language*. Cambridge: Cambridge University Press.

Sperber, Dan and Deirdre Wilson. (1986/95). *Relevance: Communication and Cognition*. First Edition 1986, Second Edition 1995. Oxford: Blackwell.

Sperber, Dan and Deirdre Wilson. (1998). The mapping between the mental and the public lexicon. In P. Carruthers and J. Boucher (eds.), *Language and Thought: Interdisciplinary Themes*. 184–200. Cambridge: Cambridge University Press.

Stalnaker, Robert. (1973). Presuppositions. *The Journal of Philosophical Logic* 2, 447–457.

Stalnaker, Rorbert. (1974). Pragmatic presuppositions. In Milton Munitz and Peter Unger (eds.), *Semantics and Philosophy*, 197–214. New York: New York University Press.

Stalnaker, Robert. (1978). Assertion. In P. Cole (ed.), *Syntax and Semantics 9: Pragmatics*. 315–332. New York: Academic Press.

Stanley, Jason. (2000). Context and logical form. *Linguistics and Philosophy* 23, 391–434.

Stanley, Jason. (2002). Nominal restriction. in G. Preyer and G. Peter (eds.), *Logical Form and Language*, 365–88. Oxford: Oxford University Press.

Stanley, Jason. (2005). Semantics in context. in G. Preyer and G. Peter (eds.), *Contextualism in Philosophy: Knowledge, Meaning, and Truth*, 221–253. Oxford: Oxford University Press.

Stanley, Jason and Zoltan Szabó. (2000). On quantifier domain restriction. *Mind and Language* 15, 219–61.

Stapleton, Gem. (2005). A survey of reasoning systems based on Euler diagrams. *Electronic Notes in Theoretical Computer Science* 134, Elsevier, 127–151.

Stapleton, Gem, John Howse, Peter Rodgers, and Leishi Zhang. (2008). Generating Euler diagrams from existing layouts. *Layout of (Software) Engineering Diagrams 2008, Electronic Communications of the EASST*, Volume 13, 16–31.

Stenlund, Sören. (1973). *The Logic of Description and Existence*. Philosophical Studies, Philosophical Society and the Department of Philosophy. Uppsala: University of Uppsala.

Stenlund, Sören. (1975). Descriptions in intuitionistic logic. In S. Kanger (ed.), *Proceedings of the Third Scandinavian Logic Symposium*. 197–212.

Strawson, Peter, F. (1950). On referring. *Mind* 59. 320–344.

Strawson, Peter, F. (1952). *Introduction to Logical Theory*. London: Methuen.

Sundholm, Göran. (1986). Proof theory and meaning. In D. Gabbay and F. Guenthner (eds.), *Handbook of philosophical logic, volume 3*, 471–506. Dordrecht: Reidel.

Sundholm, Göran. (1989). Constructive generalized quantifiers. *Synthese* 79(1), 1–12.

Swoboda, Nik and Gerard Allwein (2005). Heterogeneous reasoning with Euler/Venn diagrams containing named constants and FOL. *Electronic Notes in Theoretical Computer Science* 134, 153–187.

Szabó, Zoltán Gendler. (2000). Descriptions and uniqueness. *Philosophical Studies* 101. 29–57.

Szabó, Zoltán Gendler. (2003). On qualification. In J. Hawthorne (ed.), *Philosophical Perspectives 17: Language and Philosophical Linguistics*. 409–438.

Szabó, Zoltán Gendler. (2005). Loss of uniqueness. *Mind* 114, 1185–1222.

Taylor, Kenneth A. (2001). Sex, breakfast, and descriptus interruptus. *Synthese* 128, 45–61.

Troelstra, A. S. and H. Schwichtenberg. (2000). *Basic Proof Theory*. Second edition. Cambridge: Cambridge University Press.

van Benthem, Johan. (1986). *Essays in Logical Semantics*. Dordrecht: Reidel.

van Benthem, Johan. (2008). A brief history of natural logic, Technical Report PP-2008-05, Institute for Logic, Language and Computation, University of Amsterdam.

van Eijck, Jan. (1994). Presupposition failure—a comedy of errors. *Formal Aspects of Computing* 6A, 766–787.

van Eijck, Jan. (2007). Natural logic for natural language. *Lecture Notes in Computer Science*, Volume 4363, Springer, 216–230.

van Eijck, Jan. (2010). The language of social software. *Synthese* 177, 77–96.

van der Sandt, Rob A. (1992). Presupposition projection as anaphora resolution. *Journal of Semantics* 9. 333–377.

van Rooij, Robert. (2007). Strengthening conditional presuppositions. *Journal of Semantics* 24. 289–304.

Veltman, Frank. (1996). Defaults in update semantics. *Journal of Philosophical Logic* 25, 221–261.

Venn, John. (1881). *Symbolic Logic*. London: Macmillan.

von Plato, Jan. (2008). Gentzen's proof of normalization for natural deduction. *The Bulletin of Symbolic Logic* 14(2), 240–257.

Westerståhl, Dag. (1989). Aristotelian syllogisms and generalized quantifiers. *Studia Logica* XLVIII, 577–585.

Williams, Edwin. (1983). Semantic vs. syntactic categories. *Linguistics and Philosophy* 6, 423–446.

Wilson, Deirdre and Dan Sperber. (2002). Truthfulness and relevance. *Mind* 111, 583–632.

Wilson, Deirdre and Dan Sperber. (2005). Pragmatics. In F. Jackson and M. Smith (eds.), *Oxford Handbook of Philosophy.* 468–501. Oxford: Oxford University Press.

Zeevat, Henk. (1992). Presupposition and accommodation in update semantics. *Journal of Semantics* 9, 379–412.